Double Fourier harmonic balance method for nonlinear oscillators by means of Bessel series

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Abstract

The standard harmonic balance method consists in expanding the displacement of an oscillator as a Fourier cosine series in time. A key modification is proposed here, in which the conservative force is additionally expanded as a Fourier sine series in space. As a result, the steady-state oscillation frequency can be expressed in terms of a Bessel series, and the sums of many such series are known or can be developed. The method is illustrated for five different physical situations, including a ball rolling inside a V-shaped ramp, an electron attracted to a charged filament, a large-amplitude pendulum, and a Duffing oscillator. As an example of the results, the predicted period of a simple pendulum swinging between $-90°$ and $+90°$ is found to be only 0.4% larger than the exact value. Even better, the predicted frequency for the V-ramp case turns out to be exact.

Keywords: Fourier expansion, harmonic balance, Bessel series, work-energy theorem, nonlinear oscillator, pendulum.

1 Introduction

Nonlinear oscillators are ubiquitous in physical and engineering systems. Approximate analytical solutions are useful for exploring the features of such systems. Simple perturbation theory, however, can give rise to secular terms (i.e., ones that diverge in time) which result in unphysical solutions. As a consequence, alternative approximations are needed. One of these is the Lindstedt-Poincaré method [1], in which both the frequency and the nonlinearity are expanded as power series in an appropriate parameter. The requirement that there should not be unbounded terms then yields a condition for the amplitude-frequency relationship. J.-H. He has developed other approaches for analyzing nonlinear oscillators, including the homotopy and the variational iteration methods [2]. More widespread is the harmonic balance method [3], wherein the displacement is expanded as a series of frequency overtones. The equation of motion imposes a relationship between the amplitude and the frequency of the system. The lowest nonzero term—the harmonic—is dominant.

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In the present article, a double Fourier method is developed that improves upon the conventional harmonic balance approximation [4, 5, 6]. Five key examples of physical importance are presented here to illustrate the method and the results are compared with solutions obtained using a work-energy principle. In the course of applying the method to these examples, novel sums of certain Bessel series are developed from special cases tabulated in the literature. In those cases when the Bessel series can be summed analytically, the new method in effect extends standard harmonic balance to include all overtones. Even when the Bessel series cannot be analytically summed, the first few terms provide insight about the behavior of the system.

2 Approximate frequency using double Fourier harmonic balance

The goal of this section is to estimate the frequency of oscillation, an important parameter of the motion [5]. Write the equation of motion for the displacement x as a function of time t in the form

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{1}$$

where $f(x)$ is the negative of the force per unit mass and is in general a nonlinear function of x. Assume the amplitude of oscillation is A, with the displacement varying continuously over the range $-A \le x \le A$. Choosing the origin so that $f(0) = 0$, expand $f(x)$ in the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{A}\right) \tag{2}$$

where the coefficients are

$$b_n = \frac{2}{A}\int_0^A f(x)\sin\left(\frac{n\pi x}{A}\right)dx. \tag{3}$$

The equation of motion then becomes

$$\frac{d^2x}{dt^2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{A}\right) = 0. \tag{4}$$

Assume the oscillator starts out at $t = 0$ with zero initial velocity at $x = A$, has zero displacement at a quarter period $t = T/4$, just attains $x = -A$ with zero velocity at half a period, crosses back through $x = 0$ at three-quarters of a period, and the cycle starts over at $t = T$. In that case, the displacement can be written as a cosine series in time with only odd harmonics due to the symmetry,

$$x(t) = A_1 \cos\omega t + A_3\cos 3\omega t + A_5 \cos 5\omega t + ... \tag{5}$$

where the angular frequency of oscillation is $\omega = 2\pi/T$ and the total amplitude is

$$A = A_1 + A_3 + A_5 + ... \tag{6}$$

Substitution of Eq. (5) into (4) leads to

$$\sum_{\text{odd } m=1}^{\infty} A_m (m\omega)^2 \cos m\omega t = \sum_{n=1}^{\infty} b_n \sin\left[\frac{(A - A_3 - ...)n\pi\cos\omega t + A_3 n\pi \cos 3\omega t + ...}{A}\right] \tag{7}$$

where Eq. (6) was used to eliminate A_1 on the right-hand side. Defining the difference between the exact and linear-order solution as

$$\delta(t) \equiv \sum_{\text{odd } m=3}^{\infty} \frac{A_m}{A} \left(\cos m\omega t - \cos \omega t\right) = \frac{x(t)}{A} - \cos \omega t, \tag{8}$$

