Infinite Products

E. Valdebenito
January , 2015

Abstract
Infinite products,Constants classic,Riemann zeta function,Radicals,...

Resumen-Abstract

En esta nota mostramos algunos productos infinitos que involucran constantes clásicas:
π=3.1415...,e=2.7182...,γ=0.5772...,ϕ=1.6180..., A=1.2824..., G=0.9159..., ln 2=0.6931...,

Esta nota está inspirada en las referencias (4) y (6).

Introducción

Esta nota mostramos una colección de productos infinitos que involucran constantes clásicas, como son:

- La constante Pi: $\pi=\sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+1}$,
- La constante e $=\sum_{n=0}^{\infty}\frac{1}{n!}$,
- La constante gamma de Euler-mascheroni: $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n)\right)$,
- La constante ϕ , Golden Ratio, $\phi=\frac{1+\sqrt{5}}{2}$,
- La constante de Glaisher-Kinkelin: $A=\lim_{n\to\infty}\frac{1^1\,2^2\,3^3\ldots\,n^n}{e^{-n^2/4}\,n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}}$,
- La constante de Catalan: $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\,n+1)^2}$,
- La constante $\ln(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

Algunos productos infinitos clásicos son :

$$\frac{\pi}{2}=\frac{1}{1}\cdot\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\ldots$$

$$\frac{\pi^2}{6}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\ldots$$

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ldots$$

$$\frac{e^{\pi}-e^{-\pi}}{2\,\pi}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$$

Productos Infinitos

$$\pi\,e^{\gamma}=2\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{2^n+n+1}{2^n\,(n+1)}}=\left(\frac{2}{1}\right)^{\frac{1}{1}}\left(\frac{2^2}{1\cdot3}\right)^{\frac{7}{12}}\left(\frac{2^3\cdot4}{1\cdot3^3}\right)^{\frac{3}{8}}\ldots \tag{1}$$

$$e^{-\gamma+\sum_{k=1}^{m}\frac{1}{k}}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+m+1)^{(-1)^k\binom{n}{k}}\right)^{\frac{1}{n+1}},\; m=0,1,2,3,\;\ldots \tag{2}$$

$$e^{-\gamma-\frac{\pi}{2}}=2\prod_{n=1}^{\infty}\left(\prod_{k=1}^{n}(4\,k+1)^{(-1)^k\binom{n}{k}}\right)^{\frac{1}{n+1}}=\left(\frac{1}{5}\right)^{\frac{1}{2}}\left(\frac{1\cdot9}{5^2}\right)^{\frac{1}{3}}\left(\frac{9^3}{5^3\cdot13}\right)^{\frac{1}{4}}\ldots \tag{3}$$

$$\frac{2\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\,e^{G/\pi}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(2\,k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n-2}{2^{n+2}}}=\left(\frac{3}{1}\right)^{-\frac{1}{8}}\left(\frac{3^3\cdot 7}{1\cdot 5^3}\right)^{\frac{1}{32}}\left(\frac{3^4\cdot 7^4}{1\cdot 5^6\cdot 9}\right)^{\frac{2}{64}}\ldots \tag{4}$$

$$\frac{\pi^3}{4\,\Gamma\left(\frac{3}{4}\right)^4}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}((k+1)\,(2\,k+1))^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n}}=\left(\frac{2\cdot 3}{1^2}\right)^{\frac{1}{2}}\left(\frac{2^2\cdot 3}{1^2\cdot 5}\right)^{\frac{1}{4}}\left(\frac{2^3\cdot 4\cdot 7}{1^2\cdot 5^3}\right)^{\frac{1}{8}}\ldots \tag{5}$$

$$\frac{\Gamma\left(\frac{1}{4}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}\,e^{G/\pi}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(2\,k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n+2}{2^{n+2}}}=\left(\frac{3}{1}\right)^{\frac{3}{8}}\left(\frac{3^2}{1\cdot 5}\right)^{\frac{4}{16}}\left(\frac{3^3\cdot 7}{1\cdot 5^3}\right)^{\frac{5}{32}}\ldots \tag{6}$$

$$\frac{\pi}{\sqrt{2}\;\Gamma\left(\frac{3}{4}\right)^2}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(2\,k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{1}{2^{n+1}}}=\left(\frac{3}{1}\right)^{\frac{1}{4}}\left(\frac{3^2}{1\cdot 5}\right)^{\frac{1}{8}}\left(\frac{3^3\cdot 7}{1\cdot 5^3}\right)^{\frac{1}{16}}\ldots \tag{7}$$

