The Constant $F_{CG(i)}$

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abstract

In this note we show some formulas for the $F_{CG(i)}$ constant:

$$F_{CG(i)} = i + \cfrac{i}{i + \cfrac{i}{i + \dots}} = [i; i, i, i, i, i, \dots]$$

$F_{CG(i)}$: Complex and Algebraic number

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I. Introduction

Definition $F_{CG(i)}$

$$F_{CG(i)} = i + \cfrac{i}{i + \cfrac{i}{i + \dots}} = [i; i, i, i, i, i, \dots] \tag{1}$$

$$F_{CG(i)} = \sqrt{\frac{\sqrt{17} - 1}{8}} + i\left(\frac{1}{2} + \sqrt{\frac{2}{\sqrt{17} - 1}}\right) \tag{2}$$

$$F_{CG(i)} = 0.62481053\ldots + 1.30024259\ldots i \tag{3}$$

II. Formulas

Si $x = F_{CG(i)}$, entonces :

$$x^2 - i x - i = 0 \tag{4}$$

$$x^4 + x^2 + 2x + 1 = 0 \tag{5}$$

Sea x_n , $n \in \mathbb{N}$, la sucesión definida por :

$$x_{n+1} = i + \frac{i}{x_n} \quad , \quad x_1 = \frac{1}{2} + i \tag{6}$$

entoces:

$$\lim_{n \to \infty} x_n = F_{CG(i)} \tag{7}$$

Sea x_n , $n \in \mathbb{N}$, la sucesión definida por :

$$x_{n+1} = \frac{x_n^2 + i}{2\,x_n - i} \quad , \quad x_1 = \frac{1}{2} + i \tag{8}$$

entonces:

$$\lim_{n\to\infty} x_n = F_{\mathrm{CG}(i)} \tag{9}$$

Sea $x_n = u_n + i\,v_n$, $n \in \mathbb{N}$, entonces de la ecuación (6) :

$$u_{n+1} = \frac{v_n}{u_n^2 + v_n^2} \;,\;\; v_{n+1} = 1 + \frac{u_n}{u_n^2 + v_n^2} \;,\;\; u_1 = \frac{1}{2} \;,\;\; v_1 = 1 \tag{10}$$

y se cumple que:

$$\lim_{n\to\infty} u_n = \mathrm{Re}(F_{\mathrm{CG}(i)}) \;,\;\; \lim_{n\to\infty} v_n = \mathrm{Im}(F_{\mathrm{CG}(i)}) \tag{11}$$

Sea $x_n = u_n + i\,v_n$, $n \in \mathbb{N}$, entonces de la ecuación (8) :

$$u_{n+1} = \frac{2\,u_n(u_n^2 - v_n^2) + (2\,u_n\,v_n + 1)\,(2\,v_n - 1)}{4\,u_n^2 + (2\,v_n - 1)^2} \tag{12}$$

$$v_{n+1} = \frac{2\,u_n(2\,u_n\,v_n + 1) - (u_n^2 - v_n^2)\,(2\,v_n - 1)}{4\,u_n^2 + (2\,v_n - 1)^2} \tag{13}$$

$$u_1 = \frac{1}{2} \;,\;\; v_1 = 1 \tag{14}$$

y se cumple que:

$$\lim_{n\to\infty} u_n = \mathrm{Re}(F_{\mathrm{CG}(i)}) \;,\;\; \lim_{n\to\infty} v_n = \mathrm{Im}(F_{\mathrm{CG}(i)}) \tag{15}$$

Integral compleja para $F_{\mathrm{CG}(i)}$:

$$F_{\mathrm{CG}(i)} = i + \frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{2\,e^{3\,i\,t} + i\,e^{2\,i\,t}}{e^{2\,i\,t} + i\,e^{i\,t} - i}\,dt \tag{16}$$

$$\mathrm{Re}(F_{\mathrm{CG}(i)}) = \frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{3\cos(t) - \cos(2\,t) + 2\sin(2\,t) - 2\sin(3\,t)}{3 + 2\sin(t) - \cos(t)\,(2 + 4\sin(t))}\,dt \tag{17}$$

$$\mathrm{Im}(F_{\mathrm{CG}(i)}) = 1 + \frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{1 - 2\cos(2\,t) + 2\cos(3\,t) + 3\sin(t) - \sin(2\,t)}{3 + 2\sin(t) - \cos(t)\,(2 + 4\sin(t))}\,dt \tag{18}$$

