Double Conformal Geometric Algebras

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Abstract

This paper gives an overview of two different, but closely related, double conformal geometric algebras. The first is the $\mathcal{G}_{8,2}$ *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA), and the second is the $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA). DCSTA is a straightforward extension of DCGA. The double conformal geometric algebras that are presented in this paper have a large set of operations that are valid on general quadric surface entities. These operations include rotation, translation, isotropic dilation, spacetime boost, anisotropic dilation, differentiation, reflection in standard entities, projection onto standard entities, and intersection with standard entities. However, the quadric surface entities and other "non-standard entities" cannot be intersected with each other.

Keywords: Clifford algebra, conformal geometric algebra, space-time algebra

Mathematics Subject Classification: 15A66, 51N25, 51B20, 83A05

1 Introduction

This paper[1] gives an overview of two different, but closely related, double conformal geometric algebras. The first is the $\mathcal{G}_{8,2}$ *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA) [5][3][4], and the second is the $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) [6]. DCSTA is a straightforward extension of DCGA.

2 G(8,2) Double Conformal Geometric Algebra

The $\mathcal{G}_{8,2}$ *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA) is a doubling of the $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) [2][9][10][11][12]. The $\mathcal{G}_{8,2}$ Geometric Algebra contains two subalgebras of the $\mathcal{G}_{4,1}$ CGA. The first CGA subalgebra, called CGA1 and denoted $\mathcal{C}^1$, is the algebra of the vector elements

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & : \ i = j, 1 \leq i \leq 4 \\ -1 & : \ i = j = 5 \\ 0 & : \ i \neq j. \end{cases} \tag{1}$$

The second CGA subalgebra, called CGA2 and denoted $\mathcal{C}^2$, is the algebra of the vector elements

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & : \ i = j, 6 \leq i \leq 9 \\ -1 & : \ i = j = 10 \\ 0 & : \ i \neq j. \end{cases} \tag{2}$$

1. First version v1, *February 20, 2016*. Submitted to http://vixra.org/author/robert_b_easter.

There are two subalgebras of the $\mathcal{G}_3$ Algebra of Physical Space (APS) in DCGA. The first APS, called APS1 and denoted $\mathcal{S}^1$, has the basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The second APS, called APS2 and denoted $\mathcal{S}^2$, has the basis $\mathbf{e}_6$, $\mathbf{e}_7$, and $\mathbf{e}_8$.

2.1 CGA1 point embedding

The $\mathcal{G}_{4,1}$ CGA1 null 1-vector *point* $\mathbf{P}_{\mathcal{C}^1}$ embedding of $\mathcal{G}_3$ APS1 vector $\mathbf{p}_{\mathcal{S}^1}$ is defined as

$$\mathbf{P}_{\mathcal{C}^1} = \mathcal{C}(\mathbf{p}_{\mathcal{S}^1}) = \mathbf{p}_{\mathcal{S}^1} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^1}^2\mathbf{e}_{\infty 1} + \mathbf{e}_{o1}. \tag{3}$$

The CGA1 point at the origin is defined as

$$\mathbf{e}_{o1} = \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5). \tag{4}$$

The CGA1 point at infinity is defined as

$$\mathbf{e}_{\infty 1} = \mathbf{e}_4 + \mathbf{e}_5. \tag{5}$$

2.2 CGA2 point embedding

The $\mathcal{G}_{4,1}$ CGA2 null 1-vector *point* $\mathbf{P}_{\mathcal{C}^2}$ embedding of $\mathcal{G}_3$ APS2 vector $\mathbf{p}_{\mathcal{S}^2}$ is defined as

$$\mathbf{P}_{\mathcal{C}^2} = \mathcal{C}(\mathbf{p}_{\mathcal{S}^2}) = \mathbf{p}_{\mathcal{S}^2} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^2}^2\mathbf{e}_{\infty 2} + \mathbf{e}_{o2}. \tag{6}$$

The CGA2 point at the origin is defined as

$$\mathbf{e}_{o2} = \frac{1}{2}(-\mathbf{e}_9 + \mathbf{e}_{10}). \tag{7}$$

The CGA2 point at infinity is defined as

$$\mathbf{e}_{\infty 2} = \mathbf{e}_9 + \mathbf{e}_{10}. \tag{8}$$

2.3 DCGA point embedding

The $\mathcal{G}_{8,2}$ DCGA null 2-vector *point* $\mathbf{P}_{\mathcal{D}}$ embedding of a $\mathcal{G}_3$ APS vector $\mathbf{p}_{\mathcal{S}}$ is defined as

$$\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p}_{\mathcal{S}}) = \mathbf{P}_{\mathcal{C}^1} \wedge \mathbf{P}_{\mathcal{C}^2} \tag{9}$$

where the APS vector $\mathbf{p}_{\mathcal{S}}$ is embedded into both CGA1 and CGA2 points as

$$\mathbf{P}_{\mathcal{C}^1} = \mathcal{C}(\mathbf{p}_{\mathcal{S}^1}) \tag{10}$$
$$\mathbf{P}_{\mathcal{C}^2} = \mathcal{C}(\mathbf{p}_{\mathcal{S}^2}). \tag{11}$$

The vector $\mathbf{p}_{\mathcal{S}}$ is implicitly transformed into the spaces of APS1 and APS2, respectively. The embedding function $\mathcal{C}$ is piecewise defined with a case for embedding an APS1 vector $\mathbf{p}_{\mathcal{S}^1}$ and a case for embedding an APS2 vector $\mathbf{p}_{\mathcal{S}^2}$ that embeds vectors into CGA1 or CGA2 points, respectively. The inverse embedding, or projection, $\mathcal{C}^{-1}(\mathbf{P}_{\mathcal{D}})$ can be defined to return a vector $\mathbf{p}_{\mathcal{S}^1}$ in APS1.