Eq. (7) can be rewritten as

$$\omega^2 \sum_{k=0}^{\infty} A_{2k+1}(2k+1)^2 \left[\cos(2k+1)\omega t\right] = \sum_{n=1}^{\infty} b_n \sin(n\pi \cos \omega t + n\pi\delta). \tag{9}$$

This expression is exact, but now make the approximation that δ can be neglected in the context of Eq. (9). Expression 9.1.45 in Abramowitz and Stegun [7] is

$$\sin\left(z \cos \theta\right) = 2 \sum_{k=0}^{\infty} \left(-1\right)^k J_{2k+1}\left(z\right) \cos\left[(2k+1)\theta\right] \tag{10}$$

in which J is a Bessel function of integer order. (This expression is a real-valued Jacobi-Anger expansion.) Consequently

$$\sin\left(n\pi \cos \omega t\right) = 2 \sum_{k=0}^{\infty} \left(-1\right)^k J_{2k+1}\left(n\pi\right) \cos\left[(2k+1)\omega t\right]. \tag{11}$$

Substituting this series into the right-hand side of Eq. (9) with $\delta = 0$ and equating coefficients of the time harmonics leads to

$$\omega^2 A_{2k+1} = \frac{2(-1)^k}{(2k+1)^2} \sum_{n=1}^{\infty} b_n J_{2k+1}(n\pi). \tag{12}$$

Summing both sides over integer values of k from 0 to ∞ gives

$$\omega^2 = \frac{2}{A} \sum_{n=1}^{\infty} b_n \left[J_1(n\pi) - \frac{1}{9} J_3(n\pi) + \frac{1}{25} J_5(n\pi) - ...\right] \tag{13}$$

using Eq. (6). Recalling that b_n is determined for a given force law by Eq. (3), this expression predicts the angular frequency of oscillation, but it is approximate because of the neglect of δ in Eq. (9). To confirm the accuracy of this approximation, the prediction can be compared to the exact frequency for various physically interesting systems.

3 Exact frequency calculated using the work-kinetic-energy theorem

The acceleration $a \equiv d^2x/dt^2$ is related to the velocity $\upsilon \equiv dx/dt$ by

$$a = \frac{d\upsilon}{dt} = \frac{dx}{dt}\frac{d\upsilon}{dx} = \upsilon \frac{d\upsilon}{dx} \tag{14}$$

and therefore Eq. (1) becomes

$$\upsilon d\upsilon = -f(x)dx. \tag{15}$$

Integrating this expression from $\upsilon = 0$ at $x = A$ to velocity υ at displacement x gives

$$\tfrac{1}{2}\upsilon^2 = \int_x^A f(x')dx'. \tag{16}$$

The left-hand side is the change in kinetic energy and the right-hand side is the work, both per unit mass. (For clarity, primes have been added to the dummy variable of integration.) Rewrite this equation as

$$\frac{dx}{dt} = \sqrt{2\int_x^A f(x')dx'}. \tag{17}$$

Since it requires one-quarter of a period T for the oscillator to move from $x = 0$ to $x = A$,

$$\frac{T}{4} = \int_0^A \frac{dx}{\sqrt{2\int_x^A f(x')dx'}} \tag{18}$$

and therefore the exact angular frequency is

$$\omega_{\text{exact}} \equiv 2\pi/T = 2^{-1/2}\pi \bigg/ \int_0^A \frac{dx}{\sqrt{\int_x^A f(x')dx'}}. \tag{19}$$

4 Physical examples applying the method

4.1 The simple harmonic oscillator

Substituting $f(x) = \omega_0^2 x$ into Eq. (19) and changing variables to θ according to $x = A\cos\theta$, the exact angular frequency is found to be $\omega_{\text{exact}} = \omega_0$ as expected. Here, for example, $\omega_0^2 = k/m$ for a particle of mass m oscillating on a spring of stiffness k.

