Pi Formulas

Part 8: Integrals , Series , Infinite Products

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abstract

In this note we give some formulas related to the constant Pi

ALGUNAS REPRESENTACIONES QUE INVOLUCRAN LA CONSTANTE π

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Resumen. Se muestran algunas fórmulas que involucran la constante π .

1. INTRODUCCIÓN.

En esta nota se muestran fórmulas, integrales, series, relacionadas con el número π .

2. FÓRMULAS.

2.1.

$$\frac{\pi}{2}+1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!(2n-1)}=Im\left(\int_{1}^{i}e^{1/x}dx\right) \quad ,i=\sqrt{-1}$$

$$\int_{1}^{i}e^{1/x}dx=(i-1)\int_{0}^{1}e^{(1+(i-1)x)^{-1}}dx=(i-1)\int_{0}^{\infty}\frac{e^{(x+1)/(x+i)}}{(x+1)^{2}}dx$$

2.2.

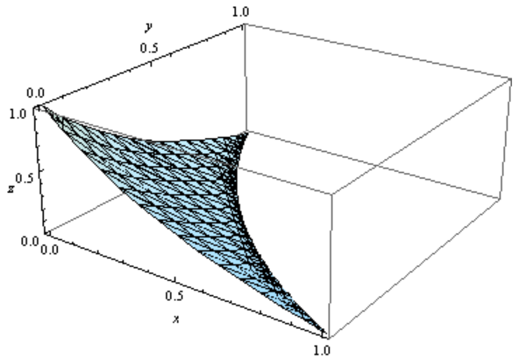
$$\frac{\pi}{4}=\int_{0}^{1}\int_{0}^{1}\frac{(xy+i)^{2}i+x}{(i-x)(xy+i)^{2}}dydx=\int_{0}^{1}\int_{0}^{1}\frac{(xy-i)^{2}i-x}{(i+x)(xy-i)^{2}}dydx \quad ,i=\sqrt{-1}$$

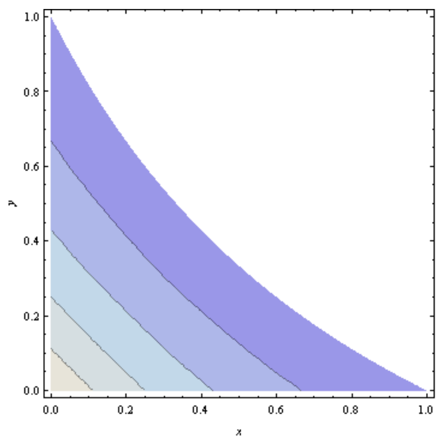
2.3.

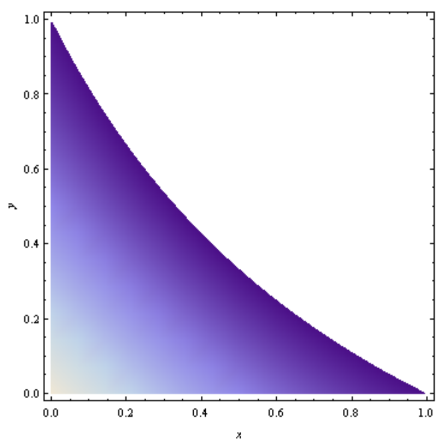
$$\frac{\pi}{4}=tan^{-1}(x)+tan^{-1}(y)+tan^{-1}(z)$$

$$z=\frac{1-xy-x-y}{1-xy+x+y}$$

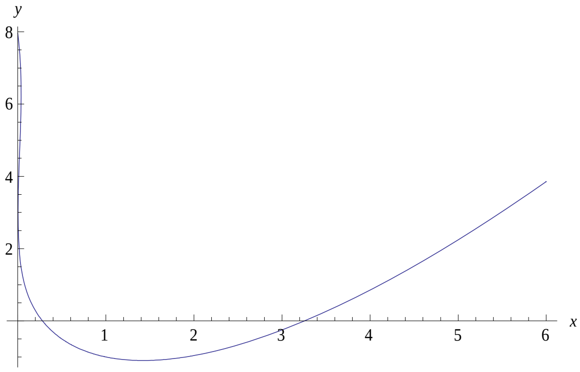
$$0 < x < 1 \,, 0 < y < 1 \,, 0 < \frac{x+y}{1-xy} < 1 \,, 0 < z < 1$$







2.4. La función $y(x)=\left(x-\frac{1}{2}\right)ln(x)-x \quad ,x>0$, tiene dos ceros reales en el intervalo $(0,\infty)$:



Sean a y b los ceros de $y(x)$ en $(0,\infty)$, entonces se verifica:

$$\pi=\frac{1}{2}(\Gamma(a))^2 exp\left(-2\int_0^\infty \frac{e^{-ay}}{y}\left(\frac{1}{e^y-1}-\frac{1}{y}+\frac{1}{2}\right)dy\right)$$

$$\pi=\frac{1}{2}(\Gamma(b))^2 exp\left(-2\int_0^\infty \frac{e^{-by}}{y}\left(\frac{1}{e^y-1}-\frac{1}{y}+\frac{1}{2}\right)dy\right)$$

Los valores para a y b son:

$$a=0.28001714796651...$$

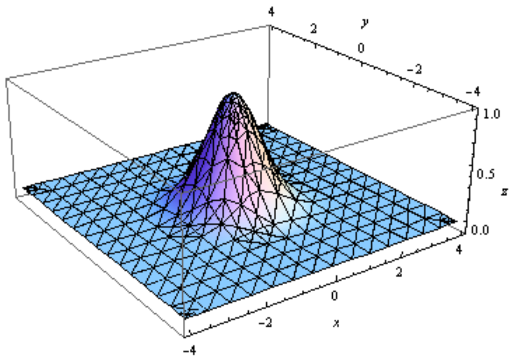
$$b=3.25847574678106...$$

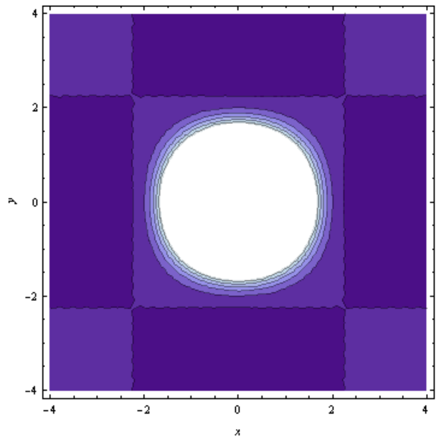
2.5. Sea $a=e^{-\frac{1}{2}e^{-\frac{1}{2}e^{-\cdots}}}=0.70346742...$, el número a satisface la ecuación: $a=e^{-a/2}$.
Se tiene:

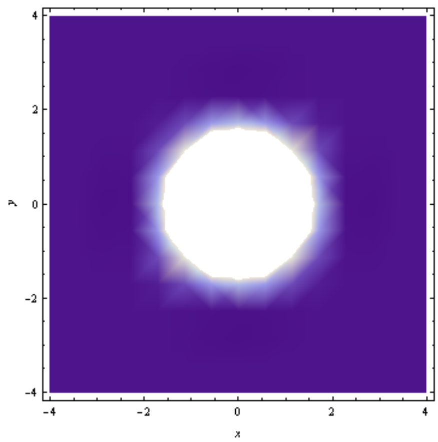
$$\sqrt{\pi}=\int_{-\infty}^{\infty} e^{-ax^2}\cos(ax)dx$$

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a\left(x^2+y^2\right)} cos\left(ax\right)cos\left(ay\right)dxdy$$

Gráficos de la función: $z\left(x,y\right)=e^{-a\left(x^2+y^2\right)} cos\left(ax\right)cos\left(ay\right) \ ,\left(x,y\right)\in\mathbb{R}^2$







2.6.

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-(ex)^2-2ex}dx$$

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-e^2\left(x^2+y^2\right)-2e(x+y)}dxdy$$

2.7. Sea $a = e^{-e^{-e^{-\cdots}}} = 0.567143...$,el número a,satisface la ecuación: $a = e^{-a}$.
Se tiene:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-\left((ax)^2+\frac{1}{(2x)^2}\right)}dx$$

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a\left(x^2+y^2\right)} cos\left(\left(2a\cos\theta\right)x\right)cos\left(\left(2asen\theta\right)y\right)dxdy$$

$$0 < \theta < \frac{\pi}{2}$$

2.8. Sea $a = e^{-2e^{-2e^{-2e^{-\cdots}}}} = 0.436302...$,el número a,satisface la ecuación: $a = e^{-2a}$.
Se tiene:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-\left((ax)^2+\frac{1}{x^2}\right)}dx$$