$$\frac{A^{12}}{\sqrt[3]{2}\;e\,\pi}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{n\,2^{-n}}=\left(\frac{2}{1}\right)^{\frac{1}{2}}\left(\frac{2^2}{1\cdot 3}\right)^{\frac{2}{4}}\left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{\frac{3}{8}}\ldots \tag{8}$$

$$\frac{A^{12}}{2\,\sqrt[3]{2}}\,e^{-1+(7\,\zeta(3))/4\,\pi^2}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n^2+9\,n+8}{2^{n+3}}}=\left(\frac{2}{1}\right)^{\frac{9}{8}}\left(\frac{2^2}{1\cdot 3}\right)^{\frac{15}{16}}\left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{\frac{22}{32}}\ldots \tag{9}$$

$$\pi\,e^{7\,\zeta(3)/4\,\pi^2}=2\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n^3+n+8}{2^{n+3}}}=\left(\frac{2}{1}\right)^{\frac{5}{8}}\left(\frac{2^2}{1\cdot 3}\right)^{\frac{7}{16}}\left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{\frac{10}{32}}\ldots \tag{10}$$

$$e^{-\gamma+\frac{\pi}{2}}=6\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(4\,k+3)^{(-1)^{k}\binom{n}{k}}\right)^{\frac{1}{n+1}}=\left(\frac{3}{7}\right)^{\frac{1}{2}}\left(\frac{3\cdot 11}{7^2}\right)^{\frac{1}{3}}\left(\frac{3\cdot 11^3}{7^3\cdot 15}\right)^{\frac{1}{4}}\ldots \tag{11}$$

$$\frac{1}{\sqrt{3}}\,e^{-\gamma-\frac{\pi\sqrt{3}}{6}}=\prod_{n=1}^{\infty}\left(\prod_{k=0}^{n}(3\,k+1)^{(-1)^{k}\binom{n}{k}}\right)^{\frac{1}{n+1}}=\left(\frac{1}{4}\right)^{\frac{1}{2}}\left(\frac{1\cdot 7}{4^2}\right)^{\frac{1}{3}}\left(\frac{1\cdot 7^3}{4^3\cdot 10}\right)^{\frac{1}{4}}\ldots \tag{12}$$

$$\frac{1}{2^{4\,m+1}}\binom{2\,m}{m}^2\pi=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2\,m+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n}},\quad m=0,1,2,3,\ \ldots \tag{13}$$

$$\frac{2^{4\,m-1}}{m}\,\frac{\binom{2\,m}{m}^{-2}}{\pi}=\prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2\,m)^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n}},\quad m=1,2,3,\ \ldots \tag{14}$$

$$e^{-\gamma}=2\prod_{n=1}^{\infty}\left(\prod_{k=1}^{n}(2\,k+1)^{(-1)^{k}\binom{n}{k}}\right)^{\frac{1}{n+1}}=\left(\frac{1}{3}\right)^{\frac{1}{2}}\left(\frac{1\cdot 5}{3^2}\right)^{\frac{1}{3}}\left(\frac{1\cdot 5^3}{3^3\cdot 7}\right)^{\frac{1}{4}}\ldots \tag{15}$$

$$e^{\pi\sqrt{5-2\sqrt{5}}} = \prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}\left(\frac{10\,k+7}{10\,k+3}\right)^{(-1)^k\binom{n}{k}}\right)^{\frac{1}{n+1}} = \left(\frac{7}{3}\right)^{\frac{1}{1}}\left(\frac{7\cdot 13}{3\cdot 17}\right)^{\frac{1}{2}}\left(\frac{3^2\cdot 7\cdot 13^2}{1\cdot 17^2\cdot 23}\right)^{\frac{1}{3}}\ldots \tag{16}$$

$$\frac{\Gamma\left(\frac{x}{2}\right)}{\sqrt{2}\;\Gamma\left(\frac{1+x}{2}\right)} = \prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+x)^{(-1)^{k+1}\binom{n}{k}}\right)^{2^{-n-1}}\quad,\ x>0 \tag{17}$$

$$\frac{1}{\pi}\sqrt{\left(1-\frac{1}{4}\left(\sqrt[3]{\frac{1+i\sqrt{3}}{2}}+\sqrt[3]{\frac{1-i\sqrt{3}}{2}}\right)^2\right)} = \frac{1}{9}\prod_{n=1}^{\infty}\left(1-\frac{1}{81\,n^2}\right) \tag{18}$$