Equation (13) becomes

$$\omega^2 = \frac{4\omega_0^2}{A^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sum_{n=1}^{\infty}J_{2k+1}(n\pi)\int_0^A x\sin\left(\frac{n\pi x}{A}\right)dx \tag{20}$$

using Eq. (3), where the Fourier series is that of a sawtooth wave, and thus

$$\omega^2 = 4\omega_0^2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}J_{2k+1}(n\pi). \tag{21}$$

However, Eq. (58) in Appendix A shows that only the $k = 0$ term is nonzero, leading to

$$\omega^2 = 4\omega_0^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}J_1(n\pi). \tag{22}$$

Equation (55) in Appendix A implies that this sum equals $\frac{1}{4}$ and consequently

$$\omega = \omega_0. \tag{23}$$

A simple harmonic oscillator has no overtones and thus $A_3 = A_5 = ... = 0$, so that $\delta = 0$ according to Eq. (8). As a result, the frequency obtained using the first-order double Fourier harmonic balance method is exact.

4.2 Motion down two intersecting planes

Consider a ball starting on a ramp inclined at angle ϕ relative to the horizontal, rolling down to the origin, and then rolling up another ramp inclined at the same angle to the horizontal, so that the two ramps form a V-shape. In the absence of rolling friction or air drag, the ball rolls a distance A along each ramp. Defining $\omega_0^2 \equiv g \sin\phi/A$, where g is the gravitational field, the restoring force divided by the mass of the ball is the step function $f(x) = A\omega_0^2 \text{sgn}(x)$. Inserting this force law into Eq. (19) and changing variables to θ according to $x = A\cos^2\theta$, the exact angular frequency is found to be

$$\omega_{\text{exact}} = \frac{\pi}{2\sqrt{2}}\omega_0 \approx 1.111\omega_0. \tag{24}$$

Since ω_0 depends on A, as will also be the case for every remaining example treated in this article, the frequency of all but the linear oscillator of Sec. 4.1 in general depends on the amplitude of motion.

Retaining only the J_1 term, Eqs. (3) and (13) predict

$$\omega^2 = \frac{4\omega_0^2}{A}\sum_{n=1}^{\infty} J_1(n\pi)\int_0^A \sin\left(\frac{n\pi x}{A}\right)dx \tag{25}$$

and the Fourier integral for this square wave gives

$$\omega^2 = \frac{8\omega_0^2}{\pi}\sum_{\text{odd } n=1}^{\infty}\frac{J_1(n\pi)}{n}. \tag{26}$$

According to Eq. (69) in Appendix B, the sum is $\frac{1}{2}$ and so the approximate frequency is

$$\omega = \frac{2}{\sqrt{\pi}}\omega_0 \approx 1.128\omega_0 \tag{27}$$

which is only 1.5% larger than the exact answer. By retaining all of the terms in the Bessel series in Eq. (13) and using Eq. (74) in Appendix C, one obtains

$$\omega^2 = \frac{4\omega_0^2}{\pi}\left[1 - \frac{1}{9}\cdot\frac{1}{3} + \frac{1}{25}\cdot\frac{1}{5} - ...\right]. \tag{28}$$

Since the sum of the alternating reciprocals of the odd cubes is $\beta(3) = \pi^3/32$, where β is the Dirichlet beta function, the square root of Eq. (28) is equal to the result in Eq. (24). This result illustrates that in the present example $\delta(t)$ can be exactly dropped in Eq. (9) even though it is not

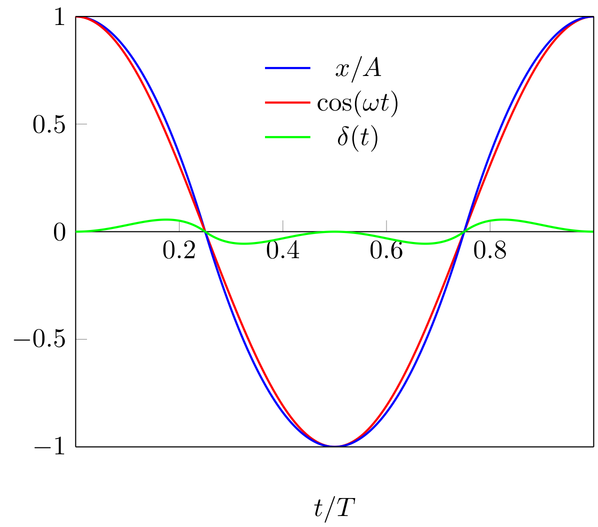

Figure 1. Graphs of the net displacement from Eq. (29), the unit-amplitude lowest-order Fourier component from Eq. (5), and the difference between them from Eq. (8).

