Pi Formulas , Part 18

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abstract

In this note we show some formulas related with the constant Pi

Número Pi , Fórmulas

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Resumen. En esta nota mostramos una colección de fórmulas que involucran a la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592\ldots$$

Notación:

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1)\,,(a)_0 = 1, n\in\mathbb{N}, a\in\mathbb{C}$$

$$Re(x+i\,y) = x\,, Im(x+i\,y) = y\,, x,y\in\mathbb{R}\,, i=\sqrt{-1}$$

Fórmulas

(1) $$\pi = 6e^{-1/\sqrt{3}}\sum_{n=1}^{\infty} a_n\left(\frac{1}{\sqrt{3}}\right)^n = 6e^{-1/\sqrt{3}}\sum_{n=1}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\sqrt{3}}\right)^n$$

(2) $$\pi = 8e^{-(\sqrt{2}-1)}\sum_{n=1}^{\infty} a_n\left(\sqrt{2}-1\right)^n = 8e^{-(\sqrt{2}-1)}\sum_{n=1}^{\infty}\frac{b_n}{n!}\left(\sqrt{2}-1\right)^n$$

(3) $$\pi = 12e^{-(2-\sqrt{3})}\sum_{n=1}^{\infty} a_n\left(2-\sqrt{3}\right)^n = 12e^{-(2-\sqrt{3})}\sum_{n=1}^{\infty}\frac{b_n}{n!}\left(2-\sqrt{3}\right)^n$$

(4) $$\pi = 6e^{1/\sqrt{3}}\sum_{n=1}^{\infty}(-1)^{n-1}a_n\left(\frac{1}{\sqrt{3}}\right)^n = 6e^{1/\sqrt{3}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{b_n}{n!}\left(\frac{1}{\sqrt{3}}\right)^n$$

(5) $$\pi = 8e^{\sqrt{2}-1}\sum_{n=1}^{\infty}(-1)^{n-1}a_n\left(\sqrt{2}-1\right)^n = 8e^{\sqrt{2}-1}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{b_n}{n!}\left(\sqrt{2}-1\right)^n$$

(6) $$\pi = 12e^{2-\sqrt{3}} \sum_{n=1}^{\infty} (-1)^{n-1} a_n \left(2-\sqrt{3}\right)^n = 12e^{2-\sqrt{3}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_n}{n!} \left(2-\sqrt{3}\right)^n$$

En las fórmulas (1)-(6) , se tiene:

$$a_{n+3} = \frac{1}{(n+1)!} + \frac{1}{n+3}\left(a_n - (n+1)a_{n+1} + a_{n+2}\right), n \in \mathbb{N}$$

$$b_{n+3} = 1 + (n+1)(n+2)(b_n - b_{n+1}) + b_{n+2}, n \in \mathbb{N},$$

$$\{a_n\} = \left\{1,1,\frac{1}{6},-\frac{1}{6},\frac{3}{40},\frac{11}{72},\ldots\right\}$$

$$\{b_n\} = \{1,2,1,-4,9,110,\ldots\}$$

$$b_n = n!\, a_n , n \in \mathbb{N}$$

(7) $$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \sum_{k=1}^{n} \frac{1}{(2k)!} + 4\sum_{n=0}^{\infty} (-1)^n I_n$$

donde

$$I_n = \sin 1 + 2n\cos 1 - 2n(2n-1)I_{n-1}, I_0 = \sin 1, n \in \mathbb{N}$$

$$I_n = p_n \sin 1 + q_n \cos 1 , n \in \mathbb{N} \cup \{0\}$$

$$p_n = 1 - 2n(2n-1)p_{n-1}, p_0 = 1, n \in \mathbb{N}$$

$$q_n = 2n - 2n(2n-1)q_{n-1}, q_0 = 0, n \in \mathbb{N}$$

(8) $$\pi = 2\sqrt{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^{-n}}{2n+1} \sum_{k=1}^{n} \frac{1}{(2k)!} + 6\sum_{n=0}^{\infty} (-1)^n I_n$$

donde

$$I_n = \left(\frac{1}{3}\right)^n \left(\sin\frac{1}{\sqrt{3}} + 2\sqrt{3}\, n\cos\frac{1}{\sqrt{3}}\right) - 2n(2n-1)I_{n-1}, I_0 = \sin\frac{1}{\sqrt{3}}, n \in \mathbb{N}$$

$$I_n = p_n \sin\frac{1}{\sqrt{3}} + q_n \cos\frac{1}{\sqrt{3}} , n \in \mathbb{N} \cup \{0\}$$

$$p_n = \left(\frac{1}{3}\right)^n - 2n(2n-1)p_{n-1}, p_0 = 1, n \in \mathbb{N}$$

$$q_n = 2\sqrt{3}\, n\left(\frac{1}{3}\right)^n - 2n(2n-1)q_{n-1}\,, q_0 = 0, n \in \mathbb{N}$$

