A Short Note on Time Inversion Symmetry And Reversibility

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Abstract. A simple mathematical proof reveals that time-inversion symmetry and reversibility are different concepts, which also resolves the Loschmidt paradox.

1. Time Reversal Symmetry

The time reversal symmetry, or $\mathcal{T}$-symmetry for short, states that every closed (i.e. energy conserving) dynamical system of N generalized coordinates $q_1, \ldots, q_N$, for every possible motion, i.e. time curve $t \mapsto (q_1(t), \ldots, q_N(t))$, $t \mapsto (q_1(-t), \ldots, q_N(-t))$ is another possible, equivalent motion. And by applying the well-known principle of Maupertuis, the time inversion symmetry is equivalent with the symmetry of mass inversion.

But then, some smart equation should capture this symmetry, and we can readily deduce that it must come in terms of squares of mass, energy, and momentum. And, there is: Every free particle satisfies: $E^2 - p^2 = m_0^2$, where $c \equiv 1$. This is an algebraic equation, and as such it has a unique, complete algebraic solution, only dictated by mathematics. It is well-known to be $m_0 = E\gamma_0 + \sum_{1 \leq j \leq 3} p_j \gamma_j$, where $\gamma_0, \ldots, \gamma_3$ are the Dirac matrices, which operate on $\mathbb{C}^4$, and therefore the solution is complete up to transformations with matrices $U \in U(4)$, where $U(4)$ denotes the group of unitary 4×4-matrices.

That symmetry group $U(4)$ evidentally contains phase symmetries for mass, energy, and momentum inversion, and therefore these guarantee that the solution is $\mathcal{T}$-invariant.

Now, that's just one particle, but we can easily generalize to any number of particles, as long as they all make up a closed (sub)system: because we know that the external forces are zero by definition and all internal forces between the particles of such a sytem do cancel out.

Hence, for any mass distribution in space-time in a closed subsystem, we

must have $\Pi_0^2(x) - \sum_{1\leq j\leq 3}\Pi_j^2(x) = \rho_0^2(x)$ with $x = (t, x_1, x_2, x_3)$ being the space-time coordinates, with Π_0^2 being the space time density of the square of the energy, and likewise Π_j^2 denoting the space-time density of the square of the j-th momentum coordinate. And the root of this equation then is given by $\Pi_0\gamma_0 + \cdots + \Pi_3\gamma_3 = \rho_0$, which again is unique modulo $U(4)$, but now even locally for each point $x \in \mathbb{R}^4$ in space-time.
The quadratic equation becomes more familiar by factoring out the mass distribution $m(x)$ and the velocity distribution $v(x)$:

$$\Pi_0^2(x) - \sum_{1\leq j\leq 3}\Pi_j^2(x) = \rho_0^2(x) = m(x)j_0(x) - \cdots m(x)v_3(x)j_3(x) = \rho_0^2(x),$$

where j_0 is the mass density in space time and $j_1, \ldots, j_3$ are the fluxes. So, $(m\gamma_0 + \cdots + mv_3\gamma_3)(j_0\gamma_0 + \cdots j_3\gamma_3) = \rho_0^2$, and cheating a bit by leaving out the left factor, we could laxly write it as: $j_0\gamma_0 + \cdots + j_3\gamma_3 = m_0$, where m_0 would a pseudo scalar, representing the rest mass density. Although not being quite correct, because we dropped the velocity factors from the fluxes and oddly turned the scalar equation into a pseudo scalar equation, it still is $\mathcal{T}$-symmetrical due to being understood to be unique modulo $U(4)$ (for each point in space-time).

2. Reversibility

As seen above, the $\mathcal{T}$-symmetry results into a phase symmetry of mass and energy. The canonical conjugate of energy is time, which therefore also must be phase symmetric. That means that for each non-zero time vector $\vec{t}$ and each $\phi \in \mathbb{R}$ $\vec{t}$ and $e^{i\phi}\vec{t}$ point to equivalent time directions. So, all time directions pointing outwards the unit circle are equivalent. And likewise all directions pointing inwards (decreasing the absolute value $|t|$) are equivalent.
But those pointing outwards by no means need to be equivalent to those pointing inwards! And this is just what a reversible system would have to yield: A reversible system is to be defined as a system which does not depend on the absolute value $|t|$ of time $\vec{t}$.