$$\frac{\sqrt{6}-\sqrt{2}}{\pi} = \frac{1}{3}\prod_{n=1}^{\infty}\left(1-\frac{1}{144\,n^2}\right) \tag{19}$$

$$\frac{\sqrt{6}+\sqrt{2}}{\pi} = \frac{5}{3}\prod_{n=1}^{\infty}\left(1-\frac{25}{144\,n^2}\right) \tag{20}$$

$$\frac{5\sqrt{3-\phi}}{2\,\pi} = \sqrt{\frac{1}{2}+\frac{\phi}{4}}\;\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\phi}{4}}}\;\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\phi}{4}}}}\;\ldots\;,\ \phi=\frac{1+\sqrt{5}}{2} \tag{21}$$

$$e^{-\gamma/\left(1-e^{-1}\right)} = e^{\left(a_m+b_m\right)/\left(1-e^{-1}\right)}\prod_{n=m}^{\infty}n^{e^{-n}}\quad,\ m=1,2,3,\ \ldots \tag{22}$$

donde $a_m=\int_0^m e^{-x}\ln(x)\,dx$, $b_m=\sum_{n=m}^{\infty}e^{-n}\int_0^1 e^{-x}\ln\left(1+\frac{x}{n}\right)dx$.

$$e^{-\gamma} = \left(\prod_{n=0}^{\infty}e^{-(-1)^n/\left(n!\,(n+1)^2\right)}\right)\left(\prod_{n=1}^{\infty}\prod_{k=1}^{n}\prod_{m=1}^{2^k-1}\left(n-k+1+m\,2^{-k}\right)^{(-1)^{m+1}\,2^{-k}\,e^{-\left(n-k+1+m\,2^{-k}\right)}}\right) \tag{23}$$

$$A = e^{\frac{1}{4}}\prod_{n=1}^{\infty}e^{\frac{2\,n+1}{4}}\left(\frac{n}{n+1}\right)^{\frac{6\,n(n+1)+1}{12}} = e^{\frac{1}{4}}\left(e^{\frac{3}{4}}\left(\frac{1}{2}\right)^{\frac{13}{12}}\right)\left(e^{\frac{5}{4}}\left(\frac{2}{3}\right)^{\frac{37}{12}}\right)\ldots \tag{24}$$

$$\frac{9}{2\,\pi} = \sqrt{\frac{a}{a-1}}\;\sqrt{\frac{1}{2}+\frac{a}{4}}\;\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{a}{4}}}\;\ldots,\ a=\sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\ldots}}} \tag{25}$$

$$\frac{1}{e} = (\mathrm{Coth}(1)-1)\prod_{n=1}^{\infty}\mathrm{Cosh}\left(\frac{1}{2^n}\right) \tag{26}$$

$$\phi = \frac{1+\sqrt{5}}{2} = \prod_{n=1}^{\infty}\frac{F_n\,F_{n+2}}{F_{n+1}^2} = \left(\frac{1\cdot 2}{1^2}\right)\left(\frac{1\cdot 3}{2^2}\right)\left(\frac{2\cdot 5}{3^2}\right)\ldots \tag{27}$$

$$\phi = \frac{1+\sqrt{5}}{2} = \prod_{n=1}^{\infty} \frac{F_{n^2}\, F_{n^2+2n+2}}{F_{n^2+1}\, F_{n^2+2n+1}} = \left(\frac{5}{3}\right)\left(\frac{33}{34}\right)\left(\frac{54\,298}{54\,285}\right) \ldots \tag{28}$$

En las fórmulas (27) y (28), $F_{n+2} = F_{n+1} + F_n\,,\; F_1 = F_2 = 1\,.$

$$e^{\pi/2\,\sqrt{3}} = \prod_{m=0}^{\infty} \prod_{n=0}^{m} \left(\prod_{k=0}^{m-n+1} (((4\,n+3)\,(4\,n+1+k\,3^{-2\,n}))/((4\,n+1)\,(4\,n+3+k\,3^{-2\,n-1})))^{(-1)^{k+1}\binom{m-n+1}{k}} \right)^{\frac{1}{m-n+1}} \tag{29}$$

$$\sqrt{2+\sqrt{2+\sqrt{2}}} = 2 \prod_{n=1}^{\infty} \frac{p_{n+1}}{p_n}\, 8^{-2^{3\,n-2}} = 2\left(\frac{63}{64}\right)\left(\frac{138\,199\,555\,793\,407}{138\,538\,465\,099\,776}\right) \ldots \tag{30}$$