Integral comleja para $1/F_{\mathrm{CG}(i)}$:

$$\frac{1}{F_{\mathrm{CG}(i)}} = \frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{2\,e^{3\,i\,t} + e^{2\,i\,t}}{e^{2\,i\,t} + e^{i\,t} + i}\,dt \tag{19}$$

$$\mathrm{Re}\left(\frac{1}{F_{\mathrm{CG}(i)}}\right) = \frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{1 + 3\cos(t) + 2\cos(2\,t) + \sin(2\,t) + 2\sin(3\,t)}{3 + 2\sin(t) + \cos(t)\,(2 + 4\sin(t))}\,dt \tag{20}$$

$$\mathrm{Im}\left(\frac{1}{F_{\mathrm{CG}(i)}}\right) = -\frac{1}{2\,\pi}\int_0^{2\,\pi} \frac{\cos(2\,t) + 2\cos(3\,t) - \sin(t)\,(3 + 4\cos(t))}{3 + 2\sin(t) + \cos(t)\,(2 + 4\sin(t))}\,dt \tag{21}$$

De la fórmula (2) se tiene:

$$\frac{1}{F_{\mathrm{CG}(i)}} = \frac{1}{4}\left(-2 + \sqrt{2\left(1 + \sqrt{17}\right)}\right) - \frac{i}{4}\sqrt{2\left(-1 + \sqrt{17}\right)} \tag{22}$$

representación usando la función hipergeometríca de Gauss:

$$F_{\mathrm{CG}(i)} = \frac{i}{2} + \left(\frac{1+i}{\sqrt{2}}\right) F(-1/2,\, 1;\, 1;\, -i/4) \tag{23}$$

$$\frac{1}{F_{\mathrm{CG}(i)}} = -\frac{1}{2} + \left(\frac{1-i}{\sqrt{2}}\right) F(-1/2,\, 1;\, 1;\, -i/4) \tag{24}$$

$$F_{\mathrm{CG}(i)} = \frac{i}{2} + \left(\frac{3+4\,i}{5}\right) F\left(-1/2,\, 1;\, 1;\, -\frac{3-4\,i}{100}\right) \tag{25}$$

una recurrencia:

$$c_{n+2} = i\, c_{n+1} + i\, c_n \;,\; c_1 = 1 \;,\; c_2 = i \tag{26}$$

$$c_n = \{1,\, i,\, -1+i,\, -2-i,\, -3\,i,\, 4-2\,i,\, 5+4\,i,\, -2+9\,i,\, -13+3\,i,\, -12-15\,i,\, ...\} \tag{27}$$

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = F_{\mathrm{CG}(i)} \tag{28}$$

algunas integrales:

$$\int_{-\infty}^{\infty} \frac{1}{t^2 - i\,t - i}\, dt = \frac{2\,i\,\pi}{2\,F_{\mathrm{CG}(i)} - i} \tag{29}$$

$$\int_{-\infty}^{\infty} \frac{1}{t^2 + t + i}\, dt = \frac{2\,i\,\pi}{2\,i\,F_{\mathrm{CG}(i)} + 1} \tag{30}$$

$$\int_{-\infty}^{\infty} \frac{1}{t^4 + t^2 + 2\,t + 1}\, dt = \sqrt{\frac{13 + 5\sqrt{17}}{34}}\;\pi \tag{31}$$

en la integral (31) se tiene:

$$t^4 + t^2 + 2\,t + 1 = (t - z)\,(t - \tilde{z})\left(t + \frac{i}{z}\right)\left(t - \frac{i}{\tilde{z}}\right) \tag{32}$$

donde $z = F_{\mathrm{CG}(i)}$, $\tilde{z}$ es el complejo conjugado de z .