(9) $$\pi = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\sqrt{2}-1\right)^{2n+1}}{2n+1}\sum_{k=1}^{n}\frac{1}{(2k)!} + 8\sum_{n=0}^{\infty}(-1)^n I_n$$

donde

$$I_n = \left(\sqrt{2}-1\right)^{2n-1}\left(\left(\sqrt{2}-1\right)\sin\left(\sqrt{2}-1\right) + 2n\cos\left(\sqrt{2}-1\right)\right) - 2n(2n-1)I_{n-1}\,, I_0 = \sin\left(\sqrt{2}-1\right), n \in \mathbb{N}$$

$$I_n = p_n \sin\left(\sqrt{2}-1\right) + q_n \cos\left(\sqrt{2}-1\right)\,, n \in \mathbb{N}\cup\{0\}$$

$$p_n = \left(\sqrt{2}-1\right)^{2n} - 2n(2n-1)p_{n-1}\,, p_0 = 1, n \in \mathbb{N}$$

$$q_n = 2n\left(\sqrt{2}-1\right)^{2n-1} - 2n(2n-1)q_{n-1}\,, q_0 = 0, n \in \mathbb{N}$$

(10) $$\pi = 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(2-\sqrt{3}\right)^{2n+1}}{2n+1}\sum_{k=1}^{n}\frac{1}{(2k)!} + 12\sum_{n=0}^{\infty}(-1)^n I_n$$

donde

$$I_n = \left(2-\sqrt{3}\right)^{2n-1}\left(\left(2-\sqrt{3}\right)\sin\left(2-\sqrt{3}\right) + 2n\cos\left(2-\sqrt{3}\right)\right) - 2n(2n-1)I_{n-1}\,, I_0 = \sin\left(2-\sqrt{3}\right), n \in \mathbb{N}$$

$$I_n = p_n \sin\left(2-\sqrt{3}\right) + q_n \cos\left(2-\sqrt{3}\right)\,, n \in \mathbb{N}\cup\{0\}$$

$$p_n = \left(2-\sqrt{3}\right)^{2n} - 2n(2n-1)p_{n-1}\,, p_0 = 1, n \in \mathbb{N}$$

$$q_n = 2n\left(2-\sqrt{3}\right)^{2n-1} - 2n(2n-1)q_{n-1}\,, q_0 = 0, n \in \mathbb{N}$$

(11) $$\pi = 3\sqrt{1+x^2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{1+x^2}{8x}\right)^{n-2k}$$

$$4-\sqrt{15} < x < 1$$

Ejemplos: $x = \frac{3}{4}\,, x = \frac{5}{12}\,, x = \frac{9}{40}$

$$\pi = \frac{15}{4}\sum_{n=0}^{\infty}\left(\frac{3}{8}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{25}{96}\right)^{n-2k}$$

$$\pi = \frac{39}{12}\sum_{n=0}^{\infty}\left(\frac{5}{24}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{169}{480}\right)^{n-2k}$$

$$\pi = \frac{123}{40}\sum_{n=0}^{\infty}\left(\frac{9}{80}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{1681}{2880}\right)^{n-2k}$$

(12) $$\pi = 2\sqrt{2(1+x^2)}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{1+x^2}{4x}\right)^{n-2k}$$

$$2-\sqrt{3} < x < 1$$

Ejemplo: $x = \frac{5}{12}$

$$\pi = \frac{13}{6}\sqrt{2}\sum_{n=0}^{\infty}\left(\frac{5}{24}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{169}{240}\right)^{n-2k}$$

(13) $$\pi = \frac{3}{2}\sqrt{3(1+x^2)}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{3(1+x^2)}{8x}\right)^{n-2k}$$

$$\frac{4-\sqrt{7}}{3} < x < 1$$

Ejemplo: $x = \frac{8}{15}$

$$\pi = \frac{17}{10}\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{8}{30}\right)^n \sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}\frac{(-1)^k}{2n-4k+1}\left(\frac{289}{320}\right)^{n-2k}$$