zero at all times. Specifically, $\delta(t)$ is given by Eq. (8) as graphed in Fig. 1, where the displacement of the ball is given by the slope-matched parabolae

$$\frac{x(t)}{A} = \begin{cases} 1 - \left(\frac{4t}{T}\right)^2 & \text{for } 0 \le t \le \frac{T}{4} \\ -1 + \left(\frac{4t}{T} - 2\right)^2 & \text{for } \frac{T}{4} \le t \le \frac{3T}{4} \\ 1 - \left(\frac{4t}{T} - 4\right)^2 & \text{for } \frac{3T}{4} \le t \le T \end{cases} \tag{29}$$

since the acceleration down the ramps has the constant values $\pm g \sin\phi$. Thus the coefficients of the temporal Fourier series in Eq. (5) are

$$A_{2m+1} = \frac{32(-1)^m}{\pi^3(2m+1)^3}, \tag{30}$$

thereby confirming the series found in Eq. (28) above.

4.3 Electric force inversely proportional to distance

An electron of charge $-e$ and mass m is released from rest a distance A away from a long straight plasma filament of positive linear charge density λ. Defining $\omega_0^2 \equiv 2ke\lambda/mA^2$ where k is the Coulomb constant, the attractive force per unit mass on the electron is $f(x) = A^2\omega_0^2/x$. Inserting this force law into Eq. (19) and changing variables to $u = \sqrt{\ln(A/x)}$ to get a Gaussian integral, the exact angular frequency is found to be

$$\omega_{\text{exact}} = \sqrt{\frac{\pi}{2}}\omega_0 \approx 1.253\omega_0. \tag{31}$$

To apply the double Fourier harmonic balance method, Eq. (3) becomes

$$b_n = 2A\omega_0^2 \int_0^A \frac{1}{x} \sin\left(\frac{n\pi x}{A}\right) dx. \tag{32}$$

The sine integral $\text{Si}(t)$ is defined as the integral of the sinc function,

$$\text{Si}\,(t) \equiv \int_0^t \frac{\sin\theta}{\theta} d\theta \tag{33}$$

so that

$$b_n = 2A\omega_0^2 \text{Si}(n\pi). \tag{34}$$

Substituting this result into Eq. (13), the approximate frequency is

$$\omega^2 = 4\omega_0^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sum_{n=1}^{\infty} \text{Si}(n\pi) J_{2k+1}(n\pi). \tag{35}$$

The partial sums over n oscillate and approach $\frac{1}{2}$ as the upper limit goes to infinity for any non-negative integer value of k. For example, Fig. 2 plots the partial sums for the lowest order term corresponding to $k = 0$,