The DCGA point at the origin is defined as

$$\mathbf{e}_o = \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}. \tag{12}$$

The DCGA point at infinity is defined as

$$\mathbf{e}_\infty = \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \tag{13}$$

More information about point embeddings can be found in [7] and [5] by this author.

2.4 DCGA standard entities

The CGA *geometric inner product null space* (GIPNS) entities have doubled forms as the DCGA GIPNS *standard entities*. The CGA GIPNS 1-vector sphere $\mathbf{S}_{\mathcal{C}}$, 1-vector plane $\mathbf{\Pi}_{\mathcal{C}}$, 2-vector line $\mathbf{L}_{\mathcal{C}}$, 2-vector circle $\mathbf{C}_{\mathcal{C}}$, and 3-vector point pair $\mathbf{2}_{\mathcal{C}}$ are doubled as the corresponding DCGA standard entities. For example, the DCGA GIPNS 2-vector *standard sphere* $\mathbf{S}_{\mathcal{D}}$ is defined as

$$\mathbf{S}_{\mathcal{D}} \;=\; \mathbf{S}_{\mathcal{C}^1} \wedge \mathbf{S}_{\mathcal{C}^2}. \tag{14}$$

The other DCGA GIPNS standard entities are defined similarly. The DCGA GIPNS standard entities can be intersected with nearly all other DCGA GIPNS entities.

2.5 DCGA extraction elements

The embedding of an APS test vector $\mathbf{t} = \mathbf{t}_{\mathcal{S}} = x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3$, where the $\boldsymbol{\gamma}_i$ are defined here as an APS vector basis, is the test point

$$\mathbf{T}_{\mathcal{D}} \;=\; \mathcal{D}(\mathbf{t}_{\mathcal{S}}) = \mathcal{C}(\mathbf{t}_{\mathcal{S}^1}) \wedge \mathcal{C}(\mathbf{t}_{\mathcal{S}^2}). \tag{15}$$

$T_x = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_6)$	$T_y = \frac{1}{2}(\mathbf{e}_2 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7)$	$T_z = \frac{1}{2}(\mathbf{e}_3 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_8)$
$T_{xy} = \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_2)$	$T_{yz} = \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_3 + \mathbf{e}_8 \wedge \mathbf{e}_2)$	$T_{zx} = \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_3)$
$T_{x^2} = \mathbf{e}_6 \wedge \mathbf{e}_1$	$T_{y^2} = \mathbf{e}_7 \wedge \mathbf{e}_2$	$T_{z^2} = \mathbf{e}_8 \wedge \mathbf{e}_3$
$T_{x\mathbf{t}^2} = (\mathbf{e}_1 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_6)$	$T_{y\mathbf{t}^2} = (\mathbf{e}_2 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_7)$	$T_{z\mathbf{t}^2} = (\mathbf{e}_3 \wedge \mathbf{e}_{o2}) + (\mathbf{e}_{o1} \wedge \mathbf{e}_8)$
$T_1 = -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}) = -\mathbf{e}_\infty$	$T_{\mathbf{t}^2} = -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_{\infty 2})$	$T_{\mathbf{t}^4} = -4(\mathbf{e}_{o1} \wedge \mathbf{e}_{o2}) = -4\mathbf{e}_o$

Table 1. DCGA 2-vector extraction elements

The DCGA 2-vector *extraction elements* T_s are defined in Table 1. The value s is extracted from a DCGA point $\mathbf{T}_{\mathcal{D}}$ as $s = T_s \cdot \mathbf{T}_{\mathcal{D}}$. The extraction element T_s represents the value s. Linear combinations of the extraction elements can represent implicit surface functions as DCGA GIPNS 2-vector entities.

For example, a DCGA GIPNS 2-vector spherical quadric surface entity can be written

$$\boldsymbol{S} = T_{x^2} + T_{y^2} + T_{z^2} - r^2 T_1. \tag{16}$$

The DCGA point $\mathbf{T}_{\mathcal{D}}$ is on the surface of $\boldsymbol{S}$ if $\mathbf{T}_{\mathcal{D}} \cdot \boldsymbol{S} = 0$.

2.6 DCGA GIPNS 2-vector entities

The most general DCGA GIPNS 2-vector entity formed as a linear combination of DCGA extraction elements is a *Darboux cyclide* entity $\mathbf{\Omega}$, which is explained in much more detail in [5]. DCGA also stands for *Darboux Cyclide Geometric Algebra*. Degenerate forms of the Darboux cyclide include Dupin cyclides, parabolic cyclides, and general quadric surfaces.

All DCGA entities can be rotated, translated, and isotropically dilated using versor operations. The general quadric surfaces can represent quadric surfaces that are arbitrarily rotated, translated, and *anisotropically* dilated, but the anisotropic dilation operation or versor has not been found in DCGA by this author.

