Pi Formulas , Part 22

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Abstract

In this paper we show some formulas for the constant Pi

Resumen

En este artículo mostramos algunas fórmulas para la constante Pi

Introduction

The number Pi is defined by the series:

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-1} = 4\left(1-3^{-1}+5^{-1}-7^{-1}+...\right) = 3.141592... \qquad (1)$$

In this note there appear some formulae that involve the constant Pi

Formulas

$$\frac{\pi}{4\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n a_n}{(2n+1)!}\left(\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)\right)^{2n+1} \qquad (2)$$

$$\frac{\pi}{6\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n a_n}{(2n+1)!}\left(\ln\left(\frac{1+\sqrt{7}}{\sqrt{6}}\right)\right)^{2n+1} \qquad (3)$$

$$\frac{\pi}{8\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n a_n}{(2n+1)!}\left(\ln\left(\frac{-1+\sqrt{2}+\sqrt{5-2\sqrt{2}}}{\sqrt{2}}\right)\right)^{2n+1} \qquad (4)$$

$$\frac{\pi}{12\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n a_n}{(2n+1)!}\left(\ln\left(\frac{2-\sqrt{3}+\sqrt{9-4\sqrt{3}}}{\sqrt{2}}\right)\right)^{2n+1} \qquad (5)$$

En las fórmulas (2),(3),(4),(5), los números $a_n \in \mathbb{N} \cup \{0\}$, se definen por:

$$a_n = (-1)^n - \sum_{k=1}^{n} (-1)^k \, 2^{2k} \binom{2n}{2k} a_{n-k} \; , a_0 = 1, n \in \mathbb{N} \tag{6}$$

Para $0 < p < 1, 1 < b < 1 + 2a, a > 0$, se tiene:

$$\frac{\pi}{\sin(p\pi)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k} \, b^{n-k+p}}{n-k+p} + \sum_{n=0}^{\infty} \frac{(-1)^n \, b^{-(n+1-p)}}{n+1-p} \tag{7}$$

Para $a \geq 2, z = \dfrac{a\sqrt{a^2-4}+4-a^2}{2}$, se tiene:

$$\pi = 2\sum_{n=0}^{\infty} a^n \left(\frac{a-\sqrt{a^2-4}}{2} \right)^{n+1} \frac{(n!)^2}{(2n+1)!} F\left(n+1, n+1; 2n+2; z\right) \tag{8}$$

$F\left(\alpha, \beta; \gamma; z\right)$, es la clásica función hipergeometrica.

En la fórmula (8) , con $a = 2$, se tiene:

$$\pi = \sum_{n=0}^{\infty} \frac{2^{n+1} (n!)^2}{(2n+1)!} \tag{9}$$

Para $m \in \mathbb{N}$, se tiene:

$$\begin{aligned} \pi &= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{m+1}} + 8\sum_{k=1}^{m}\sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n+1)^{k+1}} = \\ &= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{m+1}} + 8\sum_{n=1}^{\infty} (-1)^n n \sum_{k=1}^{m} \frac{1}{(2n+1)^{k+1}} \end{aligned} \tag{10}$$

Para $m \in \mathbb{N} - \{1\}$, se tiene:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{mn+1} + 4(m-2)\sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n+1)(mn+1)} \tag{11}$$

$$\pi\sqrt{3} = 6\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \sum_{m=0}^{k} \frac{3^{-(2n-k+m)}}{m!(k-m)!(4n-2k+2m+1)} \tag{12}$$

$$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!}\sum_{m=0}^{k}\frac{\left(2^{1/2}-1\right)^{4n-2k+2m+1}}{m!(k-m)!(4n-2k+2m+1)} \tag{13}$$

Para $n \in \mathbb{N} \cup \{0\}, a > 0$, se tiene:

$$\frac{(2n)!(1+a)^{1/2}}{2^{6n+2}(n!)^2}\pi = \sum_{k=0}^{\infty}\frac{(-1)^k (2k)!}{2^{2k} k!(1+a)^k}\sum_{m=0}^{k}\frac{(-a)^{k-m} 2^{2m}\left((2n+m)!\right)^2}{m!(k-m)!(4n+2m+1)!} \tag{14}$$