2.9. Para $m \in \mathbb{N}$, se tiene:

$$\frac{\pi}{6} = tan^{-1}\left(\frac{\sqrt{3}Li_m\left(-\frac{1}{3}\right)}{\Phi\left(-\frac{1}{3},m,\frac{1}{2}\right)}\right) + tan^{-1}\left(\frac{\sum_{n=1}^{\infty}(-1)^{n-1}H_{2n}^{m}3^{-n}}{\sqrt{3}\sum_{n=1}^{\infty}(-1)^{n-1}H_{2n-1}^{m}3^{-n}}\right)$$

$$H_n^m=\sum_{k=1}^{n}k^{-m}\ ,\ Li_m\left(-\frac{1}{3}\right)=\sum_{n=1}^{\infty}\frac{(-1/3)^n}{n^m}\ ,\ \Phi\left(-\frac{1}{3},m,\frac{1}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1/3)^n}{\left(n+\frac{1}{2}\right)^m}$$

2.10. Sea $m\in\mathbb{N}$, y $Li_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$, $|x|<1$, la función Polylogaritmo. Se tiene:

$$\frac{\pi^2}{6}-\frac{\pi}{2^{m+1}}tan^{-1}\left(\frac{\underbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}_{m-radicales}}{4-\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}}\right)=Re\left(Li_2\left(\frac{1}{2}e^{i\pi 2^{-m-1}}\right)\right)+$$

$$+Re\left(Li_2\left(1-\frac{1}{2}e^{i\pi 2^{-m-1}}\right)\right)-\frac{ln(2)}{2}ln\left(\frac{5}{4}-\frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}\right)$$

$$\frac{\pi}{2^{m+2}}ln\left(\frac{5}{4}-\frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}\right)=-Im\left(Li_2\left(\frac{1}{2}e^{i\pi 2^{-m-1}}\right)\right)$$

$$-Im\left(Li_2\left(1-\frac{1}{2}e^{i\pi 2^{-m-1}}\right)\right)-ln(2)tan^{-1}\left(\frac{\underbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}_{m-radicales}}{4-\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}}\right)$$

2.11.

$$\frac{1}{\sqrt{3}}+\frac{\pi}{6}=tan^{-1}\left(\frac{\sum_{n=0}^{\infty}(-1)^n h_{2n+1}3^{-n}}{\sqrt{3}\sum_{n=0}^{\infty}(-1)^n h_{2n}3^{-n}}\right)\quad ,\ h_n=\sum_{k=0}^{n}\frac{1}{k!}$$

2.12.

$$\frac{\pi}{6}+tan^{-1}\left(th\left(\frac{1}{\sqrt{3}}\right)\right)=tan^{-1}\left(\frac{\sum_{n=0}^{\infty}(-1)^n c_{2n+1}3^{-n}}{\sqrt{3}\sum_{n=0}^{\infty}(-1)^n c_{2n}3^{-n}}\right)$$

$$\frac{\pi}{6}+tan^{-1}(a)=tan^{-1}\left(\frac{\sum_{n=0}^{\infty}(-1)^n c_{2n+1}a^{2n+1}}{\sum_{n=0}^{\infty}(-1)^n c_{2n}a^{2n}}\right)$$

$$a=\frac{1}{2}ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$c_{2n}=-\frac{(-1)^n}{(2n+1)!}+\sum_{k=0}^{n}(-1)^k\left(\frac{1}{(2k)!}+\frac{1}{(2k+1)!}\right)\ ,n=0,1,2,...$$

$$c_{2n+1}=\sum_{k=0}^{n}(-1)^k\left(\frac{1}{(2k)!}+\frac{1}{(2k+1)!}\right)\ ,n=0,1,2,...$$

2.13.

$$\pi\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(a_1)_n\cdots(a_p)_n}{(n!)^2(b_1)_n\cdots(b_q)_n}a^n=2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{\left(\frac{1}{2}\right)_{n-m}}{(n-m)!}\frac{a^m}{m!(2n+1)}$$

$$p\le q+1\ ,|a|<1$$

$a_i\ ,i=1...p\ ,b_j\ ,j=1...q$, reales tales que las series son convergentes.