Sea $C_n = A_n / B_n$, $n \in \mathbb{N}$, los convergentes de la fracción continua que define $a\, F_{\mathrm{CG}(i)}$, se tiene :

$$A_{n+2} = i\, A_{n+1} + i\, A_n \;,\; A_0 = 1 \;,\; A_1 = i \tag{33}$$

$$B_{n+2} = i\, B_{n+1} + i\, B_n \;,\; B_0 = 0 \;,\; B_1 = 1 \tag{34}$$

$$C_n \to F_{\mathrm{CG}(i)} \;,\; n \to \infty \tag{35}$$

$$C_n = \left\{i,\, 1+i,\, \frac{1+3\,i}{2},\, \frac{3+6\,i}{5},\, \frac{2+4\,i}{3},\, \frac{6+13\,i}{10},\, \frac{26+53\,i}{41},\, ...\right\} \tag{36}$$

Otra interesante recurrencia es:

$$x_{n+1} = \frac{1 + 11\,x_n + x_n^2 + x_n^4}{9} \;,\; x_1 = \frac{1}{2} + i \tag{37}$$

$$x_n \to F_{\mathrm{CG}(i)} \;,\; n \to \infty \tag{38}$$

Sea a_n , $n \in \mathbb{N}$, la sucesión definida por :

$$a_{n+4} = a_{n+3} + 2\,a_{n+2} + a_{n+1} - a_n \;,\; a_1 = 1,\; a_2 = 3,\; a_3 = 6,\; a_4 = 12 \tag{39}$$

se tiene:

$$\frac{a_{n+1}}{a_n} \to |F_{CG(i)}|^2 \ , \ n \to \infty \tag{40}$$

Sea a_n , $n \in \mathbb{N}$, la sucesión definida por :

$$a_{n+8} = 8\,a_{n+7} - 27\,a_{n+6} + 50\,a_{n+5} - 53\,a_{n+4} + 28\,a_{n+3} - 8\,a_{n+1} + 2\,a_n \tag{41}$$

$$a_1 = 1,\ a_2 = 8,\ a_3 = 37,\ a_4 = 130,\ a_5 = 388,\ a_6 = 1048,\ a_7 = 2671,\ a_8 = 6610 \tag{42}$$

se tiene:

$$\frac{a_{n+1}}{a_n} \to 1 + |F_{CG(i)}| \ , \ n \to \infty \tag{43}$$

Potencias de $F_{CG(i)}$:

$$(F_{CG(i)})^n = z_n\,F_{CG(i)} + w_n \ , \ n \in \mathbb{N} \tag{44}$$

$$z_{n+1} = i\,z_n + w_n \ , \ w_{n+1} = i\,z_n \ , \ z_1 = 1,\ w_1 = 0 \tag{45}$$

$$(F_{CG(i)})^n = (a_n\,i + b_n)\,F_{CG(i)} + c_n\,i + d_n \ , \ n \in \mathbb{N} \tag{46}$$

$$a_{n+1} = b_n + c_n \ , \ b_{n+1} = -a_n + d_n \ , \ c_{n+1} = b_n \ , \ d_{n+1} = -a_n \ , \ a_1 = 0,\ b_1 = 1,\ c_1 = 0,\ d_1 = 0 \tag{47}$$

La forma matricial de las ecuaciones (46) es:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} , \ \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{48}$$

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \ n \in \mathbb{N} \tag{49}$$

La forma matricial de las ecuaciones (44) es:

$$\begin{pmatrix} z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} i & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} z_n \\ w_n \end{pmatrix} , \ \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{50}$$

$$\begin{pmatrix} z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} i & 1 \\ i & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \ n \in \mathbb{N} \tag{51}$$