En (14) con $n = 0, a = 1$, se tiene:

$$\frac{\pi\sqrt{2}}{4} = \sum_{k=0}^{\infty}\frac{(2k)!}{2^{3k} k!}\sum_{m=0}^{k}\frac{(-1)^m 2^{2m} m!}{(k-m)!(2m+1)!} \tag{15}$$

$$\pi = \lim_{n\to\infty}\left(4n\left(\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}\right)^2\right) \tag{16}$$

Para $m \in \mathbb{N} \cup \{0\}$, se tiene:

$$\pi = \left(\frac{2^{2m+1} m!}{(2m)!}\right)^2 \lim_{n\to\infty}\left(n^{2m+1}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+2m+1}\right)^2\right) \tag{17}$$

Para $m \in \mathbb{N} - \{1\}, a = 1, 2, 3, \ldots, m^2 - 1$, se tiene:

$$\pi = \frac{2\sqrt{2m^2 - a}}{m}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n} m^{2n} (n!)^2 (2n+1)}\sum_{k=0}^{n}\binom{2n+1}{2k}\left(2m^2 - a\right)^{n-k} a^k \tag{18}$$

$$\pi = 3 + 2\sin^{-1}\left(\cos\left(\frac{3}{2}\right)\right) = 3 + 2\sin^{-1}\left(\cos\left(1 - \frac{1}{2!}\left(\frac{3}{2}\right)^2 + \frac{1}{4!}\left(\frac{3}{2}\right)^4 - \ldots\right)\right) \tag{19}$$

Para $n \in \mathbb{N}$, se tiene:

$$\pi = 4\sum_{k=0}^{n}\frac{(-1)^k}{2k+1} + 2\sin^{-1}\left(\cos\left(2\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right)\right) \tag{20}$$

Para $m \in \mathbb{N} \cup \{0\}$, se tiene:

$$\pi = \sum_{n=0}^{m}\frac{a_n}{10^n} + 2\sin^{-1}\left(\cos\left(\frac{1}{2}\sum_{n=0}^{m}\frac{a_n}{10^n}\right)\right) \tag{21}$$

donde $a_n = \{3,1,4,1,5,9,2,6,\ldots\}$, son los digitos de π en base 10 .

Para $p,q\in\mathbb{N}, p\le q$, se tiene:

$$\pi = 4\sum_{n=1}^{\infty}\frac{a_n\left(p,q\right)}{\left(2n-1\right)q^{2n-1}} \tag{22}$$

donde

$$a_{n+2}\left(p,q\right)=\left(2q^2-4p^2\right)a_{n+1}\left(p,q\right)-q^4a_n\left(p,q\right) \quad ,n\in\mathbb{N} \tag{23}$$

$$a_1\left(p,q\right)=p \quad , \quad a_2\left(p,q\right)=3pq^2-4p^3 \tag{24}$$

En la fórmula (22), con $p=q=1$, se tiene $a_n\left(1,1\right)=\left(-1\right)^{n-1}$ y

$$\pi = 4\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{2n-1} \tag{25}$$

$$\pi = 8\sigma e^{-\sigma e^{-\sigma e^{-\sigma\cdots}}} \quad , \quad \pi = 8\lambda e^{\lambda e^{\lambda e^{\lambda\cdots}}} \tag{26}$$

donde

$$\sigma = \frac{1}{4}\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}e^{1/(4n-3)(4n-1)} = \frac{\pi}{8}e^{\pi/8} \tag{27}$$

$$\lambda = \frac{1}{4}\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}e^{-1/(4n-3)(4n-1)} = \frac{\pi}{8}e^{-\pi/8} \tag{28}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\left(-1\right)^k\binom{n}{2k+1}\left(c^{n-2k-1}s^{2k+1}+s^{n-2k-1}c^{2k+1}\right) \tag{29}$$

donde

$$c=\frac{\sqrt{7}+1}{4} \quad , \quad s=\frac{\sqrt{7}-1}{4} \tag{30}$$

La función $\left[x\right]= mayor\, entero \le x$.