2.14.

$$\frac{\pi^2}{6}=\sum_{n=2}^{\infty}ln(n)ln\left(\frac{n^2}{n^2-1}\right)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m\,n^m}\int_0^1\frac{x^m}{(x+n)(x+n-1)}dx$$

2.15.

$$\begin{pmatrix}\frac{\pi}{4} & 0\\ 0 & \frac{\pi}{4}\end{pmatrix}=tan^{-1}\left(\begin{pmatrix}\frac{1}{2} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{2}\end{pmatrix}\right)+tan^{-1}\left(\begin{pmatrix}\frac{31}{77} & -\frac{24}{77}\\ -\frac{24}{77} & \frac{31}{77}\end{pmatrix}\right)$$

$$\frac{\pi}{4}=tan^{-1}\left(\frac{1}{6}\right)+tan^{-1}\left(\frac{5}{7}\right)$$

$$\frac{\pi}{4}=tan^{-1}\left(\frac{5}{6}\right)+tan^{-1}\left(\frac{1}{11}\right)$$

2.16.

$$\begin{pmatrix}\frac{\pi}{4} & 0\\ 0 & \frac{\pi}{4}\end{pmatrix}=tan^{-1}\left(\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{5}\end{pmatrix}\right)+tan^{-1}\left(\begin{pmatrix}\frac{41}{103} & -\frac{30}{103}\\ -\frac{40}{103} & \frac{77}{103}\end{pmatrix}\right)$$

$$\frac{\pi}{4}=tan^{-1}\left(\frac{7}{20}+\frac{\sqrt{381}}{60}\right)+tan^{-1}\left(\frac{59}{103}-\frac{2\sqrt{381}}{103}\right)$$

$$\frac{\pi}{4}=tan^{-1}\left(\frac{7}{20}-\frac{\sqrt{381}}{60}\right)+tan^{-1}\left(\frac{59}{103}+\frac{2\sqrt{381}}{103}\right)$$

2.17.

$$\begin{pmatrix}\frac{\pi}{4} & 0 & 0\\ 0 & \frac{\pi}{4} & 0\\ 0 & 0 & \frac{\pi}{4}\end{pmatrix}=tan^{-1}\left(\begin{pmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{6}\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6}\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6}\end{pmatrix}\right)+tan^{-1}\left(\begin{pmatrix}\frac{11}{23} & -\frac{6}{23} & -\frac{4}{23}\\ -\frac{12}{23} & \frac{17}{23} & -\frac{4}{23}\\ -\frac{12}{23} & -\frac{6}{23} & \frac{19}{23}\end{pmatrix}\right)$$

$$\frac{\pi}{4}=tan^{-1}\left(\frac{1}{23}\right)+tan^{-1}\left(\frac{11}{12}\right)$$

2.18. Para $0<a<\frac{1}{3}$, se tiene:

$$a^2\pi^2=2\sum_{n=1}^{\infty}\frac{\left(1-cos\left(a\pi\right)+i\,sen\left(a\pi\right)\right)^n+\left(1-cos\left(a\pi\right)-i\,sen\left(a\pi\right)\right)^n}{n^2}$$

2.19. Para $m\in\mathbb{N}_0=\{0,1,2,3,...\}$, se tiene:

$$\frac{\pi}{2^{m+1}}ln\left(2sen\left(\frac{\pi}{2^{m+2}}\right)\right)-2sen\left(\frac{\pi}{2^{m+2}}\right)F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\left(sen\left(\frac{\pi}{2^{m+2}}\right)\right)^{2}\right)=$$
$$=-\sum_{n=1}^{\infty}\frac{1}{n^{2}}sen\left(\frac{n\pi}{2^{m+1}}\right)$$

2.20. Para $0<a<1$, se tiene:

$$\frac{\pi^{2}}{12}=-ln(a)ln(1+a)+\sum_{n=0}^{\infty}\frac{(-1)^{n}a^{n+1}}{(n+1)^{2}}-\int_{a}^{1}\frac{ln(x)}{1+x}dx$$

2.21. Para $0\le a<\frac{\pi}{2}$, se tiene:

$$\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)B_{n}a^{2n-1}}{(2n)!(2n-1)}+\int_{a}^{\infty}\frac{tan(x)}{x}dx$$

B_n números de Bernoulli

2.22. Para $a>0,b>0,0\le u\le 1$, se tiene:

$$\frac{\pi}{2}ln\frac{a}{b}=\sum_{n=0}^{\infty}\frac{(-1)^{n}\left(a^{2n+1}-b^{2n+1}\right)u^{2n+1}}{(2n+1)^{2}}+\int_{u}^{\infty}\frac{tan^{-1}(ax)-tan^{-1}(bx)}{x}dx$$

2.23. Para $a\ge 0,0<p<1$, se tiene:

$$\frac{\pi}{2\Gamma(p)sen\left(\frac{p\pi}{2}\right)}=\sum_{n=0}^{\infty}\frac{(-1)^{n}a^{2n+2-p}}{(2n+1)!(2n+2-p)}+\int_{a}^{\infty}\frac{sen(x)}{x^{p}}dx$$

$$\frac{\pi}{2\Gamma(p)cos\left(\frac{p\pi}{2}\right)}=\sum_{n=0}^{\infty}\frac{(-1)^{n}a^{2n+1-p}}{(2n)!(2n+1-p)}+\int_{a}^{\infty}\frac{cos(x)}{x^{p}}dx$$