Sean a_n , b_n , $n \in \mathbb{N}$, las sucesiones definidas por :

$$a_{n+1} = i\,a_n + i\,b_n \ , \ b_{n+1} = a_n \ , \ a_1 = -3 - i \ , \ b_1 = -1 + 2\,i \tag{52}$$

Sea L_n , $n \in \mathbb{N}$, la sucesión definida por :

$$L_n = \frac{1 + 2\,i + a_1 + a_2 + \ldots + a_n}{2 + b_1 + b_2 + \ldots + b_n} \tag{53}$$

se tiene:

$$L_n \to F_{CG(i)} \ , \ n \to \infty \tag{54}$$

por (52) y (53) podemos escribir:

$$F_{CG(i)} = \frac{(1 + 2\,i) + (-3 - i) + (-1 - 4\,i) + (5 - 4\,i) + \ldots}{(2 + 0\,i) + (-1 + 2\,i) + (-3 - i) + (-1 - 4\,i) + \ldots} \tag{55}$$

Fracciónes continuas alternativas para $F_{CG(i)}$:

$$F_{CG(i)} = \frac{1}{2} + i - \frac{1-2i}{4+4i-}\ \frac{1-2i}{1+i-}\ \frac{1-2i}{4+4i-}\ \frac{1-2i}{1+i-} \ldots \tag{56}$$

$$F_{CG(i)} = \frac{1+i}{2} - \frac{1-2i}{2-}\ \frac{1-2i}{1-}\ \frac{1-2i}{2-}\ \frac{1-2i}{1-} \ldots \tag{57}$$

$$F_{CG(i)} = \frac{1+3i}{2} + \frac{1}{2+4i+}\ \frac{1}{1+2i+}\ \frac{1}{2+4i+}\ \frac{1}{1+2i+} \ldots \tag{58}$$

$$F_{CG(i)} = \frac{3+6i}{5} - \frac{(3-4i)/5}{6+7i-}\ \frac{3-4i}{6+7i-}\ \frac{3-4i}{6+7i-}\ \frac{3-4i}{6+7i-} \ldots \tag{59}$$

Ecuaciones diferenciales relacionadas:

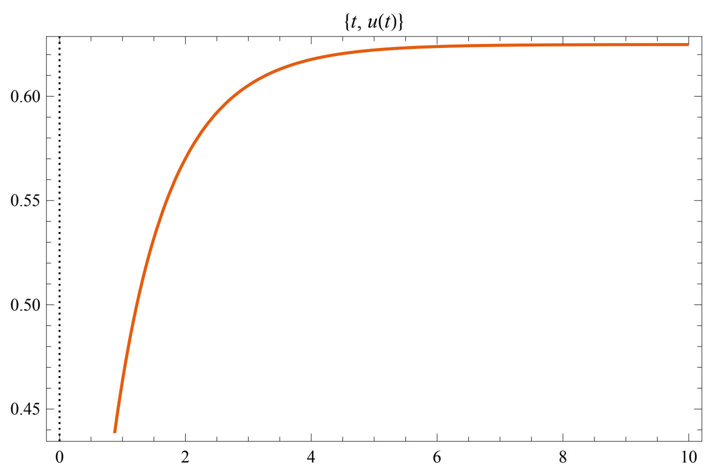
$$\frac{dx}{dt} = \frac{i\,e^{-t}}{2x-i}\ ,\ x(0) = i \tag{60}$$

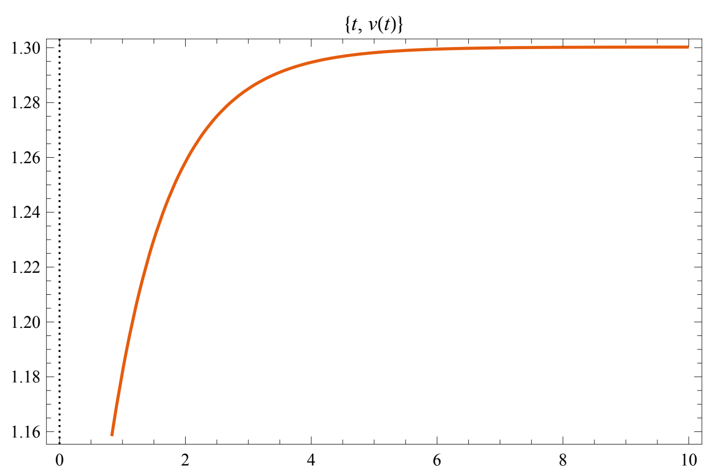
$$\lim_{t\to\infty} x(t) = F_{CG(i)} \tag{61}$$