In the extension of DCGA, called the $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA), an anisotropic dilator is found to be the boost operator with *imaginary* natural speed β or rapidity φ. The introduction of imaginary or complex number scalars into DCSTA may seem unfortunate, but there may be ways to reformulate to use only real number scalars.

While the DCGA standard entities can be intersected with almost any other DCGA entity to form a valid intersection entity, the DCGA GIPNS 2-vector entities formed as linear combinations of the DCGA extraction elements, which can be called *non-standard entities*, generally *cannot* be intersected with each other. In general, the wedge of two non-standard entities forms an *invalid* or incorrect intersection entity.

2.7 DCGA differential elements

The DCGA 2-vector *differential elements* are defined as

$$D_x = 2T_xT_{x^2}^{-1} \tag{17}$$
$$D_y = 2T_yT_{y^2}^{-1} \tag{18}$$
$$D_z = 2T_zT_{z^2}^{-1}. \tag{19}$$

With the commutator product $\times$, a unit magnitude linear combination of the differential elements forms an $\mathbf{n}$-direction derivative operator as

$$\begin{aligned}\partial_{\mathbf{n}} = \frac{\partial}{\partial \mathbf{n}} &= D_{\mathbf{n}} \times \\ &= (n_xD_x + n_yD_y + n_zD_z) \times .\end{aligned} \tag{20}$$

Any DCGA GIPNS 2-vector entity $\mathbf{\Omega}$ can be differentiated as

$$\partial_{\mathbf{n}}\mathbf{\Omega} = D_{\mathbf{n}} \times \mathbf{\Omega}. \tag{21}$$

More information about the DCGA differential operators can be found in the paper [4].

2.8 DCGA versors

The CGA 2-versors, the *rotor* R, *translator* T, and *isotropic dilator* D, can each be doubled into the corresponding DCGA 4-versor. For example, the DCGA 4-versor *rotor* $R_{\mathcal{D}}$ is defined as

$$R_{\mathcal{D}} = R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2}. \tag{22}$$

and the rotor *versor operation* on any DCGA entity A is

$$A' = R_{\mathcal{D}} A R_{\mathcal{D}}^{\sim}. \tag{23}$$

The notation $R^{\sim}$ is the reverse, but the inverse R^{-1} can also be used instead. More information about the CGA versors can be found in numerous papers and books, including [7] and [5] by this author.

2.9 DCGA conics

By intersecting any DCGA GIPNS 2-vector *quadric surface* $\mathbf{Q}$ with a DCGA 2-vector *standard plane* $\mathbf{\Pi}$, a DCGA 4-vector *conic* entity $\boldsymbol{\kappa}$ is formed. The quadric surface $\mathbf{Q}$ can be a cone $\mathbf{K}$ and *conic sections* can be formed. A conic entity $\boldsymbol{\kappa}$ can be projected orthographically or perspectively onto a DCGA 2-vector *standard plane* $\mathbf{\Pi}$.

The orthographic projection $\boldsymbol{\kappa}_{\text{ortho}}$ of a DCGA GIPNS 4-vector *conic* entity $\boldsymbol{\kappa} = \mathbf{Q} \wedge \boldsymbol{\Pi}_{\boldsymbol{\kappa}}$ onto a DCGA GIPNS 2-vector *standard plane* $\boldsymbol{\Pi}$ is defined as

$$\boldsymbol{\kappa}_{\text{ortho}} = (\boldsymbol{\kappa} \cdot \boldsymbol{\Pi})\boldsymbol{\Pi}^{-1} \tag{24}$$

which is the algebraic projection of $\boldsymbol{\kappa}$ onto $\boldsymbol{\Pi}$.

The perspective projection $\boldsymbol{\kappa}_{\text{persp}}$ of a DCGA GIPNS 4-vector *conic* entity $\boldsymbol{\kappa}$ onto a DCGA GIPNS 2-vector *standard plane* $\boldsymbol{\Pi}$ from the viewpoint $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ represented by a DCGA GIPNS 2-vector *standard sphere* $\mathbf{S}$ with center $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ and radius $r = 1$ can be defined as

$$\begin{aligned} \boldsymbol{\kappa}_{\text{persp}} &= (((\boldsymbol{\kappa} \cdot \mathbf{S})\mathbf{S}^{-1}) \cdot \mathbf{S}) \wedge \boldsymbol{\Pi} && (25)\\ &= ((\mathbf{S}\boldsymbol{\kappa}\mathbf{S}^{-1}) \cdot \mathbf{S}) \wedge \boldsymbol{\Pi} && (26)\\ &= \mathbf{K}_{\mathbf{p}} \wedge \boldsymbol{\Pi} \end{aligned}$$

where $\mathbf{K}_{\mathbf{p}}$ is the DCGA 2-vector *cone* of the perspective projection with vertex or eye point at $\mathbf{p}$. The radius r of $\mathbf{S}$ is arbitrary, but $r = 1$ is a good choice. More information about conics in DCGA can be found in the paper [3] by this author.