3. Irreversibility of Classical Electromagnetism

These are Maxwell's equations in covariant form:

$$\Box A_\mu = j_\mu, 0 \leq \mu \leq 3,$$

where $\Box = \partial_0^2 - \cdots - \partial_3^2$ is the wave operator, j is the 4-vector current, and A is the electromagnetic field.
Because A is known to be (globally) phase symmetric, the current j must be either, and the whole theory becomes (globally) $\mathcal{T}$-symmetric.

Remark 3.1. The fact that j is (globally) phase symmetric suggests some care when it comes to the addition of positive and negative charges/fluxes: Instead, we might be better off by summing up their squares.

It is common sense to interpret the current j to be the source of the electromagnetic field A. However, in order to be a reversible system, the current must equivalently be the target of the source A, either.
We can exclude the latter, as long as we base physics on a concept of spatial distance: the longer the spatial distance between two objects, the longer the time span for light to cross that distance; and with that, we can identify a source as the point in space-time that immediately reacts to the acceleration by either the inertial force or the radiation of the electromagnetic field.

A thought experiment illustrates this:
When a driver of a car, D say, speeds up, he immediately feels the inertial force, the particles in his body begin to shake, and he experiences a raise of temperature. At the same time, as he is measuring the electromagnetic field around him, he will notice the radiation. In a time reverted perspective, he would equivalently see the whole universe accelerating at the same time, each stellar object causing a huge radiation that superimposes such that he is becoming the target of a resultant radiation that lets him feel the force of inertia and heats him up slightly. Sofar, everything is alright. The only thing is, that the distant objects must have begun their mutual acceleration a while before or after he pressed the gas throttle, and all that immediately stops, at the moment he takes his foot from the throttle. (Maybe one half of the radiation is coming from future, the other from the past, and the instance will become a mixture of both? No: D in his time frame sees only radiation going out, and if half of that is going to the past, half to the future, their share to the force of inerta will be the opposite of eachother, so add to zero.)

Likewise, two observers E and F at different radial distances to D may also measure the electromagnetic field (and exchange their results). With that, E and F can trace the field back or forward in time up to the space time curve of D, where the radiation is contracted to points in space-time by looking for the center of the radiation. That curve is what steers the whole radiation due to its instantaneous reaction to acceleration. (So, once radiation reaches E and F, both could tell themselves not to be the accelerating source, because at that moment they do not stay in the center of the radiation, and they are not subjected to a steadily displacing force of inertia.) In all, D, E, and F will determine D to be the source of the radiation.

4. Wrapping Up

Given a closed subsystem of charged particles, we are naturally given a partial ordering of time based on any source in the subsystem:
A source is to be described as a continuous time curve of charge $q : \mathbb{R} \ni t \mapsto q(t) \in \mathbb{R}^3$, which is inert, i.e. resists to acceleration, and therefore moves at a speed strictly slower than light. For each $t \in \mathbb{R}$ let $\Gamma(q(t))$ be the forward light cone with vertex $(t, q(t)) \in \mathbb{R}^4$. Then for each $x' \in \mathbb{R}^4$ there is a unique $t \in \mathbb{R}$, such that $x' \in \Gamma(q(t))$, and the time x'_0 can be replaced with $t + r/c$, where r is the spatial Euclidean distance between $q(t)$ and (x'_1, x'_2, x'_3), and c

is the speed of light. In that time frame, seen from the charge $q : t \mapsto q(t)$, the radiation of all sources, including its own, is radially spreading out in the positive time direction. That time scale is per source only. It is not strictly transitive: the time difference of two points $x, y \in \mathbb{R}^4$ may be or zero w.r.t. q, but will generally be unequal zero w.r.t. a different source $q' : t' \mapsto q'(t')$, which is caused by the fact that the sources q and q' will see eachother at a later time (given they are spatially apart); the two time systems are directionally shifted, but they are not flipped, and they converge over large distances and small velocities. That said, they maintain the order.
Yet, on top of that ordering, we still have the (global) phase symmetry, telling us that for any phase $e^{i\phi}$ the ordered time axis $\vec{t}$ is equivalent with $e^{i\phi}\vec{t}$.

Note, that all this is in harmony with Loschmidt's argument: If entropy is to grow in a (globally) $\mathcal{T}$-symmetric, closed subsystem along a (positive) time direction $\vec{t}$, say, then it also grows along $e^{i\phi}\vec{t}$ for any $\phi \in \mathbb{R}$. Yet, due to irreversibilty, no dynamically equivalent path would need to exist, along which that entropy decreases.

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