$$S(N) \equiv \sum_{n=1}^{N} \text{Si}(n\pi) J_1(n\pi), \tag{36}$$

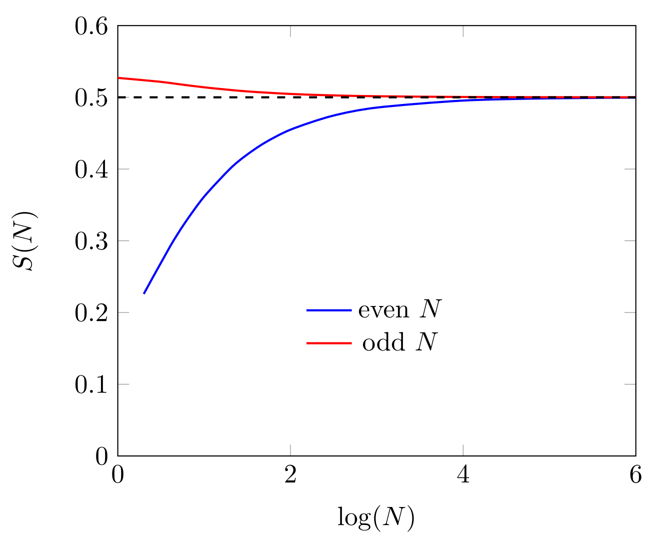

Figure 2. Graphs of the partial sums in Eq. (36), corresponding to $k = 0$ in Eq. (35), for values of N ranging from one to a million. The sums approach $\frac{1}{2}$ from above for odd values of N and from below for even values of N. Similar behavior is observed for positive even values of k. For positive odd values of k, the partial sums for odd values of N approach $\frac{1}{2}$ from below, while the sums for even values of N approach it from above. For negative integer values of k, the partial sums approach $-\frac{1}{2}$ instead.

for even and odd values of N up to a million. Assuming $S(N) \to 1/2$ as $N \to \infty$ for all $k \geq 0$, then

$$\omega = \sqrt{2G}\omega_0 \approx 1.353\omega_0 \tag{37}$$

where the Catalan constant is $G = \beta(2) = 1 - 1/9 + 1/25 - ... \approx 0.91597$. This approximate frequency is 8% larger than the exact value, which is an improvement over the standard harmonic balance estimate obtained as follows. Equation (1) can be rewritten for the inverse-distance force law as

$$x\frac{d^2x}{dt^2} = -A^2\omega_0^2. \tag{38}$$

Substitute $x = A\cos\omega t$ to get

$$\tfrac{1}{2}\omega^2(1 + \cos 2\omega t) = \omega_0^2, \tag{39}$$

using the double-angle formula for cosine, and balance the dc terms to find

$$\omega = \sqrt{2}\omega_0 \approx 1.414\omega_0 \tag{40}$$

which is 13% larger than the exact value in Eq. (31).

4.4 The large-amplitude simple pendulum

A point mass m is suspended from an ideal string of length L whose angular displacement is θ relative to the vertical. Defining $\omega_0^2 \equiv g/L$, the restoring torque divided by the moment of inertia of the bob (both about the pivot) is the trigonometric function $f(\theta) = \omega_0^2 \sin\theta$. Inserting this torque into Eq. (19) with x replaced by θ, the exact angular frequency is found to be

$$\omega_{\text{exact}} = \frac{2^{-1/2}\pi\,\omega_0}{\int_0^A \frac{d\theta}{\sqrt{\cos\theta - \cos A}}} \tag{41}$$

which can be expressed in terms of an elliptic integral. Evaluating it for an angular amplitude of $A = \pi/2$ gives

$$\omega_{\text{exact}} = \frac{2\pi^{3/2}}{\Gamma^2(1/4)}\omega_0 \approx 0.8472\omega_0 \tag{42}$$

where Γ is the gamma function. (The reciprocal of this frequency implies that the period of a pendulum swinging between −90° and +90° is 18% longer than it would be at small angular amplitudes.)

Equation (3) becomes

$$b_n = \frac{2\pi n\,(-1)^{n+1}\sin A}{n^2\pi^2 - A^2}\omega_0^2. \tag{43}$$

Equation (13) then predicts

$$\omega^2 = \frac{4\omega_0^2 \sin A}{\pi A}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n\, J_{2k+1}(n\pi)}{n^2 - A^2/\pi^2}. \tag{44}$$

Expression 19 on page 679 of Prudnikov, Brychkov, and Marichev [8] is

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n\, J_{2k+1}(nx)}{n^2 - a^2} = \frac{\pi}{2}\csc(a\pi)J_{2k+1}(ax). \tag{45}$$

Evaluating it at $a = A/\pi$ and $x = \pi$, and substituting the result into Eq. (44), one obtains

$$\omega^2 = \frac{2\omega_0^2}{A} \sum_{k=0}^{\infty} \frac{(-1)^k J_{2k+1}(A)}{(2k+1)^2} \tag{46}$$

which reproduces Eq. (25) of Ref. [9]. Retaining only the first term in this series, the period of oscillation is found to be

$$T \approx T_0 \sqrt{\frac{A}{2J_1(A)}} \tag{47}$$

where $T_0 \equiv 2\pi\sqrt{L/g}$ is the familiar period for small amplitudes. This result agrees with the standard harmonic balance method, Eq. (34) of Ref. [10]. Equation (46) can be thought of as its extension to all overtones. For the large-amplitude case $A = \pi/2$, it becomes

$$\omega^2 = \frac{4\omega_0^2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k J_{2k+1}(\pi/2)}{(2k+1)^2}. \tag{48}$$