donde $p_{n+1} = -2^{\frac{3}{7}\, 2^{3\,n+1} - \frac{41}{7}} + 8^{2^{3\,n-2}}\, p_n + 2^{\frac{9 \cdot 8^n - 16}{14}}\, p_n^2 - 5 \cdot 2^{\frac{3 \cdot 8^n - 10}{7}}\, p_n^4 + 2^{\frac{3\,(8^n+6)}{14}}\, p_n^6 - p_n^8\,,\; p_1 = 2$

$$e^{\pi^2/12} \cdot e^{\mathrm{Polylog}(2,-e^{-1})} = \prod_{n=1}^{\infty} \left(\prod_{k=1}^{2^n-1} \left(1 + e^{-k\,2^{-n}}\right)^{(-1)^{k+1}} \right)^{2^{-n}} \tag{31}$$

$$e^{\pi^2/12} = \prod_{n=1}^{\infty} \prod_{k=1}^{2^n-1} (1 + k\,2^{-n})^{(-1)^{k+1}/k} \tag{32}$$

$$2^{\ln(2)} = \prod_{n=1}^{\infty} \prod_{k=1}^{2^n-1} (1 + k\,2^{-n})^{\frac{2\,(-1)^{k+1}}{2^n+k}} \tag{33}$$

$$2^{\ln(2)} = \sqrt{2} \prod_{n=0}^{\infty} \prod_{k=1}^{2^n} \left(1 + \frac{2\,k-2}{2^{n+1}}\right)^{\frac{-1}{2^{n+1}+2\,k-2}} \left(1 + \frac{2\,k-1}{2^{n+1}}\right)^{\frac{2}{2^{n+1}+2\,k-1}} \left(1 + \frac{2\,k}{2^{n+1}}\right)^{\frac{-1}{2^{n+1}+2\,k}} \tag{34}$$

$$e^{-2}\, e^{\pi/2} = \frac{1}{2} \prod_{n=1}^{\infty} \left(\prod_{k=1}^{2^n-1} (1 + k^2\, 2^{-2\,n})^{(-1)^{k+1}} \right)^{2^{-n}} \tag{35}$$

$$e^{4-\pi} = 2 \prod_{n=0}^{\infty} \left(\prod_{k=1}^{2^n} \left(1 + \left(\frac{2\,k-2}{2^{n+1}}\right)^2\right) \left(1 + \left(\frac{2\,k-1}{2^{n+1}}\right)^2\right)^{-2} \left(1 + \left(\frac{2\,k}{2^{n+1}}\right)^2\right) \right)^{2^{-n-1}} \tag{36}$$

$$\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}} = \sqrt{2}\, \sqrt{2^{-1}+2^{-3/2}}\, \sqrt{\left(\left(1+2^{-1/2}\right)^{-1} + \left(1+2^{-1/2}\right)^{-3/2}\right)}\, \sqrt{\left(\left(1+\left(1+2^{-1/2}\right)^{-1/2}\right)^{-1} + \left(1+\left(1+2^{-1/2}\right)^{-1/2}\right)^{-3/2}\right)} \ldots \tag{37}$$

$$\sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\ldots}}} = \sqrt[5]{2}\, \sqrt[5]{2^{-1}+2^{-4/5}}\, \sqrt[5]{\left(1+2^{1/5}\right)^{-1} + \left(1+2^{1/5}\right)^{-4/5}}\, \left(\left(1+\left(1+2^{1/5}\right)^{1/5}\right)^{-1} + \left(1+\left(1+2^{1/5}\right)^{1/5}\right)^{-4/5}\right) \wedge (1/5) \ldots \tag{38}$$

$$\frac{2\,\pi\,\sqrt{3}}{27} + \frac{4}{3} = \prod_{n=1}^{\infty} \left(1 + \left(\binom{0}{0}\binom{2}{1}\binom{4}{2}\binom{6}{3} \ldots \binom{2\,n-2}{n-1}\right) \Big/ \left(\binom{2\,n}{n} a_n\right)\right) \tag{39}$$

donde $a_{n+2} = \left(\binom{2\,n}{n} + \binom{2\,n+2}{n+1}\right) a_{n+1} - \binom{2\,n}{n}^2 a_n\,,\; a_1 = 1,\, a_2 = 3$

$$\frac{\pi}{\sqrt{3}} = 2\prod_{n=1}^{\infty}\left(1+\frac{(-1)^n\,(2\,n-1)!!}{a_n}\right) \tag{40}$$