(14) $$\sqrt{\pi} = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k e^{-(n-k)^2}}{k!}\sum_{m=0}^{k}\binom{k}{m}\frac{(2n-2k)^m}{2k-m+1}$$

(15) $$\frac{\pi}{2\sqrt{3}} = 1 - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{-n}}{n(2n+1)} + \sum_{n=0}^{\infty}\frac{3^{-n-2}}{2n+5}\sum_{k=0}^{n}\binom{n+1}{k}\frac{(-1)^k}{n-k+2}$$

(16) $$\frac{\pi}{8} = \sqrt{2} - 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\sqrt{2}-1\right)^{2n+1}}{n(2n+1)} + \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}-1\right)^{2n+5}}{2n+5} \sum_{k=0}^{n} \binom{n+1}{k} \frac{(-1)^k}{n-k+2}$$

(17) $$\frac{\pi}{12} = 2 - \sqrt{3} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(2-\sqrt{3}\right)^{2n+1}}{n(2n+1)} + \sum_{n=0}^{\infty} \frac{\left(2-\sqrt{3}\right)^{2n+5}}{2n+5} \sum_{k=0}^{n} \binom{n+1}{k} \frac{(-1)^k}{n-k+2}$$

En las fórmulas (18)-(74) , aparece la función hypergeométrica de Gauss:

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

(18) $$\frac{\pi\left(3+\sqrt{3}\right)}{24} + \frac{\ln 3}{8} + \frac{\sqrt{3}\,\ln\left(2+\sqrt{3}\right)}{12} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{8n+1} F\left(n, 1; 2n+\frac{5}{4}; \frac{1}{2}\right)$$

(19) $$\frac{\pi\sqrt[4]{2}\left(3+\sqrt{3}\right)}{24} + \frac{\sqrt[4]{2}\ln 3}{8} + \frac{\sqrt[4]{2}\sqrt{3}\,\ln\left(2+\sqrt{3}\right)}{12} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{8n+1} F\left(2n+\frac{1}{4}, n+\frac{5}{4}; 2n+\frac{5}{4}; \frac{1}{2}\right)$$

(20) $$\frac{\pi\left(3+\sqrt{3}\right)}{48} + \frac{\ln 3}{16} + \frac{\sqrt{3}\,\ln\left(2+\sqrt{3}\right)}{24} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{8n+1} F\left(n+\frac{5}{4}, 1; 2n+\frac{5}{4}; -1\right)$$

(21) $$\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{8n+1} F\left(n+1,1; 2n+\frac{5}{4}; \frac{1}{2}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{8n+7} F\left(n+1,1; 2n+\frac{11}{4}; \frac{1}{2}\right)$$

(22) $$\frac{\pi\sqrt[4]{2}}{2\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-2n}}{8n+1}F\left(2n+\frac{1}{4},n+\frac{1}{4};2n+\frac{5}{4};\frac{1}{2}\right) + \frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-2n}}{8n+7}F\left(2n+\frac{7}{4},n+\frac{7}{4};2n+\frac{11}{4};\frac{1}{2}\right)$$

(23) $$\frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{8n+1}F\left(n+\frac{1}{4},1;2n+\frac{5}{4};-1\right) + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{8n+7}F\left(n+\frac{7}{4},1;2n+\frac{11}{4};-1\right)$$

(24) $$\frac{\pi}{\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{4n+1}F\left(n+1,1;\frac{4n+9}{8};\frac{1}{2}\right) + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{4n+7}F\left(n+1,1;\frac{4n+15}{8};\frac{1}{2}\right)$$

(25) $$\frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{4n+1}F\left(\frac{-4n+1}{8},1;\frac{4n+9}{8};-1\right) + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{4n+7}F\left(\frac{-4n+7}{8},1;\frac{4n+15}{8};-1\right)$$

(26) $$\frac{\pi}{2\sqrt{3}} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-(4n+1)/8}}{4n+1}F\left(\frac{4n+1}{8},\frac{-4n+1}{8};\frac{4n+9}{8};\frac{1}{2}\right) + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{-(4n+7)/8}}{4n+7}F\left(\frac{4n+7}{8},\frac{-4n+7}{8};\frac{4n+15}{8};\frac{1}{2}\right)$$