3 G(4,8) Double Conformal Space-Time Algebra

The $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) [6] is a doubling of the $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra* (CSTA) [1]. The $\mathcal{G}_{4,8}$ Geometric Algebra contains two subalgebras of the $\mathcal{G}_{2,4}$ CSTA. The first CSTA subalgebra, called CSTA1 and denoted $\mathcal{C}^1$, is the algebra of the vector elements

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & : \ i = j, i \in \{1, 5\} \\ -1 & : \ i = j, i \in \{2, 3, 4, 6\} \\ 0 & : \ i \neq j. \end{cases} \tag{27}$$

The second CSTA subalgebra, called CSTA2 and denoted $\mathcal{C}^2$, is the algebra of the vector elements

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & : \ i = j, i \in \{7, 11\} \\ -1 & : \ i = j, i \in \{8, 9, 10, 12\} \\ 0 & : \ i \neq j. \end{cases} \tag{28}$$

There are two subalgebras of the $\mathcal{G}_{0,3}$ Space Algebra (SA) in DCSTA. The first SA, called SA1 and denoted $\mathcal{S}^1$, has the basis $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. The second SA, called SA2 and denoted $\mathcal{S}^2$, has the basis $\mathbf{e}_8$, $\mathbf{e}_9$, and $\mathbf{e}_{10}$. A vector in SA is denoted $\mathbf{p}_{\mathcal{S}}$, in SA1 as $\mathbf{p}_{\mathcal{S}^1}$, and in SA2 as $\mathbf{p}_{\mathcal{S}^2}$.

There are two subalgebras of the $\mathcal{G}_{1,3}$ Space-Time Algebra (STA) [8] in DCSTA. The first STA, called STA1 and denoted $\mathcal{M}^1$, has the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$. The second STA, called STA2 and denoted $\mathcal{M}^2$, has the basis $\mathbf{e}_7$, $\mathbf{e}_8$, $\mathbf{e}_9$, and $\mathbf{e}_{10}$. A vector in STA is denoted $\boldsymbol{p}_{\mathcal{M}}$, in STA1 as $\boldsymbol{p}_{\mathcal{M}^1}$, and in STA2 as $\boldsymbol{p}_{\mathcal{M}^2}$. A vector in STA can be denoted on the Dirac gammas as

$$\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3 = p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}. \tag{29}$$

An *observer* position in spacetime can be denoted as

$$\boldsymbol{o}t \;=\; w\boldsymbol{\gamma}_0 = ct\boldsymbol{\gamma}_0 \tag{30}$$

where c is the speed of light and t is time.

3.1 CSTA1 point embedding

The $\mathcal{G}_{2,4}$ CSTA1 null 1-vector *point* $\mathbf{P}_{\mathcal{C}^1}$ embedding of $\mathcal{G}_{1,3}$ STA1 vector $\boldsymbol{p}_{\mathcal{M}^1}$ is defined as

$$\boldsymbol{P}_{\mathcal{C}^1} \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1}) = \boldsymbol{p}_{\mathcal{M}^1} + \frac{1}{2}\boldsymbol{p}^2_{\mathcal{M}^1}\mathbf{e}_{\infty 1} + \mathbf{e}_{o1}. \tag{31}$$

The CSTA1 point at the origin is defined as

$$\mathbf{e}_{o1} \;=\; \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5). \tag{32}$$

The CSTA1 point at infinity is defined as

$$\mathbf{e}_{\infty 1} \;=\; \mathbf{e}_4 + \mathbf{e}_5. \tag{33}$$

3.2 CSTA2 point embedding

The $\mathcal{G}_{2,4}$ CSTA2 null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}^2}$ embedding of $\mathcal{G}_{1,3}$ STA2 vector $\boldsymbol{p}_{\mathcal{M}^2}$ is defined as

$$\boldsymbol{P}_{\mathcal{C}^2} \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}) = \boldsymbol{p}_{\mathcal{M}^2} + \frac{1}{2}\boldsymbol{p}^2_{\mathcal{M}^2}\mathbf{e}_{\infty 2} + \mathbf{e}_{o2}. \tag{34}$$

The CSTA2 point at the origin is defined as

$$\mathbf{e}_{o2} \;=\; \frac{1}{2}(-\mathbf{e}_{11} + \mathbf{e}_{12}). \tag{35}$$

The CSTA2 point at infinity is defined as

$$\mathbf{e}_{\infty 2} \;=\; \mathbf{e}_{11} + \mathbf{e}_{12}. \tag{36}$$

3.3 DCSTA point embedding

The $\mathcal{G}_{4,8}$ DCSTA null 2-vector *point* $\boldsymbol{P}_{\mathcal{D}}$ embedding of a $\mathcal{G}_{1,3}$ STA vector $\boldsymbol{p}_{\mathcal{M}}$ is defined as

$$\boldsymbol{P}_{\mathcal{D}} \;=\; \mathcal{D}(\boldsymbol{p}_{\mathcal{M}}) = \boldsymbol{P}_{\mathcal{C}^1} \wedge \boldsymbol{P}_{\mathcal{C}^2} \tag{37}$$

where the STA vector $\boldsymbol{p}_{\mathcal{M}}$ is embedded into both CSTA1 and CSTA2 points as

$$\boldsymbol{P}_{\mathcal{C}^1} \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1}) \tag{38}$$

$$\boldsymbol{P}_{\mathcal{C}^2} \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}). \tag{39}$$

The vector $\boldsymbol{p}_{\mathcal{M}}$ is implicitly transformed into the spaces of STA1 and STA2, respectively. The embedding function $\mathcal{C}$ is piecewise defined with a case for embedding an STA1 vector $\boldsymbol{p}_{\mathcal{M}^1}$ and a case for embedding an STA2 vector $\boldsymbol{p}_{\mathcal{M}^2}$ that embeds vectors into CSTA1 or CSTA2 points, respectively. The inverse embedding, or projection, $\mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{D}})$ can be defined to return a vector $\boldsymbol{p}_{\mathcal{M}^1}$ in STA1.