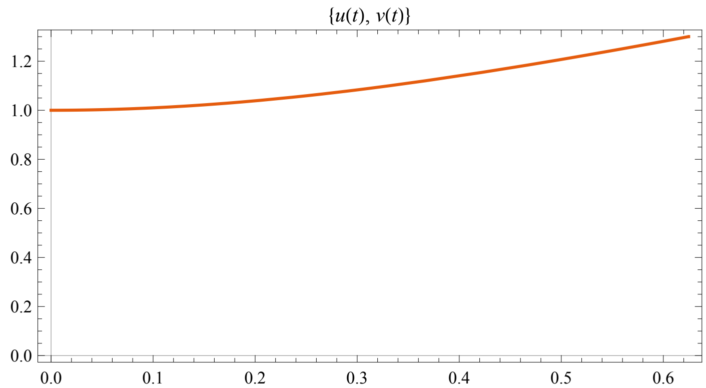
con $x(t) = u(t) + i\,v(t)$, se tiene :

$$\frac{du}{dt} = \frac{(2v-1)\,e^{-t}}{4u^2 + (2v-1)^2}\ ,\ \frac{dv}{dt} = \frac{2u\,e^{-t}}{4u^2 + (2v-1)^2}\ ,\ u(0) = 0,\ v(0) = 1 \tag{62}$$

$$\lim_{t\to\infty} u(t) = \mathrm{Re}(F_{CG(i)})\ ,\ \lim_{t\to\infty} v(t) = \mathrm{Im}(F_{CG(i)}) \tag{63}$$





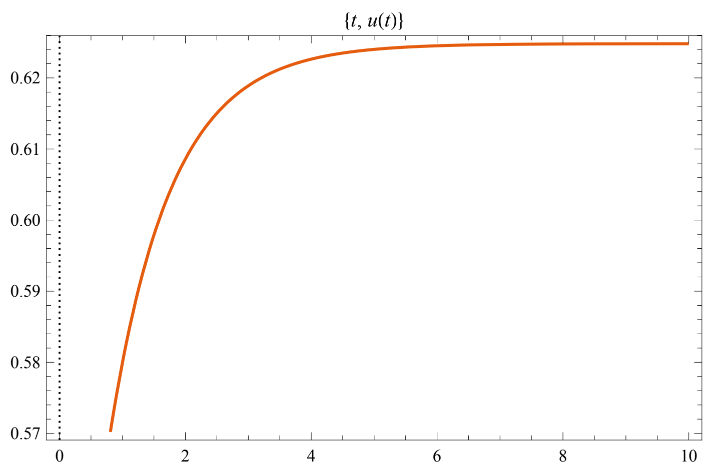


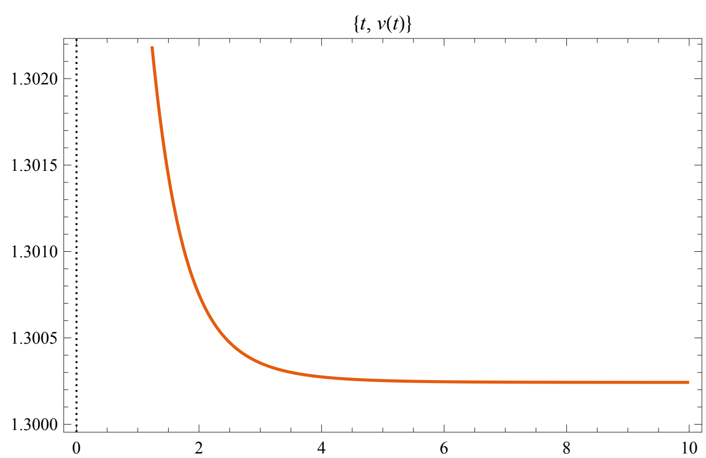
Otra ecuación diferencial es:

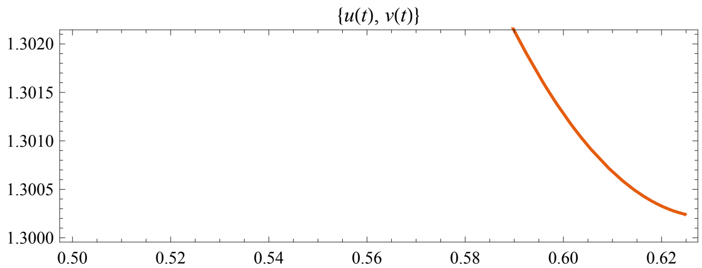
$$\frac{dx}{dt} = -\frac{e^{-t}}{4x^3 + 2x + 2}\ ,\ x(0) = \frac{1 + i\sqrt{7}}{2} \tag{64}$$

$$\lim_{t\to\infty} x(t) = F_{CG(i)} \tag{65}$$

con $x(t) = u(t) + i\, v(t)$, se tiene : $\lim_{t\to\infty} u(t) = \text{Re}(F_{CG(i)})$, $\lim_{t\to\infty} v(t) = \text{Im}(F_{CG(i)})$.







Serie general para $F_{CG(i)}$:

sea $x_0 \in \mathbb{C}$, $0 < \epsilon \ll 1$, tal que : $|x_0 - F_{CG(i)}| < \epsilon$, se tiene :

$$F_{CG(i)} = x_0 + \sum_{n=1}^{\infty} c_n \left(i + i\, x_0 - x_0^2\right)^n \tag{66}$$

donde $c_1 = \dfrac{1}{2\,x_0 - i}$, y para $n = 1, 2, 3, \ldots$:

$$c_{n+1} = -\frac{2\,c_1}{n+1} \sum_{k=0}^{n-1} (k+1)\, c_{k+1}\, c_{n-k} \tag{67}$$

algunos valores para x_0 son :

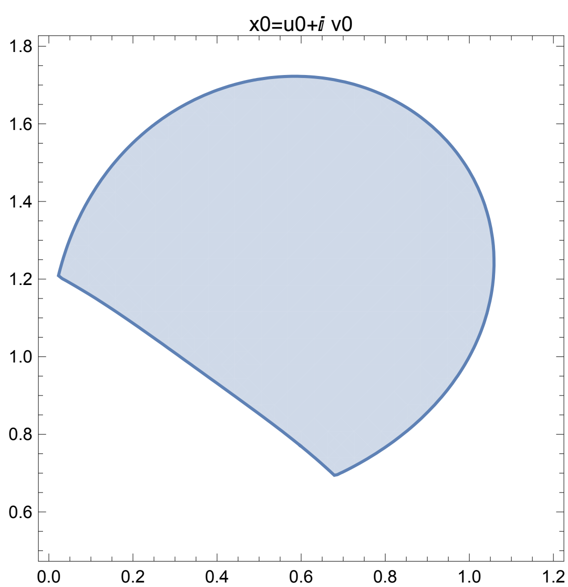
$$x_0 = \left\{ \frac{1+3\,i}{2}, \frac{3+6\,i}{5}, \frac{2+4\,i}{3}, \frac{6+13\,i}{10}, \frac{26+53\,i}{41}, \ldots \right\} \tag{68}$$