(27) $$\frac{\pi}{\sqrt{3}} = \frac{3}{2}F\left(1,\frac{1}{3};\frac{4}{3};-1\right) + \frac{3}{4}F\left(1,\frac{2}{3};\frac{5}{3};-1\right)$$

(28) $$\frac{\pi}{\sqrt{3}} = \frac{3}{4}F\left(1,1;\frac{4}{3};\frac{1}{2}\right) + \frac{3}{8}F\left(1,1;\frac{5}{3};\frac{1}{2}\right)$$

(29) $$\frac{\pi}{\sqrt{3}} = \frac{3}{2\sqrt[3]{2}}F\left(\frac{1}{3},\frac{1}{3};\frac{4}{3};\frac{1}{2}\right) + \frac{3\sqrt[3]{2}}{8}F\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};\frac{1}{2}\right)$$

(30) $$\frac{\pi}{\sqrt{3}}=\frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{n+1}\,F\left(n+1,1;\frac{n+3}{2};\frac{1}{2}\right)$$

(31) $$\frac{\pi}{\sqrt{3}}=3\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-n}}{n+1}\,F\left(-\frac{n-1}{2},1;\frac{n+3}{2};-1\right)$$

(32) $$\frac{\pi}{\sqrt{3}}=3\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-(n+1)/2}}{n+1}\,F\left(\frac{n+1}{2},-\frac{n-1}{2};\frac{n+3}{2};\frac{1}{2}\right)$$

(33) $$\frac{\pi}{\sqrt{3}}=\frac{3}{4}\sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}\,F\left(n+1,1;\frac{n+3}{2};\frac{1}{2}\right)$$

(34) $$\frac{\pi}{\sqrt{3}}=\frac{3}{2}\sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}\,F\left(-\frac{n-1}{2},1;\frac{n+3}{2};-1\right)$$

(35) $$\frac{\pi}{\sqrt{3}}=\frac{3}{2}\sum_{n=0}^{\infty}\frac{2^{-(n+1)/2}}{n+1}\,F\left(\frac{n+1}{2},-\frac{n-1}{2};\frac{n+3}{2};\frac{1}{2}\right)$$

(36) $$\frac{2\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 2^{-k-1}}{n+k+1}\,F\left(n+1,1;n+k+2;\frac{1}{2}\right)$$

(37) $$\frac{2\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 2^{-2k-1}}{n+k+1}\,F\left(n+k+1,k+1;n+k+2;\frac{1}{2}\right)$$

(38) $$\frac{2\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 2^{-k}}{n+k+1}\,F(k+1,1;n+k+2;-1)$$

(39) $$\pi=4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{n-k}(1+a)^{-n-1}}{n+k+1}\,F\left(n+1,1;n+k+2;\frac{a}{1+a}\right)$$

$$a>0$$

(40) $$\pi=4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{n-k}(1+a)^{-n-k-1}}{n+k+1}\,F\left(n+k+1,k+1;n+k+2;\frac{a}{1+a}\right)$$

$$a>0$$

(41) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k a^{n-k}(1+a)^{-n}}{n+k+1}\,F(k+1,1;n+k+2;-a)$$

$$a > 0$$

(42) $$\pi = 4a\,F\left(1,\frac{1}{2};\frac{3}{2};-a^2\right)+\frac{4(1-a)}{1+a}\,F\left(1,\frac{1}{2};\frac{3}{2};-\left(\frac{1-a}{1+a}\right)^2\right)$$

$$0 < a < 1$$

(43) $$\pi = \frac{4a}{1+a^2}\,F\left(1,1;\frac{3}{2};\frac{a^2}{1+a^2}\right)+\frac{2(1-a^2)}{1+a^2}\,F\left(1,1;\frac{3}{2};\frac{(1-a)^2}{2(1+a^2)}\right)$$

$$0 < a < 1$$

(44) $$\pi = \frac{4a}{\sqrt{1+a^2}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{a^2}{1+a^2}\right)+\frac{2\sqrt{2}(1-u)}{\sqrt{1+a^2}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{(1-a)^2}{2(1+a^2)}\right)$$