$$\begin{aligned}\pi &= 12\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}2^{n/2}\sin\left(n\pi/4\right)}{n\left(1+3^{1/2}\right)^n}= \\ &= 12\sum_{n=1}^{\infty}\left(-1\right)^{n-1}\left(\frac{2}{\left(1+3^{1/2}\right)^2}\right)^{2n-1}\left(\frac{1+3^{1/2}}{2\left(4n-3\right)}-\frac{1}{4n-2}+\frac{1}{\left(1+3^{1/2}\right)\left(4n-1\right)}\right)\end{aligned} \tag{31}$$

$$\pi = 6\sum_{n=1}^{\infty}\frac{2^{n/2}\sin(n\pi/4)}{n\left(1+\sqrt{3}\right)^n} =$$

$$= 6\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{\left(1+\sqrt{3}\right)^2}\right)^{2n-1}\left(\frac{1+\sqrt{3}}{2(4n-3)}+\frac{1}{4n-2}+\frac{1}{\left(1+\sqrt{3}\right)(4n-1)}\right) \tag{32}$$

$$\pi = 4\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1/2^{1/2}}{(4n-3)(4n-3)!}-\frac{1}{(4n-2)(4n-2)!}+\frac{1/2^{1/2}}{(4n-1)(4n-1)!}\right)+$$
$$+4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(e^{-1/2^{1/2}}\sum_{m=0}^{2n}\frac{2^{-m/2}}{m!}\right) \tag{33}$$

$$\pi = 6\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1/2}{(6n-5)(6n-5)!}-\frac{\sqrt{3}/2}{(6n-4)(6n-4)!}+\right.$$
$$\left.+\frac{1}{(6n-3)(6n-3)!}-\frac{\sqrt{3}/2}{(6n-2)(6n-2)!}+\frac{1/2}{(6n-1)(6n-1)!}\right)+ \tag{34}$$
$$+6\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{2n+1}\left(e^{-\sqrt{3}/2}\sum_{m=0}^{2n}\frac{1}{m!}\left(\frac{\sqrt{3}}{2}\right)^m\right)$$

Para $p,q\in\mathbb{N}, 0<\frac{p}{q}<\frac{\sqrt{3}}{2}$, se tiene:

$$\pi\sqrt{3} = 6\frac{p^2}{q^2}-6\frac{p^2}{q^2}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{p}{q}\right)^{2n}\sum_{k=0}^{n}\binom{n}{k}\frac{3^{-k}}{2k+1}+$$
$$+\frac{6q^2}{p^2+q^2}\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\left(\frac{1}{3q^2\left(p^2+q^2\right)}\right)^n \tag{35}$$

donde

$$c_{n+2} = -\left(2p^2+q^2\right)q^2 c_{n+1} - p^2q^4\left(p^2+q^2\right)c_n \quad ,n\in\mathbb{N}\cup\{0\} \tag{36}$$

$$c_0 = 1 \quad , \quad c_1 = -\left(2p^2+q^2\right)q^2 \tag{37}$$

En (35) con $p=1, q=2$, se tiene:

$$\pi\sqrt{3} = \frac{3}{2}-\frac{3}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\,2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{3^{-k}}{2k+1}+\frac{24}{5}\sum_{n=0}^{\infty}\frac{c_n}{60^n\,(2n+1)} \tag{38}$$

donde

$$c_{n+2} = -24c_{n+1} - 80c_n \quad ,c_0 = 1, c_1 = -24, n \in \mathbb{N} \cup \{0\} \tag{39}$$

Para $0 \le a \le \frac{1}{2}$, se tiene:

$$\pi^2 = \frac{2}{a^2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{I_n(a)}{2^{2n}(2n+1)} \tag{40}$$

donde

$$I_n(a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1-(\cos(a\pi))^{2k+1}}{2k+1} \tag{41}$$

Para $0 \le a \le \frac{1}{2}$, se tiene:

$$\pi^2 = \frac{2}{a(1-a)} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{I_n(a)}{2^{2n}(2n+1)} \tag{42}$$

donde

$$I_n(a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\sin(a\pi))^{2k+1}}{2k+1} \tag{43}$$

