Pi Formulas , Part 17

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abstract

In this note we show some formulas related with the constant Pi

Fórmulas Para La Constante Pi

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Resumen. En esta nota mostramos algunas fórmulas para la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Introducción. En esta nota mostramos algunas fórmulas para la constante Pi ,todas las fórmulas se han tomado de la referencia (5).

Recordamos una típica fórmula para Pi , debida a Newton (1666):

$$\pi = \frac{3}{4}\sqrt{3} + 24\int_0^{1/4}\sqrt{x-x^2}\;dx = \frac{3}{4}\sqrt{3} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

La cual se obtiene usando las siguientes relaciones:

$$\int\sqrt{x-x^2}\;dx = \frac{1}{4}(2x-1)\sqrt{x-x^2} - \frac{1}{8}\sin^{-1}(1-2x)$$

$$\int\sqrt{x-x^2}\;dx = \frac{2}{3}x^{3/2} - \frac{1}{5}x^{5/2} - \frac{1}{28}x^{7/2} - \frac{1}{72}x^{9/2} - \cdots$$

Fórmulas

(1) $$\pi = 5\sqrt{5-2\sqrt{5}}\left(\frac{5}{6} + \sum_{n=1}^{\infty}\left(5-2\sqrt{5}\right)^{2n}\left(\frac{1}{4n+1} - \frac{1}{8n+6} - \frac{1}{40n-10}\right)\right)$$

(2) $$\pi = 5\sqrt{5-2\sqrt{5}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(5-2\sqrt{5}\right)^n$$

(3) $$\pi = \frac{5}{4}\sqrt{2\left(5-\sqrt{5}\right)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{5-\sqrt{5}}{8}\right)^n$$

(4) $$\pi = \frac{5}{8}\sqrt{2\left(5+\sqrt{5}\right)}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{5-\sqrt{5}}{8}\right)^n$$

(5) $$\pi = 10\sqrt{1-\frac{2}{\sqrt{5}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(1-\frac{2}{\sqrt{5}}\right)^n$$

(6) $$\pi = 5\sqrt{\frac{3-\sqrt{5}}{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{3-\sqrt{5}}{8}\right)^n$$

(7) $$\pi = \frac{5}{4}\sqrt{2\left(5-\sqrt{5}\right)}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{3-\sqrt{5}}{8}\right)^n$$

(8) $$\pi = \frac{5}{12}\sqrt{2\left(5+\sqrt{5}\right)}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{3+\sqrt{5}}{8}\right)^n$$

(9) $$\pi = \frac{5}{3}\sqrt{\frac{3+\sqrt{5}}{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{3+\sqrt{5}}{8}\right)^n$$

(10) $$\pi = 4\sqrt{2-\sqrt{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{2-\sqrt{2}}{4}\right)^n$$

(11) $$\pi = 4\sqrt{2-\sqrt{2}}\left\{\left(2-\sqrt{2}\right)\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{a_n}{4^n}\right)-\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{b_n}{4^n}\right)\right\}$$

donde

$$a_{n+1} = 4a_n - b_n \ , b_{n+1} = 2a_n \ , a_0 = 0, b_0 = -1, n \in \mathbb{N} \cup \{0\}$$

$$\{(a_n, b_n): n \in \mathbb{N} \cup \{0\}\} = \{(0,-1), (1,0), (4,2), (14,8), (48,28), \dots\}$$

(12) $$\pi = 6\sqrt{2-\sqrt{3}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{2-\sqrt{3}}{4}\right)^n$$

(13) $$\pi = 6\sqrt{2-\sqrt{3}}\left\{(2-\sqrt{3})\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{a_n}{4^n}\right)-\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{b_n}{4^n}\right)\right\}$$

donde

$$a_{n+1} = 4a_n - b_n \ , b_{n+1} = a_n \ , a_0 = 0, b_0 = -1, n \in \mathbb{N}\cup\{0\}$$

$$\{(a_n, b_n): n \in \mathbb{N}\cup\{0\}\} = \{(0,-1),(1,0),(4,1),(15,4),(56,15),\dots\}$$

(14) $$\pi = \frac{5}{2}\sqrt{\frac{5-\sqrt{5}}{2}}\left\{(5-\sqrt{5})\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{a_n}{8^n}\right)-\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{b_n}{8^n}\right)\right\}$$

donde

$$a_{n+1} = 10a_n - b_n \ , b_{n+1} = 20a_n \ , a_0 = 0, b_0 = -1, n \in \mathbb{N}\cup\{0\}$$

$$\{(a_n, b_n): n \in \mathbb{N}\cup\{0\}\} = \{(0,-1),(1,0),(10,20),(80,200),(600,1600),\dots\}$$