The DCSTA point at the origin is defined as

$$\mathbf{e}_o \;=\; \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}. \tag{40}$$

The DCSTA point at infinity is defined as

$$\mathbf{e}_\infty \;=\; \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \tag{41}$$

3.4 DCSTA standard entities

The CSTA *geometric inner product null space* (GIPNS) entities have doubled forms as the DCSTA GIPNS *standard entities*. The CSTA GIPNS 1-vector *hypercone* $\boldsymbol{P}_\mathcal{C} = \boldsymbol{K}_\mathcal{C}$, 1-vector *hyperplane* $\boldsymbol{E}_\mathcal{C}$, 1-vector *hyperhyperboloid of one sheet* $\boldsymbol{\Sigma}_\mathcal{C}$, 1-vector *hyperhyperboloid of two sheets* $\boldsymbol{\Xi}_\mathcal{C}$, 2-vector *plane* $\boldsymbol{\Pi}_\mathcal{C}$, 2-vector *(imaginary) pseudosphere or sphere* $\boldsymbol{S}_\mathcal{C}$, 3-vector *line* $\boldsymbol{L}_\mathcal{C}$, 4-vector *pseudocircle or circle* $\boldsymbol{C}_\mathcal{C}$, 4-vector *null line* $\mathcal{L}_\mathcal{C}$, 4-vector *point pair* $\mathbf{2}_\mathcal{C}$, 4-vector *flat point* $\mathbb{P}_\mathcal{C}$, and 5-vector *point* $\boldsymbol{P}^\star_\mathcal{C}$ are doubled as the corresponding DCSTA standard entities. For example, the DCSTA GIPNS 4-vector *standard plane* $\boldsymbol{\Pi}_\mathcal{D}$ is defined as

$$\boldsymbol{\Pi}_\mathcal{D} \;=\; \boldsymbol{\Pi}_{\mathcal{C}^1} \wedge \boldsymbol{\Pi}_{\mathcal{C}^2}. \tag{42}$$

The other DCSTA GIPNS standard entities are defined similarly. The DCSTA GIPNS standard entities can be intersected with nearly all other DCSTA GIPNS entities. The CSTA GIPNS and GOPNS entities are explained some more in [6] by this author.

3.5 DCSTA extraction elements

The embedding of an STA test vector $\boldsymbol{t} = \boldsymbol{t}_\mathcal{M} = w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3$, where the $\boldsymbol{\gamma}_i$ are defined here as an STA vector basis, is the test point

$$\boldsymbol{T}_\mathcal{D} \;=\; \mathcal{D}(\boldsymbol{t}_\mathcal{M}) = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}^1}) \wedge \mathcal{C}(\boldsymbol{t}_{\mathcal{M}^2}). \tag{43}$$

The value w may be identified as the product of light speed c and time t as $w = ct$.

$T_x = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1})$	$T_y = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{\infty 1})$	$T_z = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{\infty 1})$
$T_{x^2} = \mathbf{e}_8 \wedge \mathbf{e}_2$	$T_{y^2} = \mathbf{e}_9 \wedge \mathbf{e}_3$	$T_{z^2} = \mathbf{e}_{10} \wedge \mathbf{e}_4$
$T_{xy} = \frac{1}{2}(\mathbf{e}_9 \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_3)$	$T_{yz} = \frac{1}{2}(\mathbf{e}_{10} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_4)$	$T_{zx} = \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_2)$
$T_{x\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{o1}$	$T_{y\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{o1}$	$T_{z\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{o1}$
$T_1 = -\mathbf{e}_\infty$	$T_{\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{o1}$	$T_{\boldsymbol{t}^4} = -4\mathbf{e}_o$
$T_w = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7)$	$T_{w^2} = \mathbf{e}_7 \wedge \mathbf{e}_1$	$T_{w\boldsymbol{t}^2} = \mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_7$
$T_{wx} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_8 + \mathbf{e}_2 \wedge \mathbf{e}_7)$	$T_{wy} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_9 + \mathbf{e}_3 \wedge \mathbf{e}_7)$	$T_{wz} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{10} + \mathbf{e}_4 \wedge \mathbf{e}_7)$
$T_t = \frac{1}{c}T_w$	$T_{t^2} = \frac{1}{c^2}T_{w^2}$	$T_{t\boldsymbol{t}^2} = \frac{1}{c}T_{w\boldsymbol{t}^2}$
$T_{tx} = \frac{1}{c}T_{wx}$	$T_{ty} = \frac{1}{c}T_{wy}$	$T_{tz} = \frac{1}{c}T_{wz}$

Table 2. DCSTA 2-vector extraction elements

The DCSTA 2-vector *extraction elements* T_s are defined in Table 2. The value s is extracted from a DCSTA point $\boldsymbol{T}_\mathcal{D}$ as $s = T_s \cdot \boldsymbol{T}_\mathcal{D}$. The extraction element T_s represents the value s. Linear combinations of the extraction elements can represent implicit surface functions as DCSTA GIPNS 2-vector entities.