Para $0 \le a \le \frac{1}{4}$, se tiene:

$$\pi^2 = \frac{2}{a^2} \sum_{n=0}^{\infty} \frac{I_n(a)}{2n+1} \tag{44}$$

donde

$$I_n(a) = \ln\left(\frac{1}{\cos(a\pi)}\right) + \sum_{k=1}^{n} \frac{(-1)^k}{2k} \binom{n}{k} \left(\left(\frac{1}{\cos(a\pi)}\right)^{2k} - 1\right) \tag{45}$$

$$\pi = 4\tan^{-1}\left(\frac{\tanh(1/2)}{\tan(1/2)}\right) + 4\tan^{-1}\left(\tan\left(\frac{1}{2}\right)\tanh\left(\frac{1}{2}\right)\right) + 4\sum_{n=1}^{\infty} \frac{(-1)^n a_n}{(2n+1)!} \tag{46}$$

donde

$$a_n = \mathrm{Im}\left(i\int_0^1 \frac{(1+ix)^{2n}}{\sin(1+ix)}\,dx \right) = \mathrm{Im}\left(\int_1^{1+i} \frac{x^{2n}}{\sin x}\,dx \right) \tag{47}$$

$$\pi = 4\tan^{-1}\left(\frac{\tan(1/2)}{\tanh(1/2)} \right) - 4\tan^{-1}\left(\tan\left(\frac{1}{2}\right)\tanh\left(\frac{1}{2}\right) \right) + 4\sum_{n=1}^{\infty} \frac{a_n}{(2n+1)!} \tag{48}$$

donde

$$a_n = \mathrm{Im}\left(i\int_0^1 \frac{(1+ix)^{2n}}{\sinh(1+ix)}\,dx \right) = \mathrm{Im}\left(\int_1^{1+i} \frac{x^{2n}}{\sinh x}\,dx \right) \tag{49}$$

$$\pi = 4\tan^{-1}\left(\frac{\tanh(1)}{\tan(1)} \right) + 4\tan^{-1}\left(\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(4n+3)!}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(4n+1)!}} \right) \tag{50}$$

$$\pi = 4\tan^{-1}\left(\frac{1-\cos(1)\cosh(1)}{\sin(1)\sinh(1)} \right) + 4\tan^{-1}\left(\frac{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n-1}\left(2n(4n-1)-1\right)}{(4n)!}}{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n-1}\left(2n(4n-1)+1\right)}{(4n)!}} \right) \tag{51}$$

$$\pi = 4\tan^{-1}\left(\frac{e\cos(1)-1}{e\sin(1)} \right) + 4\tan^{-1}\left(\frac{\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1}}{\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{2k}} \right) \tag{52}$$

$$\pi = 4\tan^{-1}\left(\frac{\ln 5}{4\tan^{-1}\left(\frac{\sqrt{5}-1}{2} \right)} \right) + 4\tan^{-1}\left(\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\,2^{2n+1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)\,2^{2n}}} \right) \tag{53}$$

Para $0 < b < a$, se tiene:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n \left(a^{n+1} - b^{n+1}\right)}{(a-b)\,a^{n+1}} \left(\frac{a}{2n+1} + \frac{b}{2n+3} \right) \tag{54}$$

Para $0<b<a\sqrt{3}, a\neq b$, se tiene:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(a^{n+1}-b^{n+1}\right)}{(a-b)(3a)^{n+1}}\left(\frac{3a}{2n+1}+\frac{b}{2n+3}\right) \tag{55}$$

Para $0<b<2a, 0<d<3c, a\neq b, c\neq d$, se tiene:

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{\left(a^{n+1}-b^{n+1}\right)}{(a-b)a^{n+1}2^{2n+3}}\left(\frac{4a}{2n+1}+\frac{b}{2n+3}\right)+\right.$$
$$\left.+\frac{\left(c^{n+1}-d^{n+1}\right)}{(c-d)c^{n+1}3^{2n+3}}\left(\frac{9c}{2n+1}+\frac{d}{2n+3}\right)\right) \tag{56}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{8+2\sqrt{3}}-\sqrt{3}+1}{\sqrt{2}\left(1+\sqrt{3}\right)}\right)^n\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{3}\right) \tag{57}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\sqrt{3}-1\right)^n\cos\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi}{3}\right) \tag{58}$$

$$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\sqrt{20+2\sqrt{3}}+\sqrt{3}-3}{2\left(1+\sqrt{3}\right)}\right)^n\cos\left(\frac{n\pi}{6}\right)\sin\left(\frac{n\pi}{3}\right) \tag{59}$$

