Metallic Numbers , Mathematical Constants

Edgar Valdebenito

Abstract

Formulas involving metallic numbers shown

Constantes Matemáticas , Números Metálicos

Edgar Valdebenito V
(2005)

Resumen. Se muestran fórmulas que involucran los números metálicos ϕ_n y algunas constantes clásicas, como son : la constante π, $ln(2)$, G (Catalan) .En algunas fórmulas aparece la función zeta de Riemann $\zeta(x)$.

1 Introducción

Los números metálicos se definen por:

$$\phi_n = \frac{n+\sqrt{n^2+4}}{2} \quad , n \in \mathbb{N} \tag{1.1}$$

y satisfacen la ecuación:

$$\phi_n^2 - n\phi_n - 1 = 0 \quad , n \in \mathbb{N} \tag{1.2}$$

Algunas representaciones para ϕ_n son:

$$\phi_n = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \dots}}} \tag{1.3}$$

$$\phi_n = \sqrt{1+n\sqrt{1+n\sqrt{1+\dots}}} \tag{1.4}$$

$$\phi_n = \sqrt[3]{n+\left(n^2+1\right)\sqrt[3]{n+\left(n^2+1\right)\sqrt[3]{n+\dots}}} \tag{1.5}$$

$$\phi_n = \sqrt[4]{n^2+1+n\left(n^2+2\right)\sqrt[4]{n^2+1+n\left(n^2+2\right)\sqrt[4]{n^2+1+\dots}}} \tag{1.6}$$

2 La Sucesión Generalizada de Fibonacci

La sucesión generalizada de Fibonacci $u_{n,k}$, $n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}$ se define por:

$$u_{n,k}=\frac{\phi_n^k-(-\phi_n)^{-k}}{\sqrt{n^2+4}} \quad ,k=0,1,2,3,... \tag{2.1}$$

recurrencia para $u_{n,k}$:

$$u_{n,k+2}=nu_{n,k+1}+u_{n,k} \quad ,u_{n,0}=0\,,u_{n,1}=1 \tag{2.2}$$

la sucesión $u_{n,k}$ satisface la relación:

$$\lim_{k\to\infty}\frac{u_{n,k+1}}{u_{n,k}}=\phi_n \quad ,n\in\mathbb{N} \tag{2.3}$$

3 Fórmulas

3.1. Para $m\in\mathbb{N}$, se tiene:

$$\phi_n^m-A_m\phi_n-B_m=0 \tag{3.1}$$

$$A_1=1\,,B_1=0 \tag{3.2}$$

$$A_{m+1}=nA_m+B_m\,,B_{m+1}=A_m \tag{3.3}$$

$$A_2=n\,,B_2=1 \tag{3.4}$$

$$A_3=n^2+1\,,B_3=n \tag{3.5}$$

$$A_4=n^3+2n\,,B_4=n^2+1 \tag{3.6}$$

$$A_5=n^4+3n^2+1\,,B_5=n^3+2n \tag{3.7}$$

3.2. Para $m\in\mathbb{N}$, se tiene:

$$\phi_n^m=\frac{a_{n,m}+b_{n,m}\sqrt{n^2+4}}{2^m} \tag{3.8}$$

$$a_{n,m+1}=na_{n,m}+\left(n^2+4\right)b_{n,m} \tag{3.9}$$

$$b_{n,m+1}=a_{n,m}+nb_{n,m} \tag{3.10}$$

$$a_{n,1}=n\,,b_{n,1}=1 \tag{3.11}$$

3.3. Para $n\in\mathbb{N}$, se tiene:

$$ln(\phi_n)=\int_0^{n/2}\frac{1}{\sqrt{x^2+1}}dx \qquad (3.12)$$

$$ln(\phi_n)=\int_2^{\infty}\frac{n}{x\sqrt{n^2+x^2}}dx \qquad (3.13)$$

$$\frac{n}{4\phi_n}+\frac{1}{2}ln(\phi_n)=\int_0^{n}\frac{1}{x+\sqrt{x^2+4}}dx \qquad (3.14)$$

$$ln(\phi_n)=\int_0^{n}\frac{1}{\sqrt{x^2+4}}dx \qquad (3.15)$$

$$\pi\, ln(\phi_n)=\int_0^{1}\frac{ln\left(n^2+4x^2\right)}{\sqrt{1-x^2}}dx \qquad (3.16)$$