donde $a_{n+2} = \frac{4\,(n+2)\,(2\,n+5)}{2\,n+3}\,a_{n+1} + 3\,(2\,n+1)\,(2\,n+5)\,a_n$, $a_1 = 9$, $a_2 = 120$; $(2\,n-1)!! = 1\cdot 3\cdot 5\cdots(2\,n-1)$

$$\pi = 8\left(\sqrt{2}-1\right)\prod_{n=1}^{\infty}\left(1+\frac{(-1)^n\,(2\,n-1)!!}{a_n}\right) \tag{41}$$

donde $a_{n+2} = \frac{\left(4\,n+8+(4\,n+6)\sqrt{2}\right)(2\,n+5)}{2\,n+3}\,a_{n+1} + (2\,n+1)\,(2\,n+5)\left(3+2\sqrt{2}\right)a_n$, $a_1 = 9+6\sqrt{2}$, $a_2 = 240+170\sqrt{2}$

$$\sqrt[3]{2} = \prod_{n=1}^{\infty}\frac{a_{n+1}}{3\,a_n^3} \tag{42}$$

donde $a_{n+2} = 2\,a_{n+1}^3 + 27\,a_n^6\,a_{n+1} - 54\,a_n^9$, $a_1 = 1$, $a_2 = 4$

$$\sqrt[3]{2} = 1 + \frac{1}{3}\prod_{n=1}^{\infty}\frac{a_{n+1}^2}{a_n\,a_{n+2}} = 1 + \frac{1}{3}\left(\frac{3^2}{1\cdot 12}\right)\left(\frac{12^2}{3\cdot 46}\right)\left(\frac{46^2}{12\cdot 177}\right)\ldots \tag{43}$$

donde $a_{n+3} = 3\,a_{n+2} + 3\,a_{n+1} + a_n$, $a_1 = 1$, $a_2 = 3$, $a_3 = 12$

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}} = \frac{1}{-1+9\prod_{n=1}^{\infty}\frac{a_n\,a_{n+2}}{a_{n+1}^2}} = 1\Big/\left(-1+9\left(\frac{1\cdot 45}{9^2}\right)\left(\frac{9\cdot 168}{45^2}\right)\left(\frac{45\cdot 531}{168^2}\right)\ldots\right) \tag{44}$$

donde $a_{n+9} = 9\,a_{n+8} - 36\,a_{n+7} + 87\,a_{n+6} - 144\,a_{n+5} + 171\,a_{n+4} - 141\,a_{n+3} + 72\,a_{n+2} - 18\,a_{n+1} + 2\,a_n$,
$a_1 = 1$, $a_2 = 9$, $a_3 = 45$, $a_4 = 168$, $a_5 = 531$, $a_6 = 1521$, $a_7 = 4107$, $a_8 = 10\,710$, $a_9 = 27\,414$

$$\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1} = \frac{1}{2}\left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{125}\right) = 6\prod_{n=1}^{\infty}\frac{a_n\,a_{n+2}}{a_{n+1}^2} = 6\left(\frac{1\cdot 30}{6^2}\right)\left(\frac{6\cdot 150}{30^2}\right)\left(\frac{30\cdot 755}{150^2}\right)\ldots \tag{45}$$

donde $a_{n+4} = 6\,a_{n+3} - 6\,a_{n+2} + 6\,a_{n+1} - a_n$, $a_1 = 1$, $a_2 = 6$, $a_3 = 30$, $a_4 = 150$

$$\sqrt{\sqrt[3]{28}-3} = \frac{1}{3}\left(\sqrt[3]{98}-\sqrt[3]{28}-1\right) = \frac{1}{5}\prod_{n=1}^{\infty}\frac{a_{n+1}^2}{a_n\,a_{n+2}} = \frac{1}{5}\left(\frac{5^2}{1\cdot 26}\right)\left(\frac{26^2}{5\cdot 136}\right)\left(\frac{136^2}{26\cdot 711}\right)\ldots \tag{46}$$

donde $a_{n+3} = 5\,a_{n+2} + a_{n+1} + a_n$, $a_1 = 1$, $a_2 = 5$, $a_3 = 26$

$$-\frac{1-i\sqrt{3}}{2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}} - \frac{1}{2}\left(\frac{1}{2}\left(-1+i\sqrt{3}\right)\right)^{1/3}\left(1+i\sqrt{3}\right) =$$