(15) $$\pi = 2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{2n+1} + \sqrt{2(\sqrt{5}-1)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{n}$$

(16) $$\pi = \sqrt{2(\sqrt{5}-1)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{n} + \sqrt{2(3-\sqrt{5})}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{3-\sqrt{5}}{2}\right)^{n}$$

(17) $$\pi = 2\sqrt{2-\sqrt{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}(2-\sqrt{2})^{n} + 2\sqrt{\sqrt{2}-1}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}(\sqrt{2}-1)^{n}$$

(18) $$\pi = 2\sqrt{\sqrt{3}-1}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}(\sqrt{3}-1)^{n} + 2\sqrt{2-\sqrt{3}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}(2-\sqrt{3})^{n}$$

(19) $$\pi = 2\sqrt{1-\frac{1}{\sqrt{5}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(1-\frac{1}{\sqrt{5}}\right)^n + 2\sqrt{\frac{1}{\sqrt{5}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{1}{\sqrt{5}}\right)^n$$

(20) $$\pi = 2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}c_n$$

donde

$$c_n = \sqrt{\frac{5+\sqrt{5}}{10}}\left(\frac{5+\sqrt{5}}{10}\right)^n + \sqrt{\frac{5-\sqrt{5}}{10}}\left(\frac{5-\sqrt{5}}{10}\right)^n \quad ,n \in \mathbb{N}\cup\{0\}$$

$$c_n = \sqrt{\frac{1}{5^n}\left(s_n+\frac{2}{\sqrt{5}}\right)} \quad ,s_{n+2} = 3s_{n+1}-s_n\ ,s_0 = 1, s_1 = 2$$

(21) $$\pi = 2\sqrt{\frac{5+2\sqrt{5}}{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{a_n}{10^n}\right) + 2\sqrt{5-2\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{b_n}{10^n}\right)$$

donde

$$a_{n+1} = 5a_n + 5b_n\ \ ,b_{n+1} = a_n + 5b_n\ \ ,a_0 = 1, b_0 = 0$$

(22) $$\pi = \frac{200}{25+25\sqrt{2}+8\sqrt{5(5+2\sqrt{5})}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}r_n$$

donde

$$r_n = \left(3-2\sqrt{2}\right)^n + \left(5-2\sqrt{5}\right)^n \quad ,n \in \mathbb{N}\cup\{0\}$$

$$r_{n+2} = \left(8-2\sqrt{2}-2\sqrt{5}\right)r_{n+1} - \left(15-10\sqrt{2}-6\sqrt{5}+4\sqrt{10}\right)r_n$$

$$r_0 = 2\ \ ,r_1 = 8-2\sqrt{2}-2\sqrt{5}$$

$$r_n = a_n + b_n\sqrt{2} + c_n\sqrt{5}$$

$$a_{n+2} = 8a_{n+1} - 4b_{n+1} - 10c_{n+1} - 15a_n + 20b_n + 30c_n$$

$$b_{n+2} = -2a_{n+1} + 8b_{n+1} + 10a_n - 15b_n - 20c_n$$

$$c_{n+2} = -2a_{n+1} + 8c_{n+1} + 6a_n - 8b_n - 15c_n$$

$$a_0 = 2, a_1 = 8, b_0 = 0, b_1 = -2, c_0 = 0, c_1 = -2$$

(23) $$\pi = \frac{200}{25 + 25\sqrt{2} - 8\sqrt{5\left(5 + 2\sqrt{5}\right)}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} r_n$$

donde

$$r_n = \left(3 - 2\sqrt{2}\right)^n - \left(5 - 2\sqrt{5}\right)^n \ , n \in \mathbb{N} \cup \{0\}$$

$$r_{n+2} = \left(8 - 2\sqrt{2} - 2\sqrt{5}\right) r_{n+1} - \left(15 - 10\sqrt{2} - 6\sqrt{5} + 4\sqrt{10}\right) r_n$$