2.24. Para $m\in\mathbb{N}$,se tiene:

$$\frac{\pi}{\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}}=2^{2m+1}\sum_{n=0}^{\infty}\frac{1}{\left(2^{m+2}n+2^{m}+1\right)\left(2^{m+2}n+3\cdot 2^{m}+1\right)}+$$
$$+\int_{\pi/4}^{\pi/2}\left(tan(x)\right)^{2^{-m}}dx$$

2.25. Para $0<u<1,0<v\le 1,-1<m<1,0<\theta<\pi$, se tiene:

$$\frac{\pi}{sen(m\pi)}\frac{sen(m\theta)}{sen(\theta)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}sen(n\theta)}{sen(\theta)}\left(\frac{u^{n+m}}{n+m}+\frac{v^{n-m}}{n-m}\right)+$$

$$+\int_{v}^{1/u}\frac{x^{-m}}{1+2x\cos(\theta)+x^2}dx$$

2.26. Para $0<u\le 1,-1<m<1,0<\theta<\pi$, se tiene:

$$\frac{\pi}{sen(m\pi)}\frac{sen(m\theta)}{sen(\theta)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}sen(n\theta)}{sen(\theta)}\left(\frac{u^{n+m}}{n+m}\right)+$$

$$+\int_{u}^{\infty}\frac{x^{m}}{1+2x\cos(\theta)+x^2}dx$$

2.27. Para $m=0,1,2,3,...$, se tiene:

$$\frac{(2m)!}{2^{2m}(m!)^2}\sqrt{\frac{\pi}{2}}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2m+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n-1}}$$

2.28. Para $m=1,2,3,...$, se tiene:

$$\frac{(m-1)!\,m!\,2^{2m}}{(2m)!\sqrt{2\pi}}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2m)^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n-1}}$$

2.29. Para $m=0,1,2,3,...$, se tiene:

$$\pi^{2^m}=2^{2^m}a_m\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n+m}(k+1)^{(-1)^{k+1}\binom{n+m}{k}}\right)^{2^{-n}}$$

$$a_0=1,a_1=2,a_2=2^4 3^{-1},a_3=2^{13}3^{-5},a_4=2^{38}3^{-16}5^{-1},....$$

$$a_{m+1}=b_m a_m^2$$

$$b_m = 2^{\binom{m+1}{1}} 3^{-\binom{m+1}{2}} 4^{\binom{m+1}{3}} 5^{-\binom{m+1}{4}} \ldots (m+2)^{(-1)^m}$$

2.30. Para $0 < x < \frac{1}{2}$, se tiene:

$$exp\left(\pi \tan(\pi x)\right) = \prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} \left(\frac{2k+2x+1}{2k-2x+1} \right)^{(-1)^k \binom{n}{k}} \right)^{\frac{1}{n+1}}$$

2.31.

$$\frac{\pi}{2} \frac{e^{\sqrt{2}} - e^{-\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \, sen(n)}{n^2+2}$$

$$\frac{\pi}{2} \frac{e^{2\sqrt{2}} - e^{-2\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \, sen(2n)}{n^2+2}$$

$$\frac{\pi}{2} \frac{e^{3\sqrt{2}} - e^{-3\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \, sen(3n)}{n^2+2}$$

2.32.

$$\frac{\pi}{2\sqrt{2}} \frac{e^{\sqrt{2}} + e^{-\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{n^2+2}$$

$$\frac{\pi}{2\sqrt{2}} \frac{e^{2\sqrt{2}} + e^{-2\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n)}{n^2+2}$$

$$\frac{\pi}{2\sqrt{2}} \frac{e^{3\sqrt{2}} + e^{-3\sqrt{2}}}{e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}}} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(3n)}{n^2+2}$$

2.33. Para $m \in \mathbb{N}, 0 < a_k < 1, k = 1 \ldots m$, se tiene:

$$\frac{1}{\pi}\prod_{k=1}^{m}\left(sen\left(a_k\pi\right)\right)^{1/m}=\left(\prod_{k=1}^{m}a_k^{1/m}\right)exp\left(-\frac{1}{m}\sum_{n=1}^{\infty}\frac{\zeta\left(2n\right)}{n}\sum_{k=1}^{m}a_k^{2n}\right)$$

3. REFERENCIAS.

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