3.4. Para $s>0$, se tiene:

$$\sum_{n=1}^{\infty}\frac{1}{n^{s+1}\phi_{2n}}<\frac{1}{2}\zeta(s+2) \qquad (3.17)$$

3.5. Para $m\in\mathbb{N}$, se tiene:

$$\frac{\pi}{4}=tan^{-1}\left(\frac{\phi_n}{m+\phi_n}\right)+\sum_{k=1}^{m}tan^{-1}\left(\frac{\phi_n}{2+k(k-1)+(2k+2n-1)\phi_n}\right) \qquad (3.18)$$

3.6.

$$\sum_{n=1}^{\infty}\frac{1}{n\phi_{2n}}=\frac{\zeta(2)}{2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)!\,\zeta(2n+2)}{2^{2n}(n!)^2(2n+2)} \qquad (3.19)$$

3.7.

$$ln(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\phi_n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\phi_n^2} \qquad (3.20)$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\phi_{2n-1}}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)\phi_{2n-1}^{2}} \tag{3.21}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{\phi_{n^2}}+\sum_{n=1}^{\infty}\left(\frac{1}{n\phi_{n^2}}\right)^2 \tag{3.22}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n\phi_n}+\sum_{n=1}^{\infty}\left(\frac{1}{n\phi_n}\right)^2 \tag{3.23}$$

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\phi_{2n-1}}+\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)\phi_{2n-1}}\right)^2 \tag{3.24}$$

$$G=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)\phi_{2n-1}}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{(2n-1)\phi_{2n-1}}\right)^2 \tag{3.25}$$

3.8. Para $n\in\mathbb{N}$, se tiene:

$$\phi_n=n+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{u_{n,k}u_{n,k+2}} \tag{3.26}$$

3.9.

$$ln(\phi_1)=\frac{1}{2}-\frac{1}{2}\frac{1}{3\cdot 2^3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5\cdot 2^5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7\cdot 2^7}+... \tag{3.27}$$

Para $n=2,3,4,...$, se tiene:

$$ln(\phi_n)=ln(n)+\frac{1}{2}\frac{2^2}{2n^2}-\frac{1\cdot 3}{2\cdot 4}\frac{2^4}{4n^4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{2^6}{6n^6}-... \tag{3.28}$$

3.10. Para $n\in\mathbb{N}$, se tiene:

$$\frac{\pi}{4}=\int_0^{\phi_n}\frac{2x-n}{1+n^2x^2-2nx^3+x^4}dx \tag{3.29}$$

3.11. Para $n,k\in\mathbb{N}$, se tiene:

$$\frac{u_{n,2k}}{u_{n,2k-1}} < \phi_n < \frac{u_{n,2k+1}}{u_{n,2k}} \tag{3.30}$$

3.12. Para $n \in \mathbb{N}$, se tiene:

$$\sum_{k=1}^{\infty} \frac{\left(2^{2k}-1\right)\zeta\left(2k\right)}{2^{2k}\phi_n^{2k}} = \sum_{k=1}^{\infty} \frac{1}{\left(2k-1\right)^2 \phi_n^2 - 1} \tag{3.31}$$

$$\sum_{k=1}^{\infty}\left(e^{\phi_n/k} - e^{-1/k\phi_n} - \frac{\sqrt{n^2+4}}{k}\right) = \sqrt{n^2+4}\sum_{k=2}^{\infty}\frac{u_{n,k}\,\zeta\left(k\right)}{k!} \tag{3.32}$$

3.13. Para $n \in \mathbb{N}$, se tiene:

$$\phi_n = n+1+2\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_k}{k!}\int_0^{n/2}\frac{dx}{\left(1+x^2\right)^k} \tag{3.33}$$

$$\int_0^{n/2}\frac{dx}{\left(1+x^2\right)^k} = p_k + q_k \tan^{-1}\left(\frac{n}{2}\right) \tag{3.34}$$

$$p_{k+1} = \frac{n\,4^{k-1}}{k\left(4+n^2\right)^k} + \frac{2k-1}{2k}\,p_k \ , \ p_1 = 0 \tag{3.35}$$

$$q_{k+1} = \frac{2k-1}{2k}\,q_k \ , \ q_1 = 1 \tag{3.36}$$

3.14. Para $n \in \mathbb{N}$, se tiene:

$$\phi_n = \frac{1}{2} + \frac{1}{2F\left(-\frac{1}{2},1;1;\frac{4}{4+n^2}\right)} \tag{3.37}$$

$$\phi_n = \frac{n}{2} + \frac{n+2}{2}F\left(-\frac{1}{2},1;1;\frac{4n}{\left(n+2\right)^2}\right) \tag{3.38}$$