$$0 < a < 1$$

(45) $$\pi = 4a\,F\left(1,\frac{1}{2};\frac{3}{2};-a^2\right)-\frac{2i(1-a)}{a-i}\,F\left(1,1;2;-\frac{1-a}{a-i}\right)+\frac{2i(1-a)}{a+i}\,F\left(1,1;2;-\frac{1-a}{a+i}\right)$$

$$0 < a < 1\,, i = \sqrt{-1}$$

(46) $$\frac{\pi}{2\sqrt[4]{3}}\,F\left(\frac{1}{2},\frac{1}{2};1;\frac{2+\sqrt{3}}{4}\right) = 2\sqrt{\frac{1+\sqrt{3}}{2}}\,F\left(1,\frac{1}{2};\frac{3}{2};-a^2\right)+\frac{4(1-a)}{1+a}\,F\left(1,\frac{1}{2};\frac{3}{2};-\left(\frac{1-a}{1+a}\right)^2\right)$$

(47) $$\pi\,F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = 4\,F\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-1\right)$$

(48) $$\pi\,F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = \frac{4}{\sqrt[4]{2}}\,F\left(\frac{1}{4},\frac{3}{4};\frac{5}{4};\frac{1}{2}\right)$$

(49) $$\pi\,F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = 2\sqrt{2}\,F\left(\frac{1}{2},1;\frac{5}{4};\frac{1}{2}\right)$$

(50) $$\pi\,F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = 4\sqrt{2}\,F\left(\frac{3}{4},1;\frac{5}{4};-1\right)$$

(51) $$\pi\, F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(\sqrt{3}+1\right)^2}{8}\right)$$
$$= 2\sqrt[4]{3}\left(\sqrt{3}-1\right) F\left(\frac{1}{3},\frac{1}{2};\frac{4}{3};-\left(\sqrt{3}-1\right)^3\right) + 2\sqrt[4]{3}\sqrt[3]{a}\, F\left(\frac{1}{3},\frac{1}{2};\frac{4}{3};-a\right)$$
$$+\frac{4}{3}\sqrt[4]{3}\sqrt{1-a}\, F\left(\frac{1}{2},\frac{2}{3};\frac{3}{2};1-a\right)$$

$$0<a<1$$

(52) $$\pi\, F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(\sqrt{3}-1\right)^2}{8}\right) = 2\sqrt{2}\sqrt[4]{3}\sqrt{\sqrt{3}-1}\, F\left(\frac{1}{2},\frac{1}{6};\frac{7}{6};\left(\frac{\sqrt{3}-1}{2}\right)^3\right)$$

(53) $$\pi\, F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(\sqrt{3}-1\right)^2}{8}\right) = 3\sqrt{2}\left(\sqrt{3}-1\right) F\left(\frac{2}{3},1;\frac{7}{6};\left(\frac{\sqrt{3}-1}{2}\right)^3\right)$$

(54) $$\pi\, F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(\sqrt{3}-1\right)^2}{8}\right) = 4\sqrt{\frac{2}{3}}\, F\left(\frac{1}{2},1;\frac{7}{6};-\frac{2\sqrt{3}-3}{9}\right)$$

(55) $$\pi\, F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(\sqrt{3}-1\right)^2}{8}\right) = 2\sqrt{2}\sqrt[3]{2}\sqrt[3]{\sqrt{3}-1}\, F\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};-\frac{2\sqrt{3}-3}{9}\right)$$

(56) $$\pi = 8\sqrt[4]{a^3}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}a^n}{4n+3}\, F\left(1,\frac{4n+3}{4};\frac{4n+7}{4};-a\right)$$
$$+2\sqrt{1-a}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^n}{2n+1}\, F\left(\frac{1}{4},\frac{2n+1}{2};\frac{2n+3}{2};1-a\right)$$

$$0<a<1$$

(57) $$\pi = 8\left(\sqrt{2}-1\right)\sum_{n=0}^{\infty}\frac{(-1)^n 7^{-n-1}}{4n+1}\, F\left(n+1,1;\frac{4n+3}{2};\frac{6}{7}\right)$$
$$+8\left(\sqrt{2}+1\right)\sum_{n=0}^{\infty}\frac{(-1)^n 7^{-n-1}}{4n+3}\, F\left(n+1,1;\frac{4n+5}{2};\frac{6}{7}\right)$$