Ejemplo : $x_0 = \dfrac{3+6\,i}{5}$

$$F_{CG(i)} = \frac{3+6\,i}{5} + \sum_{n=1}^{\infty} c_n \left(\frac{-3+4\,i}{25}\right)^n \tag{69}$$

$$c_1 = \frac{6-7\,i}{17} \;,\; c_{n+1} = -\frac{2\,(6-7\,i)}{17} \sum_{k=0}^{n-1} (k+1)\, c_{k+1}\, c_{n-k} \tag{70}$$

la serie (66) converge más rapido si x_0 se encuentra en la región mostrada en la figura :



Una función generadora:

$$\frac{1}{1 - i\,x - i\,x^2} = \sum_{n=0}^{\infty} f_n\,x^n \tag{71}$$

$$f_{n+2} = i\,f_{n+1} + i\,f_n \;,\; f_0 = 1 \;,\; f_1 = i \tag{72}$$

$$f_n = \left(\frac{1 + F_{\text{CG}(i)}}{2 + F_{\text{CG}(i)}}\right)(F_{\text{CG}(i)})^n + \left(\frac{1}{2 + F_{\text{CG}(i)}}\right)\left(-\frac{i}{F_{\text{CG}(i)}}\right)^n \;,\; n = 0,\,1,\,2,\,3,\,\ldots \tag{73}$$

sea $f_n = u_n + i\,v_n$, se tiene :

$$u_{n+2} = -v_{n+1} - v_n \;,\; v_{n+2} = u_{n+1} + u_n \;,\; u_0 = 1,\, u_1 = 0,\, v_0 = 0,\, v_1 = 1 \tag{74}$$

$$u_{n+4} = -u_{n+2} - 2\,u_{n+1} - u_n \;,\; u_0 = 1,\, u_1 = 0,\, u_2 = -1,\, u_3 = 2 \tag{75}$$

$$v_{n+4} = -v_{n+2} - 2\,v_{n+1} - v_n \;,\; v_0 = 0,\, v_1 = 1,\, v_2 = 1,\, v_3 = -1 \tag{76}$$

$$f_{n+4} = -f_{n+2} - 2\,f_{n+1} - f_n \;,\; f_0 = 1,\, f_1 = i,\, f_2 = -1 + i,\, f_3 = -2 - i \tag{77}$$

$$\frac{f_{n+1}}{f_n} \to F_{\text{CG}(i)} \;,\; n \to \infty \tag{78}$$

$$\frac{f_{n+2}}{f_n} \to (F_{\text{CG}(i)})^2 \;,\; n \to \infty \tag{79}$$

$$\frac{f_{n+m}}{f_n} \to (F_{\text{CG}(i)})^m \;,\; n \to \infty \;,\; m = 0,\,1,\,2,\,3,\,\ldots \tag{80}$$

$$|f_{n+2}| \le |f_{n+1}| + |f_n| \;,\; n = 0,\,1,\,2,\,3,\,\ldots \tag{81}$$

$$|f_{n+4}| \le |f_{n+2}| + 2\,|f_{n+1}| + |f_n| \;,\; n = 0,\,1,\,2,\,3,\,\ldots \tag{82}$$

$$F_{\text{CG}(i)} = i - \sum_{n=1}^{\infty} \frac{(-i)^n}{f_{n-1}\,f_n} \tag{83}$$

$$F_{\text{CG}(i)} = i \prod_{n=1}^{\infty}\left(1 - \frac{(-i)^n}{f_n^2}\right) \tag{84}$$

Algunas relaciones : $F_{\text{CG}(i)} = F$

$$1 + F + F^2 + F^3 + F^4 = (F - 1)\,F\,(F + 1) \tag{85}$$

$$\left(F - \frac{1 + i\sqrt{7}}{2}\right)\left(F - \frac{1 - i\sqrt{7}}{2}\right)(F + 1)\,F = -1 \tag{86}$$