$$r_0 = 0 \ , r_1 = -2 - 2\sqrt{2} + 2\sqrt{5}$$

$$r_n = a_n + b_n\sqrt{2} + c_n\sqrt{5}$$

$$a_{n+2} = 8a_{n+1} - 4b_{n+1} - 10c_{n+1} - 15a_n + 20b_n + 30c_n$$

$$b_{n+2} = -2a_{n+1} + 8b_{n+1} + 10a_n - 15b_n - 20c_n$$

$$c_{n+2} = -2a_{n+1} + 8c_{n+1} + 6a_n - 8b_n - 15c_n$$

$$a_0 = 0, a_1 = -2, b_0 = 0, b_1 = -2, c_0 = 0, c_1 = 2$$

(24) $$\pi = \frac{300}{50 + 25\sqrt{3} + 12\sqrt{5\left(5 + 2\sqrt{5}\right)}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} r_n$$

donde

$$r_n = \left(7 - 4\sqrt{3}\right)^n + \left(5 - 2\sqrt{5}\right)^n \ , n \in \mathbb{N} \cup \{0\}$$

$$r_{n+2} = \left(12 - 4\sqrt{3} - 2\sqrt{5}\right) r_{n+1} - \left(35 - 20\sqrt{3} - 14\sqrt{5} + 8\sqrt{15}\right) r_n$$

$$r_0 = 2 \ , r_1 = 12 - 4\sqrt{3} - 2\sqrt{5}$$

$$r_n = a_n + b_n\sqrt{2} + c_n\sqrt{5}$$

$$a_{n+2} = 12a_{n+1} - 12b_{n+1} - 10c_{n+1} - 35a_n + 60b_n + 70c_n$$

$$b_{n+2} = -4a_{n+1} + 12b_{n+1} + 20a_n - 35b_n - 40c_n$$

$$c_{n+2} = -2a_{n+1} + 12c_{n+1} + 14a_n - 24b_n - 35c_n$$

$$a_0 = 2, a_1 = 12, b_0 = 0, b_1 = -4, c_0 = 0, c_1 = -2$$

(25) $$\pi = \frac{300}{50 + 25\sqrt{3} - 12\sqrt{5\left(5 + 2\sqrt{5}\right)}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} r_n$$

donde

$$r_n = \left(7 - 4\sqrt{3}\right)^n - \left(5 - 2\sqrt{5}\right)^n \quad , n \in \mathbb{N} \cup \{0\}$$

$$r_{n+2} = \left(12 - 4\sqrt{3} - 2\sqrt{5}\right) r_{n+1} - \left(35 - 20\sqrt{3} - 14\sqrt{5} + 8\sqrt{15}\right) r_n$$

$$r_0 = 0 \ , r_1 = 2 - 4\sqrt{3} + 2\sqrt{5}$$

$$r_n = a_n + b_n\sqrt{2} + c_n\sqrt{5}$$

$$a_{n+2} = 12a_{n+1} - 12b_{n+1} - 10c_{n+1} - 35a_n + 60b_n + 70c_n$$

$$b_{n+2} = -4a_{n+1} + 12b_{n+1} + 20a_n - 35b_n - 40c_n$$

$$c_{n+2} = -2a_{n+1} + 12c_{n+1} + 14a_n - 24b_n - 35c_n$$

$$a_0 = 0, a_1 = 2, b_0 = 0, b_1 = -4, c_0 = 0, c_1 = 2$$

(26) $$\pi = \frac{12}{11}\sqrt{\frac{4 + \sqrt{2} + \sqrt{6}}{2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left(\frac{4 + \sqrt{2} + \sqrt{6}}{8}\right)^n$$

(27) $$\pi = \frac{12}{7}\sqrt{\frac{4 - \sqrt{2} + \sqrt{6}}{2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left(\frac{4 - \sqrt{2} + \sqrt{6}}{8}\right)^n$$

(28) $$\pi = \frac{12}{5}\sqrt{\frac{4 + \sqrt{2} - \sqrt{6}}{2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left(\frac{4 + \sqrt{2} - \sqrt{6}}{8}\right)^n$$

(29) $$\pi = 12\sqrt{\frac{4 - \sqrt{2} - \sqrt{6}}{2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left(\frac{4 - \sqrt{2} - \sqrt{6}}{8}\right)^n$$