For example, a DCSTA GIPNS 2-vector spherical quadric surface entity can be written

$$\boldsymbol{S} = T_{x^2} + T_{y^2} + T_{z^2} - r^2 T_1. \tag{44}$$

The DCSTA point $\boldsymbol{T}_{\mathcal{D}}$ is on the surface of $\boldsymbol{S}$ if $\boldsymbol{T}_{\mathcal{D}} \cdot \boldsymbol{S} = 0$. It is possible to place an entity at a time in spacetime by using the extractions containing w. The DCSTA extraction elements can form all of the same 2-vector entities as in DCGA. General quadric surface entities can be formed at $w = 0$ and then boosted into a constant velocity $\mathbf{v}_{\mathcal{S}}$ using the DCSTA *boost* $B_{\mathcal{D}}$ versor operation. A boosted *quadric surface* is also contracted in length in the direction of boost velocity $\mathbf{v}_{\mathcal{S}}$ by the factor $\gamma_{\mathbf{v}}^{-1} = \sqrt{1 - \beta_{\mathbf{v}}}$, which is consistent with length contraction in the theory of special relativity.

3.6 DCGA GIPNS 2-vector entities

The most general DCSTA GIPNS 2-vector spatial geometric entity formed as a linear combination of DCSTA extraction elements is a *Darboux cyclide* entity $\boldsymbol{\Omega}$, which is explained in much more detail in [5]. Degenerate forms of the Darboux cyclide include Dupin cyclides, parabolic cyclides, and general quadric surfaces. In DCSTA, the general quadric surface entities are of the most interest since they support boost and anisotropic dilation operations, while the cyclide entities *cannot* properly support those operations.

All DCSTA entities can be rotated, translated, and isotropically dilated using versor operations. Rotation is spatial rotation. Translation may include translation in spacetime of the time w component. Isotropic dilation also dilates any non-zero time w component.

DCSTA includes a boost operation, which is a hyperbolic rotation in spacetime that is valid on all of the quadric surface entities. The boost operation also implements anisotropic dilation of the quadric surface entities. The anisotropic dilator is found to be the boost operator with *imaginary* natural speed β or rapidity φ. The introduction of imaginary or complex number scalars into DCSTA may seem unfortunate, but there may be ways to reformulate to use only real number scalars.

While the DCSTA standard entities can be intersected with almost any other DCSTA entity to form a valid intersection entity, the DCSTA GIPNS 2-vector entities formed as linear combinations of the DCSTA extraction elements, which can be called *non-standard entities*, generally *cannot* be intersected with each other. In general, the wedge of two non-standard entities forms an *invalid* or incorrect intersection entity.

3.7 DCSTA differential elements

The DCSTA 2-vector *differential elements* are defined as

$$
\begin{aligned}
D_w &= 2T_w T_{w^2}^{-1} && (45)\\
D_t &= 2T_t T_{t^2}^{-1} && (46)\\
D_x &= 2T_x T_{x^2}^{-1} && (47)\\
D_y &= 2T_y T_{y^2}^{-1} && (48)\\
D_z &= 2T_z T_{z^2}^{-1} && (49)
\end{aligned}
$$

With the commutator product $\times$, a unit magnitude linear combination of the differential elements forms an $\mathbf{n}$-direction derivative operator as

$$
\begin{aligned}
\partial_{\mathbf{n}} = \frac{\partial}{\partial \mathbf{n}} &= D_{\mathbf{n}} \times && (50)\\
&= (n_w D_w + n_x D_x + n_y D_y + n_z D_z) \times .
\end{aligned}
$$

Any DCSTA GIPNS 2-vector entity $\boldsymbol{\Omega}$ can be differentiated as

$$
\partial_{\mathbf{n}} \boldsymbol{\Omega} = D_{\mathbf{n}} \times \boldsymbol{\Omega}. \quad (51)
$$

The DCSTA time t derivative operator is

$$\partial_t = \frac{\partial}{\partial t} \;=\; D_t \times . \tag{52}$$

The time t derivative of any DCSTA GIPNS 2-vector *spacetime entity* $\mathbf{\Omega}$ is

$$\dot{\mathbf{\Omega}} = \partial_t \mathbf{\Omega} = \frac{\partial \mathbf{\Omega}}{\partial t} \;=\; D_t \times \mathbf{\Omega}. \tag{53}$$

The DCSTA 2-vector *spacetime entity* $\mathbf{\Omega}$ is the most general DCSTA GIPNS 2-vector *non-standard surface entity* that is formed as a linear combination of the DCSTA 2-vector *extraction elements* T_s.

For example, a DCSTA quadric surface entity can be formed using the DCSTA extraction elements at $w = 0$ and centered at the origin of spacetime, then boosted into a velocity $\mathbf{v}$. The time t derivative of the boosted quadric surface entity produces an entity representing the velocity $\mathbf{v}$.