(58) $$\pi = 8\frac{\sqrt{2}-1}{\sqrt{7}}\sum_{n=0}^{\infty}\frac{(-1)^n 7^{-2n}}{4n+1}\, F\left(\frac{2n+1}{2},\frac{4n+1}{2};\frac{4n+3}{2};\frac{6}{7}\right)$$
$$+8\frac{\sqrt{2}+1}{\sqrt{7}}\sum_{n=0}^{\infty}\frac{(-1)^n 7^{-2n-1}}{4n+3}\, F\left(\frac{2n+3}{2},\frac{4n+3}{2};\frac{4n+5}{2};\frac{6}{7}\right)$$

(59) $$\pi = 2(\sqrt{2}-1)\sum_{n=0}^{\infty}\frac{(1/\sqrt{2})^n}{2n+1}\ F\left(n+1,1;\frac{2n+3}{2};\frac{3}{4}\right)$$

(60) $$\pi = 4(\sqrt{2}-1)\sum_{n=0}^{\infty}\frac{(1/\sqrt{2})^n}{2n+1}\ F\left(\frac{1}{2},\frac{2n+1}{2};\frac{2n+3}{2};\frac{3}{4}\right)$$

(61) $$\frac{1}{\pi}\ F\left(\frac{1}{4},\frac{3}{4};\frac{7}{4};\frac{1}{2}\right) = \frac{3}{2\sqrt[4]{2}}\ F\left(-\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) - \frac{3}{4\sqrt[4]{2}}\ F\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right)$$

(62) $$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}\ F\left(1,n+1;\frac{3}{2};\frac{1}{3}\right)$$

(63) $$\pi = 3\sqrt{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n}\ F\left(\frac{1}{2},-\frac{2n-1}{2};\frac{3}{2};\frac{1}{3}\right)$$

(64) $$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}\ F\left(1,-\frac{2n-1}{2};\frac{3}{2};-\frac{1}{2}\right)$$

(65) $$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}(-1)^n 2^{-n}\ F\left(\frac{1}{2},n+1;\frac{3}{2};-\frac{1}{2}\right)$$

(66) $$\pi = 12(2-\sqrt{3})\sum_{n=0}^{\infty}\frac{(-1)^n 15^{-n-1}}{4n+1}\ F\left(n+1,1;\frac{4n+3}{2};\frac{14}{15}\right) + 12(2+\sqrt{3})\sum_{n=0}^{\infty}\frac{(-1)^n 15^{-n-1}}{4n+3}\ F\left(n+1,1;\frac{4n+5}{2};\frac{14}{15}\right)$$

(67) $$\pi = 12\frac{2-\sqrt{3}}{\sqrt{15}}\sum_{n=0}^{\infty}\frac{(-1)^n 15^{-2n}}{4n+1}\ F\left(\frac{2n+1}{2},\frac{4n+1}{2};\frac{4n+3}{2};\frac{14}{15}\right) + 12\frac{2+\sqrt{3}}{\sqrt{15}}\sum_{n=0}^{\infty}\frac{(-1)^n 15^{-2n-1}}{4n+3}\ F\left(\frac{2n+3}{2},\frac{4n+3}{2};\frac{4n+5}{2};\frac{14}{15}\right)$$

(68) $$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sqrt{2}-1}{2}\right)^n\ F\left(\frac{1}{2},n+1;\frac{3}{2};-\frac{\sqrt{2}-1}{2}\right)$$

(69) $$\pi = 4\sum_{n=0}^{\infty}(-1)^n(\sqrt{2}-1)^{2n}\ F\left(1,-\frac{2n-1}{2};\frac{3}{2};-\frac{\sqrt{2}-1}{2}\right)$$

(70) $$\pi = 8\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{2}-1\right)^{2n+1}\ F\left(1,n+1;\frac{3}{2};\left(\sqrt{2}-1\right)^2\right)$$

(71) $$\pi = 4\sqrt{2}\sqrt{\sqrt{2}-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sqrt{2}-1}{2}\right)^n\ F\left(\frac{1}{2},-\frac{2n-1}{2};\frac{3}{2};\left(\sqrt{2}-1\right)^2\right)$$

(72) $$\pi = \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n 3^{-n}}{2n+1}\ F\left(1,n+1;\frac{2n+3}{2};\frac{2}{3}\right)$$

(73) $$\pi = 3\sum_{n=0}^{\infty}\frac{(-1)^n 3^{-n}}{2n+1}\ F\left(\frac{1}{2},\frac{2n+1}{2};\frac{2n+3}{2};\frac{2}{3}\right)$$