Para $m\in\mathbb{N}\cup\{0\}$, se tiene:

$$\frac{\pi}{6\sqrt{3}}+\frac{\ln 3}{6} = \sum_{n=0}^{\infty}(-1)^n\binom{m+n}{n}\sum_{k=0}^{m}\binom{m}{k}\frac{2^{-3n-3k-1}}{3n+3k+1} \tag{60}$$

Para $a,b>0, 4b>a^2$, se tiene:

$$\frac{\pi}{2\sqrt{4b-a^2}} = \sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{a^{n-k}b^k}{n+k+1}\left(p^{n+k+1}-q^{n+k+1}\right) \tag{61}$$

donde

$$p=\frac{\sqrt{4b-a^2}-a}{2b} \quad , \quad q=-\frac{a}{2b} \tag{62}$$

En (61) con $a=b=1$,se tiene:

$$\frac{\pi}{2\sqrt{3}}=\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\left(\sqrt{3}-1\right)/2\right)^{n+k+1}-\left(-1/2\right)^{n+k+1}}{n+k+1} \tag{63}$$

Para $a,b>0$, se tiene:

$$\frac{\pi}{2b}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{a^2+b^2}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{a^{n-k}\left(p^{n+k+1}-q^{n+k+1}\right)}{2^k\left(n+k+1\right)} \tag{64}$$

donde

$$p=b-a \quad , \quad q=-a \tag{65}$$

En (64) con $a=b=1$, se tiene:

$$\pi=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\,2^{-k}}{n+k+1} \tag{66}$$

$$\pi=\frac{12}{5}\sum_{n=0}^{\infty}\frac{c_n}{(n+1)5^n} \tag{67}$$

donde

$$c_n=\left(\frac{3-2i}{6}\right)(2+3i)^n+\left(\frac{3+2i}{6}\right)(2-3i)^n \quad ,n\in\mathbb{N}\cup\{0\} \tag{68}$$

$$c_{n+2}=4c_{n+1}-13c_n \quad ,c_0=1,c_1=4 \tag{69}$$

$$\pi=\frac{16}{5}\sum_{n=0}^{\infty}\frac{c_n}{(n+1)5^n} \tag{70}$$

donde

$$c_n=\left(\frac{4-i}{8}\right)(1+4i)^n+\left(\frac{4+i}{8}\right)(1-4i)^n \quad ,n\in\mathbb{N}\cup\{0\} \tag{71}$$

$$c_{n+2}=2c_{n+1}-17c_n \quad ,c_0=1,c_1=2 \tag{72}$$

$$\pi=\frac{14}{5}\sum_{n=0}^{\infty}\frac{c_n}{(n+1)10^n} \tag{73}$$

donde

$$c_n=\left(\frac{7-3i}{14}\right)(3+7i)^n+\left(\frac{7+3i}{14}\right)(3-7i)^n \quad ,n\in\mathbb{N}\cup\{0\} \tag{74}$$

$$c_{n+2}=6c_{n+1}-58c_n \quad ,c_0=1,c_1=6 \tag{75}$$

$$\pi=\frac{18}{5}\sum_{n=0}^{\infty}\frac{c_n}{(n+1)10^n} \tag{76}$$

donde

$$c_n=\left(\frac{9-i}{18}\right)(1+9i)^n+\left(\frac{9+i}{18}\right)(1-9i)^n \quad ,n\in\mathbb{N}\cup\{0\} \tag{77}$$

$$c_{n+2}=2c_{n+1}-82c_n \quad ,c_0=1,c_1=2 \tag{78}$$

Para $a<0,b>0,2b>a^2$, se tiene:

$$\pi=\frac{8}{\sqrt{4b-a^2}}\sum_{n=0}^{\infty}\frac{c_n}{n+1}\left(\frac{2}{\sqrt{4b-a^2}-a}\right)^{n+1} \tag{79}$$

donde

$$c_{n+2}=-ac_{n+1}-bc_n \quad ,c_0=1,c_1=-a \tag{80}$$

Referencias

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