3.8 DCSTA versors

The CSTA 2-versors, the *rotor* R, *translator* T, *isotropic dilator* D, and *boost (hyperbolic rotor)* B can each be doubled into the corresponding DCSTA 4-versor. For example, the DCSTA 4-versor *rotor* $R_{\mathcal{D}}$ is defined as

$$R_{\mathcal{D}} \;=\; R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2}. \tag{54}$$

and the rotor *versor operation* on any DCSTA entity A is

$$A' \;=\; R_{\mathcal{D}} A \widetilde{R_{\mathcal{D}}}. \tag{55}$$

The notation $R^{\sim}$ is the reverse, but the inverse R^{-1} can also be used instead. The two-sided versor operation, also called a versor "sandwich" operation, is used to apply the operation of all versors.

3.8.1 DCSTA 4-versor spatial rotor

The CSTA1 2-versor *spatial rotor* $R_{\mathcal{C}^1}$ for a spatial rotation around the unit norm SA1 vector $\mathbf{n}_{\mathcal{S}^1}$ by an angle θ is defined as

$$R_{\mathcal{C}^1} \;=\; e^{\frac{1}{2}\theta \mathbf{n}^*_{\mathcal{S}^1}} = \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right) \mathbf{n}_{\mathcal{S}^1} \widetilde{\mathbf{I}_{\mathcal{S}^1}} \tag{56}$$

where $\mathbf{I}_{\mathcal{S}^1}$ is the SA1 unit pseudoscalar

$$\mathbf{I}_{\mathcal{S}^1} \;=\; \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \tag{57}$$

and the unit norm bivector $\mathbf{n}^*_{\mathcal{S}^1} = \mathbf{n}_{\mathcal{S}^1} \widetilde{\mathbf{I}_{\mathcal{S}^1}}$ is the SA1 dual of the axis $\mathbf{n}_{\mathcal{S}^1}$.

The CSTA2 2-versor *spatial rotor* $R_{\mathcal{C}^2}$ for a spatial rotation around the unit norm SA2 vector $\mathbf{n}_{\mathcal{S}^2}$ by an angle θ is defined as

$$R_{\mathcal{C}^2} \;=\; e^{\frac{1}{2}\theta \mathbf{n}^*_{\mathcal{S}^2}} = \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right) \mathbf{n}_{\mathcal{S}^2} \widetilde{\mathbf{I}_{\mathcal{S}^2}} \tag{58}$$

where $\mathbf{I}_{\mathcal{S}^2}$ is the SA2 unit pseudoscalar

$$\mathbf{I}_{\mathcal{S}^2} \;=\; \mathbf{e}_8 \wedge \mathbf{e}_9 \wedge \mathbf{e}_{10} \tag{59}$$

and the unit norm bivector $\mathbf{n}^*_{\mathcal{S}^2} = \mathbf{n}_{\mathcal{S}^2}\mathbf{I}^{\sim}_{\mathcal{S}^2}$ is the SA2 dual of the axis $\mathbf{n}_{\mathcal{S}^2}$.

The DCSTA 4-versor *spatial rotor* $R_{\mathcal{D}}$ for a spatial rotation around the unit norm SA vector $\mathbf{n}_{\mathcal{S}}$ by an angle θ is defined as

$$R_{\mathcal{D}} \;=\; R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2}. \tag{60}$$

The CSTA 2-versor *line rotor* that rotates directly around a CSTA GIPNS 3-vector *line* can be doubled into a DCSTA 4-versor *line rotor*, which is discussed further in the paper [6].

3.8.2 DCSTA 4-versor spacetime translator

The CSTA *translator* $T_{\mathcal{C}}$ is defined as

$$T_{\mathcal{C}} \;=\; e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} = 1 - \frac{1}{2}\boldsymbol{d}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma}. \tag{61}$$

The translation vector $\boldsymbol{d}_{\mathcal{M}}$ is an STA *spacetime displacement* vector in STA1 or STA2.

The DCSTA 4-versor *translator* $T_{\mathcal{D}}$ is defined as

$$T_{\mathcal{D}} \;=\; T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2}. \tag{62}$$

3.8.3 DCSTA 4-versor isotropic dilator

The CSTA 2-versor *isotropic dilator* $D_{\mathcal{C}}$ is defined as

$$D_{\mathcal{C}} \;=\; \frac{1}{2}(1+d) + \frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma} \wedge \mathbf{e}_{o\gamma}. \tag{63}$$

The scalar d is the *dilation factor*. The γ is either 1 or 2 for the dilator in CSTA1 or CSTA2, respectively. The CSTA isotropic dilator $D_{\mathcal{C}}$ is a *spacetime dilator*, which includes the dilation of the time and space components of an entity by the factor d.

The DCSTA 4-versor *isotropic dilator* $D_{\mathcal{D}}$ is defined as

$$D_{\mathcal{D}} \;=\; D_{\mathcal{C}^1} \wedge D_{\mathcal{C}^2}. \tag{64}$$

3.8.4 DCSTA 4-versor hyperbolic rotor (boost)

The CSTA 2-versor *boost operator* $B_{\mathcal{C}}$ for a natural speed $\beta_{\mathbf{v}}$ in the SA *unit* direction $\hat{\mathbf{v}}_{\mathcal{S}}$ is defined as

$$B_{\mathcal{C}} \;=\; e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}}\boldsymbol{\gamma}_0} \tag{65}$$

$$\;=\; \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right) + \sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}_{\mathcal{S}} \wedge \boldsymbol{\gamma}_0 \tag{66}$$

where the boost velocity is

$$\mathbf{v}_{\mathcal{S}} \;=\; \|\mathbf{v}_{\mathcal{S}}\|\hat{\mathbf{v}}_{\mathcal{S}} = \beta_{\mathbf{v}} c \hat{\mathbf{v}}_{\mathcal{S}} \tag{67}$$

and the rapidity is

$$\varphi_{\mathbf{v}} \;=\; \operatorname{atanh}(\beta_{\mathbf{v}}) = \operatorname{atanh}\left(\frac{\|\mathbf{v}_{\mathcal{S}}\|}{c}\right). \tag{68}$$