(74) $$\pi\sqrt{1+a^2}\,F\left(-\frac{1}{2},\frac{1}{2};1;\frac{a^2}{1+a^2}\right)$$

$$= 2\sqrt{b}\sum_{n=0}^{\infty}\frac{(-1)^n(-1/2)_n(a^2b)^n}{(2n+1)n!}\ F\left(\frac{1}{2},\frac{2n+1}{2};\frac{2n+3}{2};b\right)$$

$$+2\sqrt{1+a^2}\,\sqrt{1-b}\sum_{n=0}^{\infty}\frac{(-1/2)_n}{(2n+1)n!}\left(\frac{a^2(1-b)}{1+a^2}\right)^n\ F\left(\frac{1}{2},\frac{2n+1}{2};\frac{2n+3}{2};1-b\right)$$

$$0<a<1\ ,\ 0<b<1$$

(75) $$\pi = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!\,2^n}\ Im\left((1+3i)(1-i)^n\right)$$

(76) $$\pi = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!\,12^n}\ Im\left(\left(3+7\sqrt{3}\,i\right)\left(3-\sqrt{3}\,i\right)^n\right)$$

(77) $$\pi = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!\,4^n}\ Im\left(\left(2-\sqrt{2}+i\left(5\sqrt{2}-4\right)\right)\left(2-\sqrt{2}-i\left(3\sqrt{2}-4\right)\right)^n\right)$$

(78) $$\pi = \frac{3}{2}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!\,4^n}\ Im\left(\left(2-\sqrt{3}+i\left(9-4\sqrt{3}\right)\right)\left(2-\sqrt{3}-i\left(7-4\sqrt{3}\right)\right)^n\right)$$

(79) $$\pi = \frac{2^{2k}}{2^k-1}\ r_k\prod_{n=1}^{\infty}\frac{2^{2k}n(n+1)}{2^{2k}n(n+1)+2^k-1}$$

donde

$$r_{k+1} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - r_k^2}}\,, r_1 = 1\,, k \in \mathbb{N}$$

(80) $$\pi = 6\left(3\sqrt{2} + 4\right)\left(\ln\left(\frac{2 + \sqrt{3}}{2 + \sqrt{2}}\right) - \sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-4n}(2^n - 1)}{(2n)(2n+1)}\right)$$

(81) $$\pi = 3\left(\frac{\sqrt{6} - \sqrt{2}}{1 - \sqrt{6} + \sqrt{2}}\right)\left(-\ln\left(\left(2 - \sqrt{3}\right)\left(2 + \sqrt{2 + \sqrt{3}}\right)\right) + \sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-4n}\left(1 - \left(2 - \sqrt{3}\right)^n\right)}{(2n)(2n+1)}\right)$$

(82) $$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(1-a)^{k+1}a^{n-k}}{2k+1}$$

donde

$$a = -\frac{1}{3} + \sqrt[3]{\frac{17}{27} + \frac{1}{3}\sqrt{\frac{11}{3}}} + \sqrt[3]{\frac{17}{27} - \frac{1}{3}\sqrt{\frac{11}{3}}}$$

(83) $$\pi = \frac{8a^2}{1 + a^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2k+1}\left(\frac{1 - a^2}{1 + a^2}\right)^{n-k}\left(\frac{2a^2}{1 + a^2}\right)^k$$

$$0 < a < 1$$

(84) $$\pi = \sqrt[3]{r + \sqrt{64(\ln 2)^6 + r^2}} + \sqrt[3]{r - \sqrt{64(\ln 2)^6 + r^2}}$$

donde

$$r = 24\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)^3}$$

(85) $$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 3^{-k}}{2k+1}\left(\frac{\sqrt[3]{10} - 1}{3}\right)^{n-k}\left(\frac{4 - \sqrt[3]{10}}{3}\right)^{k+1}$$

(86) $$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(\sqrt{2}-1\right)^{2k+1}(1-a)^{k+1}a^{n-k}}{2k+1}$$

donde

$$a = -\frac{1}{3} + \sqrt[3]{\frac{53-36\sqrt{2}}{27} + \sqrt{\frac{283-200\sqrt{2}}{27}}} + \sqrt[3]{\frac{53-36\sqrt{2}}{27} - \sqrt{\frac{283-200\sqrt{2}}{27}}}$$