$$\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}+\ldots\right)^3\right)^3\right)^3 = \frac{1}{3}\prod_{n=1}^{\infty}\frac{a_{n+1}^2}{a_n\,a_{n+2}} = \frac{1}{3}\left(\frac{3^2}{1\cdot 9}\right)\left(\frac{9^2}{3\cdot 26}\right)\left(\frac{26^2}{9\cdot 75}\right)\ldots \tag{47}$$

donde $a_{n+3} = 3\,a_{n+2} - a_n$, $a_1 = 1$, $a_2 = 3$, $a_3 = 9$

$$\sqrt{2} = \prod_{n=0}^{\infty} \frac{a_{n+1}\, b_n}{a_n\, b_{n+1}} = \left(\frac{3}{2}\right)\left(\frac{2\cdot 7}{3\cdot 5}\right)\left(\frac{5\cdot 17}{7\cdot 12}\right) \ldots \tag{48}$$

donde $a_{n+2} = 2\, a_{n+1} + a_n$, $a_0 = 1, a_1 = 3$; $b_{n+2} = 2\, b_{n+1} + b_n$, $b_0 = 1, b_1 = 2$

$$\sqrt{3} - \sqrt{2} = \frac{1}{4} \prod_{n=1}^{\infty} \frac{a_{n+1}\, b_n\, c_{n+1}\, d_n}{a_n\, b_{n+1}\, c_n\, d_{n+1}} \tag{49}$$

donde $a_{n+2} = 2\, a_{n+1} + a_n$, $a_0 = 1, a_1 = 3$; $b_{n+2} = 2\, b_{n+1} + b_n$, $b_0 = 1, b_1 = 2$; $c_{n+1} = c_n + 4\, d_n$, $d_{n+1} = 2\, c_n + 9\, d_n$, $c_1 = 1, d_1 = 2$

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, \ldots] = 3\left(\frac{2\cdot 11}{3\cdot 7}\right)\left(\frac{3^2\cdot 7\cdot 37}{2^2\cdot 11\cdot 53}\right)\left(\frac{2\cdot 5\cdot 53\cdot 71}{3^2\cdot 37\cdot 113}\right)\left(\frac{113\cdot 103\,993}{2\cdot 3^3\cdot 5\cdot 71\cdot 613}\right) \ldots \tag{50}$$

$$\sqrt[4]{1 + \sqrt[4]{1 + \sqrt[4]{1 + \ldots}}} = \frac{16}{11} \prod_{n=1}^{\infty} \frac{a_n\, a_{n+2}}{a_{n+1}^2} = \frac{16}{11}\left(\frac{11\cdot 21}{16^2}\right)\left(\frac{16\cdot 22}{21^2}\right)\left(\frac{21\cdot 27}{22^2}\right) \ldots \tag{51}$$

donde $a_{n+4} = a_{n+1} + a_n$, $a_1 = 11, a_2 = 16, a_3 = 21, a_4 = 22$

$$\gamma = \prod_{n=1}^{\infty} \left(1 + \frac{\frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right)}{H_n - \ln(n)}\right) \tag{52}$$

donde $H_n = \sum_{k=1}^{n} \frac{1}{k}$

$$\ln(2) = \frac{\zeta(2)}{4} \prod_{n=1}^{\infty} (1 + \zeta(n+2) / (2^n\, \zeta(2) + 2^{n-1}\, \zeta(3) + \ldots + 2\, \zeta(n+1))) \tag{53}$$

donde $\sum_{k=1}^{\infty} k^{-n} = \zeta(n)$, $n > 1$, función zeta de Riemann.

$$\sqrt{2} = \prod_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{q_n^2}\right) = \left(\frac{1}{2} + 1\right)\left(\frac{1}{2} + \left(\frac{2}{3}\right)^2\right)\left(\frac{1}{2} + \left(\frac{12}{17}\right)^2\right) \ldots \tag{54}$$

donde $q_{n+1} = \frac{1}{2}\, q_n + \frac{1}{q_n}$, $q_1 = 1$

$$\sqrt{2} = \prod_{n=1}^{\infty} \frac{4}{2 + q_n^2} = \left(\frac{4}{2+1}\right)\left(\frac{4}{2 + (4/3)^2}\right)\left(\frac{4}{2 + (48/34)^2}\right) \ldots \tag{55}$$