The SA velocity $\mathbf{v}_{\mathcal{S}}$ of the CSTA boost $B_{\mathcal{C}}$ is relative to the STA observer velocity

$$\boldsymbol{o}_{\mathcal{M}} \;=\; c\boldsymbol{\gamma}_0 \tag{69}$$

and has the STA velocity

$$\boldsymbol{v}_{\mathcal{M}} \;=\; \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}}. \tag{70}$$

The boost and normalization of the STA observer velocity $\boldsymbol{o}_{\mathcal{M}}$ produces a particle velocity

$$\boldsymbol{o}'_{\mathcal{M}} = \boldsymbol{v}_{\mathcal{M}} \;=\; c\frac{B_{\mathcal{C}}\boldsymbol{o}_{\mathcal{M}}B_{\mathcal{C}}^{\sim}}{(B_{\mathcal{C}}\boldsymbol{o}_{\mathcal{M}}B_{\mathcal{C}}^{\sim})\cdot\boldsymbol{\gamma}_0} \tag{71}$$

$$\;=\; \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}} = c\boldsymbol{\gamma}_0 + \mathbf{v}_{\mathcal{S}} \tag{72}$$

moving relative to the same observer $\boldsymbol{o}_{\mathcal{M}}$.

The STA observer velocity $\boldsymbol{o}_{\mathcal{M}} = c\boldsymbol{\gamma}_0$ embedded as CSTA velocity $\boldsymbol{O}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{o}_{\mathcal{M}})$ is boosted as

$$\boldsymbol{O}'_{\mathcal{C}} \;=\; B_{\mathcal{C}}\boldsymbol{O}_{\mathcal{C}}B_{\mathcal{C}}^{\sim}. \tag{73}$$

Then $\boldsymbol{O}'_{\mathcal{C}}$ can be projected and normalized as the STA particle velocity $\boldsymbol{v}_{\mathcal{M}}$

$$\boldsymbol{o}'_{\mathcal{M}} \;=\; c\frac{\mathcal{C}^{-1}(\boldsymbol{O}'_{\mathcal{C}})}{\mathcal{C}^{-1}(\boldsymbol{O}'_{\mathcal{C}})\cdot\boldsymbol{\gamma}_0} = \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}} = \boldsymbol{v}_{\mathcal{M}}. \tag{74}$$

The DCSTA 4-versor *boost operator* $B_{\mathcal{D}}$ is defined as

$$B_{\mathcal{D}} \;=\; B_{\mathcal{C}^1} \wedge B_{\mathcal{C}^2}. \tag{75}$$

By using the reverse of $B_{\mathcal{D}}$, a change of observer or frame of reference can be made, which is similar to rotating by the negative angle to change basis.

By using a natural speed $\beta_{\mathbf{v}} = \sqrt{1-d^2}$ for a dilation factor d, the boost of a DCSTA GIPNS 2-vector quadric surface entity $\boldsymbol{Q}$ can implement an anisotropic dilation, which is a directed scaling in the direction of the boost velocity $\mathbf{v}_{\mathcal{S}}$. Dilations with $1 < d$ make $\beta_{\mathbf{v}}$ an imaginary number. After the anisotropic dilation, it may be necessary to project the result into the purely spatial part of DCSTA as

$$\boldsymbol{Q}' \;=\; ((B_{\mathcal{D}}\boldsymbol{Q}B_{\mathcal{D}}^{\sim})\cdot\mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1} \tag{76}$$

where

$$\mathbf{I}_{\mathcal{DS}} \;=\; \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6\mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12}. \tag{77}$$

The projection discards imaginary components of time. If $\boldsymbol{Q}'$ should be at some other time than $w = 0$, then a DCSTA translator operation can be used to translate $\boldsymbol{Q}'$.

4 Conclusion

This paper has given a basic overview of two different, but related, double conformal geometric algebras, $\mathcal{G}_{8,2}$ DCGA and $\mathcal{G}_{4,8}$ DCSTA. The reader is recommended to see the other much longer papers on DCGA [5][3][4] and on DCSTA [6] for more details.

The double conformal geometric algebras that have been presented in this paper have a large set of operations that are valid on general quadric surface entities. These operations include rotation, translation, isotropic dilation, spacetime boost, anisotropic dilation, differentiation, reflection in standard entities, projection onto standard entities, and intersection with standard entities. However, intersection of the quadric surfaces or other non-standard entities with each other is generally not valid and produces invalid results.

Admittedly, the double conformal geometric algebras are large and complicated and their efficient implementation for applications may be difficult. Also due to the complicated nature of these algebras, this paper and the others by this author may contain some mistakes or may have ignored important issues or results that others may notice. Nevertheless, it is hoped that this paper and the others cited provide useful ideas that can be researched further for additional results and possible applications.

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