(30) $$\pi = 8 \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{n-1}}{(2n - 2k + 1)\ b_{k-1,n-k+1}\ b_{k,n-k+1}}$$

donde

$$b_{k,n} = a_n b_{k-1,n} + b_{k-2,n} \ \ , b_{0,n} = 1 \ \ , b_{1,n} = a_n \ \ , n \in \mathbb{N}$$

$$a_n = \left(1+\sqrt{2}\right)\left(\left(3+2\sqrt{2}\right)^{n-1} - \left(3-2\sqrt{2}\right)^{n}\right)$$

$$a_{n+2} = 6a_{n+1} - a_n \ \ , a_1 = 2, a_2 = 14$$

(31) $$\pi = 12 \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(2n-2k+1)\ b_{k-1,n-k+1}\ b_{k,n-k+1}}$$

donde

$$b_{k,n} = a_n b_{k-1,n} - b_{k-2,n} \ \ , b_{0,n} = 1 \ \ , b_{1,n} = a_n \ \ , n \in \mathbb{N}$$

$$a_n = \left(2+\sqrt{3}\right)\left(7-4\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)\left(7+4\sqrt{3}\right)^n$$

$$a_{n+2} = 14a_{n+1} - a_n \ \ , a_1 = 4, a_2 = 52$$

(32) $$\pi = \left(2+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)\left(\frac{24}{11} - 528 \sum_{n=1}^{\infty} \frac{1}{(24n)^2 - 11^2}\right)$$

(33) $$\pi = \frac{528}{2+\sqrt{2}+\sqrt{3}+\sqrt{6}} \sum_{n=1}^{\infty} \frac{1}{(24n-12)^2 - 11^2}$$

(34) $$\pi = \frac{24}{11} \sum_{n=0}^{\infty} 2^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1-\left(3+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)^{-n-k-1}}{n+k+1}\right)$$

(35) $$\pi = \frac{24}{7} \sum_{n=0}^{\infty} 2^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1-\left(3-\sqrt{2}-\sqrt{3}+\sqrt{6}\right)^{-n-k-1}}{n+k+1}\right)$$

(36) $$\pi = \frac{12}{7} \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \left(2-\sqrt{2}-\sqrt{3}+\sqrt{6}\right)^{2k+1}}{2k+1}$$

(37) $$\pi = 24 \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n} \left(\frac{1}{\cos x + z\ \sin x}\right)^n$$

donde

$$z = 2+\sqrt{2}+\sqrt{3}+\sqrt{6} \ \ , -\pi < x < -\frac{\pi}{12} \ \ , 0 < x < \frac{11\pi}{12}$$

Ejemplos:

$$\pi = 12\sqrt{3}\ \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{1}{3n-2}\left(\frac{2}{1+z\sqrt{3}}\right)^{3n-2} + \frac{1}{3n-1}\left(\frac{2}{1+z\sqrt{3}}\right)^{3n-1}\right\}$$

$$\pi = 24\ \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{1/\sqrt{2}}{4n-3}\left(\frac{\sqrt{2}}{1+z}\right)^{4n-3} + \frac{1}{4n-2}\left(\frac{\sqrt{2}}{1+z}\right)^{4n-2} + \frac{1/\sqrt{2}}{4n-1}\left(\frac{\sqrt{2}}{1+z}\right)^{4n-1}\right\}$$

(38) $$\pi = \frac{5}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^{n}\left(5-2\sqrt{5}\right)^{n}\sum_{k=0}^{n}\binom{2k}{k}\left(\frac{1}{40}\right)^{k}\frac{(-1)^{k}}{2n-2k+1}$$

(39) $$\pi = \frac{5}{\sqrt{2}}\sum_{n=0}^{\infty}\left(5-2\sqrt{5}\right)^{n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\left(\frac{1}{40}\right)^{n-k}\frac{(-1)^{k}}{2k+1}$$

(40) $$\pi = \frac{1}{10}\sqrt{5\left(5+2\sqrt{5}\right)}\left(5+\sum_{n=1}^{\infty}\left(5-2\sqrt{5}\right)^{2n}\left(\frac{1}{4n-3}-\frac{10}{4n-1}+\frac{5}{4n+1}\right)\right)$$

(41) $$\pi = 5\sqrt{5-2\sqrt{5}}\ \left(1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n}\sum_{k=0}^{n}\binom{n-1}{k}\left(\frac{5-2\sqrt{5}}{5}\right)^{k}\frac{(-1)^{k}}{2n-2k+1}\right)$$

(42) $$\pi = 5\sqrt{5-2\sqrt{5}}\ \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}\left(5-2\sqrt{5}\right)^{k}}{2k+1}$$

(43) $$\pi = 5\ \sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}\left(\frac{3}{4}\right)^{2n+1}F\left(-n-\frac{1}{2},1,1,\frac{32\sqrt{5}-71}{9}\right)$$

donde $F(a,b,c,x)$, es la función hipergeométrica de Gauss.

Referencias

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