donde $q_{n+1} = \frac{4\, q_n}{2 + q_n^2}$, $q_1 = 1$

$$\sqrt{2} = \prod_{n=1}^{\infty} \left(\frac{3}{2} - \frac{1}{4}\, q_n^2\right) = \left(\frac{5}{4}\right)\left(\frac{3}{2} - \frac{1}{4}\left(\frac{5}{4}\right)^2\right)\left(\frac{3}{2} - \frac{1}{4}\left(\frac{355}{256}\right)^2\right) \ldots \tag{56}$$

donde $q_{n+1} = \frac{3}{2}\, q_n - \frac{1}{4}\, q_n^3$, $q_1 = 1$

$$\phi = \prod_{n=1}^{\infty}\left(1+\frac{1}{\phi^{2n+1}-\phi}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}+(F_{2n+1}-1)\,\phi}\right) \tag{57}$$

donde F_n , se define como en (28).

$$\sqrt[5]{2}-1 = \frac{1}{5}\prod_{n=1}^{\infty}\frac{a_{n+1}^2}{a_n\,a_{n+2}} \tag{58}$$

donde $a_{n+5} = 5\,a_{n+4} + 10\,a_{n+3} + 10\,a_{n+2} + 5\,a_{n+1} + a_n$, $a_1 = 1, a_2 = 5, a_3 = 35, a_4 = 235, a_5 = 1580$

$$1+\sqrt{2}+i = (2+2\,i)\prod_{n=1}^{\infty}\frac{a_n\,a_{n+2}}{a_{n+1}^2} = (2+2\,i)\left(\frac{3-i}{4}\right)\left(\frac{28-4\,i}{25}\right)\left(\frac{17+i}{16}\right)\ldots \tag{59}$$

donde $a_{n+2} = (2+2\,i)\,a_{n+1} + (2-2\,i)\,a_n$, $a_1 = 1, a_2 = 2+2\,i$

$$\pi = 4\prod_{n=1}^{\infty}p_n = 4\,(\mathrm{Cot}(1))\,(\mathrm{Cot}(\mathrm{Cot}(1)))\,(\mathrm{Cot}(\mathrm{Cot}(1)\,\mathrm{Cot}(\mathrm{Cot}(1))))\ \ldots \tag{60}$$

donde $p_{n+1} = \mathrm{Cot}\left(\prod_{k=1}^{n} p_n\right)$, $p_1 = \mathrm{Cot}(1)$

$$\ln(2) = \prod_{n=1}^{\infty}p_n = (2\,e^{-1})\left(2\,e^{-2\,e^{-1}}\right)\left(2\,e^{-4\,e^{-1-2\,e^{-1}}}\right)\ldots \tag{61}$$

donde $p_{n+1} = 2\exp\left(-\prod_{k=1}^{n} p_n\right)$, $p_1 = 2\,e^{-1}$

$$e = \prod_{n=1}^{\infty}p_n = 3\left(\frac{1}{\ln(3)}\right)\left(\frac{1}{\ln\left(\frac{3}{\ln 3}\right)}\right)\left(\frac{1}{\ln\left(\frac{3}{\ln(3)\,\ln\left(\frac{3}{\ln 3}\right)}\right)}\right)\ldots \tag{62}$$

donde $p_{n+1} = \left(\ln\left(\prod_{k=1}^{n} p_n\right)\right)^{-1}$, $p_1 = 3$

$$\cfrac{e^{-2\pi/5}}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\ldots}}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} = \frac{1}{2}\prod_{n=1}^{\infty}\frac{a_{n+1}^2}{a_n\,a_{n+2}} = \frac{1}{2}\left(\frac{2^2}{1\cdot 10}\right)\left(\frac{10^2}{2\cdot 30}\right)\ldots \tag{63}$$

donde $a_{n+4} = 2\,a_{n+3} + 6\,a_{n+2} - 2\,a_{n+1} - a_n$, $a_1 = 1, a_2 = 2, a_3 = 10, a_4 = 30$

$$i = \sqrt{-1} = \prod_{n=1}^{\infty}\left(\mathrm{Cos}\left(\frac{\pi}{2^{n+1}}\right) + i\,\mathrm{Sin}\left(\frac{\pi}{2^{n+1}}\right)\right) =$$

$$\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}+\frac{i\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}\right)\ldots \tag{64}$$

$$\frac{1}{2}+\frac{i\sqrt{3}}{2}=\prod_{n=1}^{\infty}\left(\mathrm{Cos}\left(\frac{\pi}{2^{2n}}\right)+i\,\mathrm{Sin}\left(\frac{\pi}{2^{2n}}\right)\right)=\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}+\frac{i\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}\right)\ldots$$