(87) $$\pi = 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\left(2-\sqrt{3}\right)^{2k+1}(1-a)^{k+1}a^{n-k}}{2k+1}$$

donde

$$a = -\frac{1}{3} + \sqrt[3]{\frac{125}{27} - \frac{8\sqrt{3}}{3} + i\sqrt{\frac{464\sqrt{3}}{9} - \frac{2411}{27}}} + \sqrt[3]{\frac{125}{27} - \frac{8\sqrt{3}}{3} - i\sqrt{\frac{464\sqrt{3}}{9} - \frac{2411}{27}}}$$

$$a = -\frac{1}{3} + \sqrt[3]{\frac{250-144\sqrt{3}}{27} + \frac{12\sqrt{3}-20}{3}\sqrt[3]{\frac{250-144\sqrt{3}}{27} + \cdots}}$$

(88) $$\pi = 2 + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)a^{2n+1}}$$

donde

$$a = 1 + \sum_{n=0}^{\infty}\frac{c_n}{n+1}$$

$$c_0 = 1\,,\, c_{2k-1} = \sum_{j=1}^{k}\frac{(-1)^{j-1}c_{2k-2j}}{(2j-1)!}\,,\, c_{2k} = \sum_{j=1}^{k}\frac{(-1)^{j-1}c_{2k-2j+1}}{(2j-1)!}\,,\, k \in \mathbb{N}$$

(89) $$\pi = 4\left(1 - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{2n-2m}{n-m}\binom{2m}{m}\frac{2^{-3n-1}}{(2n+3)(2m+1)}\right)$$

(90) $$\pi = 2\sqrt{3}\left(1 - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{2n-2m}{n-m}\binom{2m}{m}\frac{2^{-4n-2}}{(2n+3)(2m+1)}\right)$$

(91) $$\pi = 3\sqrt{3}\left(1 - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{2n-2m}{n-m}\binom{2m}{m}\frac{2^{-2n}(3/4)^{n+1}}{(2n+3)(2m+1)}\right)$$

(92) $$\pi = 12(2-\sqrt{3})\left(1 - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{2n-2m}{n-m}\binom{2m}{m}\frac{2^{-4n-2}(2-\sqrt{3})^{n+1}}{(2n+3)(2m+1)}\right)$$

(93) $$\pi = 8(\sqrt{2}-1)\left(1 - \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{2n-2m}{n-m}\binom{2m}{m}\frac{2^{-4n-2}(2-\sqrt{2})^{n+1}}{(2n+3)(2m+1)}\right)$$

(94) $$\pi = \frac{3\sqrt{3}\ln 3}{2} + \frac{3}{2}\sqrt{3(\ln 3)^2 - 12\sum_{n=0}^{\infty}\frac{(2/3)_n}{n!\,(3n+2)^2}}$$

(95) $$\pi = -\frac{3\sqrt{3}\ln 3}{2} + \frac{3}{2}\sqrt{3(\ln 3)^2 + 12\sum_{n=0}^{\infty}\frac{(1/3)_n}{n!\,(3n+1)^2}}$$

(96) $$\frac{1}{\pi} = \frac{5}{24}\sum_{n=0}^{\infty}6^{-2n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\left(\frac{3}{4}\right)^{2n-2k}(4k+1)\sum_{m=0}^{k}\binom{k}{m}^4$$

(97) $$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(2n-2k+1)2^k}$$

(98) $$\pi \ln 2 = 2\sum_{n=0}^{\infty}\frac{1}{(n+2)2^n}\sum_{k=0}^{n}\frac{c_k}{n-k+1}$$

donde

$$c_k = 2(c_{k-1} - c_{k-2})\,, c_0 = 1, c_1 = 2$$

(99) $$\pi \ln 2 = \frac{32}{9}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{c_k}{(2n-2k+1)(2n-k+2)3^{2n-k}}$$

donde

$$c_k = 2c_{k-1} - 5c_{k-2}\,, c_0 = 1, c_1 = 2$$

(100) $$\frac{1}{\pi} = \frac{\sqrt{5}}{6}\prod_{n=1}^{\infty}\frac{\sqrt{(12n-5)(12n-1)(12n+1)(12n+5)}}{(12n)^2}$$

Nota. Todas las fórmulas se han tomado de la referencia (5).

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