F , es la función hipergeométrica

3.15. Para $n \in \mathbb{N}$, se tiene:

$$ln(2) = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\phi_k^{-2n+2k-1} \tag{3.39}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\phi_{2k-1}^{-2n+2k-1} \tag{3.40}$$

$$G = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}\phi_{2k-1}^{-2n+2k-1} \tag{3.41}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\phi_{k^2}^{-2n+2k-1} \tag{3.42}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(n-k+1)\phi_k^{-2n+2k-2} \tag{3.43}$$

$$G = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}(n-k+1)\phi_{2k-1}^{-2n+2k-2} \tag{3.44}$$

3.16. Para $n \in \mathbb{N}$, se tiene:

$$\frac{\pi}{2} = \sum_{k=1}^{\infty}tan^{-1}\left(\frac{n}{u_{n,2k-1}}\right) \tag{3.45}$$

$$\frac{\pi}{4} = \sqrt{n^2+4}\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{2k+1}u_{n,2k+1}}{(2k+1)\left(\sqrt{n^2+4}+\sqrt{n^2+8}\right)^{2k+1}} \tag{3.46}$$

3.17.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\phi_{2n}}\right)^2 = \frac{1}{6} + \int_0^1\frac{4x\sqrt{1-x^2}}{e^{2\pi x}-1}dx \tag{3.47}$$

3.18. Para $n \in \mathbb{N}$, se tiene:

$$\sum_{k=1}^{\infty}\frac{1}{u_{n,k}} = \sqrt{n^2+4}\sum_{k=1}^{\infty}\frac{\phi_n^{-k}}{1-(-1)^k\phi_n^{-2k}} = \sqrt{n^2+4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\phi_n^{-2k+1}}{1-(-1)^{k-1}\phi_n^{-2k+1}} \tag{3.48}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{u_{n,k}}=\sqrt{n^2+4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\phi_n^{-k}}{1-(-1)^k\phi_n^{-2k}}=\sqrt{n^2+4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\phi_n^{-2k+1}}{1-(-1)^k\phi_n^{-2k+1}} \quad (3.49)$$

3.19.

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{u_{n,k}u_{n,k+2}}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{u_{k,n-k+1}u_{k,n-k+3}} \quad (3.50)$$

$$\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{u_{n,k}u_{n,k+2}}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{u_{k,n-k+1}u_{k,n-k+3}} \quad (3.51)$$

3.20. Para $n\in\mathbb{N}$, se tiene:

$$\sum_{k=0}^{\infty}\frac{1}{k^2-nk-1}=-\frac{1}{2\phi_n-n}\left(\psi(-\phi_n)-\psi\left(\frac{1}{\phi_n}\right)\right) \quad (3.52)$$

$\psi(x)=\frac{d}{dx}\ln\Gamma(x)$, es la función Psi

3.21.

$$\sum_{n=2k+1}^{\infty}\frac{(-1)^{n-1}}{\phi_n}<\ln(2)-\sum_{n=1}^{2k}\frac{(-1)^{n-1}}{n} \quad , \quad k\in\mathbb{N} \quad (3.53)$$

$$\sum_{n=2k}^{\infty}\frac{(-1)^{n-1}}{\phi_n}>\ln(2)-\sum_{n=1}^{2k-1}\frac{(-1)^{n-1}}{n} \quad , \quad k\in\mathbb{N} \quad (3.54)$$

4 Referencias

1) Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover , 1965.
2) I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey) , Academic Press, New York, London, and Toronto, 1980.
3) M. R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York, 1968.
4) E. Valdebenito, Pi Handbook, manuscript, unpublished, 1989 , (20000 fórmulas).
5) Vera W. de Spinadel , From The Golden Mean to Chaos , Nueva Librería , 1998.
6) Vera W. de Spinadel , The metallic means family and multifractal spectra , Nonlinear Analysis , Vol 36 , pp. 721-745 , 1999.