$$\frac{\sqrt{3}}{2}+\frac{i}{2}=\prod_{n=1}^{\infty}\left(\mathrm{Cos}\left(\frac{\pi}{2^{2n+1}}\right)+i\,\mathrm{Sin}\left(\frac{\pi}{2^{2n+1}}\right)\right)=$$
$$\left(\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)\left(\frac{1}{2}\left(\sqrt{\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}\right)+\frac{1}{2}i\sqrt{\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}\right)\ldots \tag{66}$$

$$\frac{1}{\pi}=\frac{5}{16}\prod_{n=1}^{\infty}\left(1+\binom{2n+2}{n+1}^3\cdot\frac{(42n+47)}{4096\,a_n}\right) \tag{67}$$

donde $a_{n+1}=4096\,a_n+\binom{2n+2}{n+1}^3(42n+47)$, $a_0=5$

$$\frac{1+i\sqrt{3}}{2}=\left(\frac{1}{2}+i\right)\prod_{n=1}^{\infty}\frac{u_{n+1}\,v_n}{u_n\,v_{n+1}}=\left(\frac{1}{2}+i\right)\left(\frac{2(-3+3i)}{(1+2i)(1+4i)}\right)\left(\frac{(1+4i)(-7-4i)}{(-3+3i)(-7+4i)}\right)\ldots \tag{68}$$

donde $u_{n+1}=(1+i)\,u_n-v_n$, $v_{n+1}=u_n+i\,v_n$, $u_1=1+2i$, $v_1=2$

$$\pi=3.1415926535\ldots=a_1\prod_{n=1}^{\infty}\left(1+\frac{a_{n+1}}{10\,\langle a_1\,a_2\,a_3\ldots a_n\rangle}\right)=3\left(1+\frac{1}{30}\right)\left(1+\frac{4}{310}\right)\left(1+\frac{1}{3140}\right)\ldots \tag{69}$$

donde los a_n representan los digitos de Pi: $a_1=3$, $a_2=1$, $a_3=4$, $a_4=1$, ... y $\langle a_1\,a_2\,a_3\ldots a_n\rangle$ es el número entero formado por los digitos $a_1, a_2, a_3, \ldots, a_n$.

$$\prod_{n=2}^{\infty}\frac{n^{2m}+1}{n^{2m}-1}=e^{\sum_{n=0}^{\infty}\frac{2}{2n+1}(\zeta(4mn+2m)-1)}\quad,\ m=1,2,3,\ \ldots \tag{70}$$

$$\prod_{n=1}^{\infty}\left(1+e^{-n}\right)=e^{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(e^n-1)}} \tag{71}$$

$$\prod_{n=1}^{\infty}\left(1-e^{-n}\right)=e^{-\sum_{n=1}^{\infty}\frac{1}{n(e^n-1)}} \tag{72}$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{2}\cdot\overbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}}}^{n\text{-radicales}}\right)=\exp\left(\sum_{n=1}^{\infty}\frac{a_n\,\pi^n\,2^{-2n}}{1-2^{-n}}\right) \tag{73}$$

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{2}\cdot\overbrace{\sqrt{2-\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}}}^{n\text{-radicales}}\right)=\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^n a_n \pi^n 2^{-2n}}{1-2^{-n}}\right)$$

En las formulas (73) y (74) , se tiene:

$$a_{2n-1}=\begin{cases}1 & n=1\\ \frac{E_{n-1}}{(2n-1)!} & n=2,3,\ \ldots\end{cases},\quad a_{2n}=\frac{-2^{2n}\left(2^{2n}-1\right)B_n}{(2n)(2n)!},\quad n=1,2,3,\ \ldots,$$

$\{E_n : n=1,2,\ \ldots\}=\{1,5,61,1385,\ldots\}$, números de Euler ,

$\{B_n :\ n=1,2,\ \ldots\}=\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\ \ldots\}$, números de Bernoulli.

Referencias

1. Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions, Nueva York: Dover , 1965.
2. Boros, G., and Moll, V.: Irresistible Integrals. Cambridge University Press , 2004.
3. Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals , Series and Products. 5-th ed., ed. Alan Jeffrey. Academic Press , 1994.
4. Guillera, J. and Sondow, J., Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent. e-print (2005). http://www.arxiv.org/abs/math.NT/0506319.
5. Spiegel, M.R.: Mathematical Handbook , McGraw-Hill Book Company , New York, 1968.
6. Valdebenito, E.: Pi Handbook , manuscript, unpublished, 1989, (20000 formulas).