$$F^4 + (1 + F)^2 = 0 \tag{87}$$

$$F^2 + (F + 1)\left(F^3 - F^2 + F + 1\right) = 0 \tag{88}$$

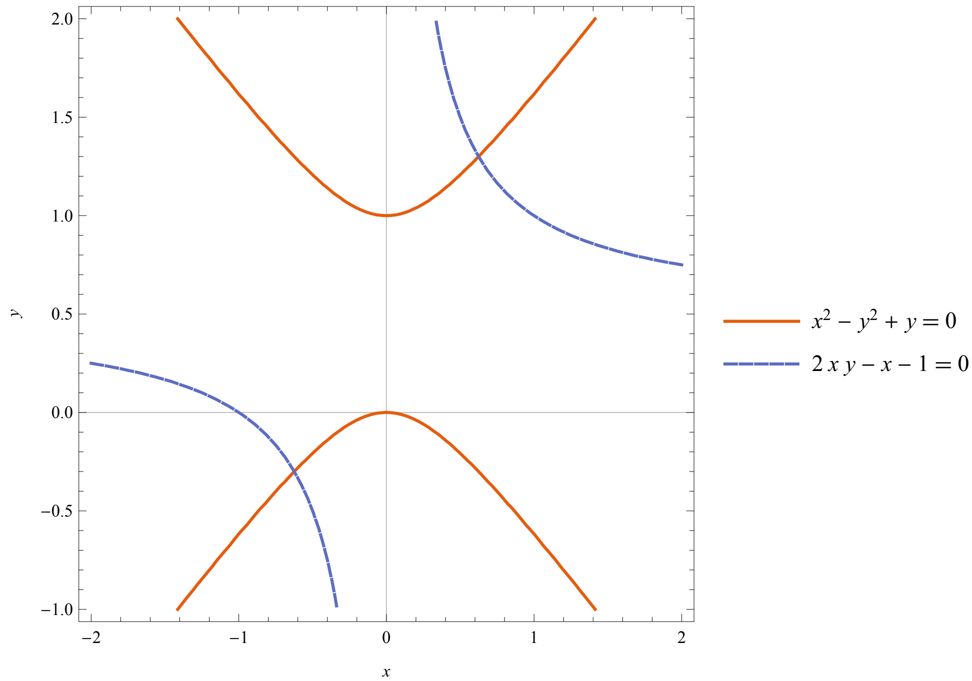
$$F + F^{-1} = -1 + (1 - i)\,F \tag{89}$$

$$F^2 + F^{-2} = 1 + 2\,i\,F \tag{90}$$

$$F^3 + F^{-3} = -2 + 2\,i - 2\,F \tag{91}$$

Curvas relacionadas:

$$\left\{x^2 - y^2 + y = 0\right\} \cap \{2\,x\,y - x - 1 = 0\} = \left\{F_{\text{CG}(i)},\, -\frac{i}{F_{\text{CG}(i)}}\right\} \tag{92}$$



Una fórmula para la constante Pi:

$$\frac{1}{\pi} = \frac{1}{16} \sum_{n=0}^{\infty} \left(\frac{1}{F_{\mathrm{CG}(i)}} \right)^n G(n) \tag{93}$$

$$G(n) = \sum_{k=[n/2]}^{n} \binom{k}{n-k} \binom{2k}{k}^3 (42\,k + 5)\, 2^{-12\,k}\, i^k \tag{94}$$

Otras representaciones:

$$\mathrm{Im}(F_{\mathrm{CG}(i)}) = \mathrm{Re}\left(\sqrt{i + \sqrt{i + \sqrt{i + \ldots}}} \right) = \mathrm{Re}\left(1 + \frac{i}{1+} \frac{i}{1+} \frac{i}{1+} \ldots \right) \tag{95}$$

$$\mathrm{Re}(F_{\mathrm{CG}(i)}) = \mathrm{Im}\left(\sqrt{i + \sqrt{i + \sqrt{i + \ldots}}} \right) = \mathrm{Im}\left(1 + \frac{i}{1+} \frac{i}{1+} \frac{i}{1+} \ldots \right) \tag{96}$$

III. References

A. Constantes matemáticas-wikipedia, la enciclopedia libre:https://es.wikipedia.org/wiki/Anexo:Constantes_matemáticas.