There is no closed form for the sum of this Neumann series. However, using the standard series expansion for a Bessel function of positive integer order, one immediately sees that $J_{2k+1}(\pi/2)$ is positive and monotonically decreasing in value as k increases. Thus Eq. (48) is an alternating series of terms of decreasing magnitude. It follows that one can cut off the series at an upper limit of $k = N$ and get an approximate sum with an error of less than the first omitted term. Choosing $N = 2$, the error is already less than $J_7(\pi/2)/49 \approx 7 \times 10^{-7}$, in which case

$$\omega \approx 0.8438\omega_0 \tag{49}$$

in good agreement with Eq. (42).

4.5 The Duffing oscillator with zero linear term

For an anharmonic oscillator having restoring force $f(x) = \alpha x^3$, define $\omega_0 = A\sqrt{\alpha}$. Using the change of variables $x = Au$, Eq. (19) gives the same frequency as Eq. (42).

On the other hand, Eq. (3) becomes

$$b_n = 2A\omega_0^2 \,(-1)^{n+1} \left[\frac{1}{n\pi} - \frac{6}{n^3\pi^3}\right] \tag{50}$$

so that Eq. (13) predicts

$$\omega^2 = 4\omega_0^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_{2k+1}(n\pi)}{n\pi} - 6\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_{2k+1}(n\pi)}{(n\pi)^3}\right]. \tag{51}$$

Using Eqs. (55), (56), (58), (59), and (65) in Appendix A, this prediction becomes

$$\omega^2 = 4\omega_0^2 \left[\frac{1}{4} - 6\left(\frac{1}{96} - \frac{1}{9}\cdot\frac{1}{96}\right)\right] \quad \Rightarrow \quad \omega = \frac{\sqrt{7}}{3}\omega_0 \approx 0.8819\omega_0 \tag{52}$$

which is 4% larger than the exact value in Eq. (42). Interestingly, the predicted frequency becomes $\sqrt{3}\,\omega_0/2$ if one retains only the J_1 terms (corresponding to $k = 0$) in the sums in Eq. (51), reducing the error to 2%.

5 Conclusions

In this paper, a double Fourier harmonic balance method has been developed, based on the expansion of the force in a Fourier sine series. By applying this new method to a number of previously studied nonlinear oscillators, approximate frequencies have been obtained that are often better than those found using other techniques [11]. The method works whenever the Fourier coefficients in Eq. (3) and the sums of the Bessel series in Eq. (13) can be calculated, although a good approximation can often be obtained by retaining only the J_1 term in the latter equation. As additional examples beyond the cases treated here, the sublinear oscillator $f(x) \propto x^{1/3}$ and the general power-law oscillator $\pm x^n$ with $n > 3$ can be expressed in terms of the incomplete gamma function. The method should also be applicable to damped oscillators, by including the full (not just the sine and odd cosine) expansions in Eqs. (2) and (5). As a consequence, this new version of the harmonic balance method is pedagogically useful in analyzing the behavior of nonlinear oscillating systems. The derivations are accessible to undergraduate students, in contrast to say the homotopy method [12, 13], especially because of the ready availability of Fourier analysis and Bessel functions in computer algebraic software.

Appendix A

In this appendix, the sum

$$S(k,p) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_k(n\pi)}{(n\pi)^p} \tag{53}$$

is evaluated, where k is any positive odd integer, and p equals either 1 or 3. There are three cases of interest as follows.

Case 1: $k = p$

Equation (2) on page 635 of Watson [14] can be rearranged as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_m(nx)}{(nx)^m} = \frac{1}{2^{m+1} m!} \tag{54}$$

where m is any positive integer and $0 < x \leq \pi$. In particular, the two needed values are

$$S(1,1) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_1(n\pi)}{n\pi} = \frac{1}{4} \tag{55}$$

and

$$S(3,3) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_3(n\pi)}{(n\pi)^3} = \frac{1}{96}. \tag{56}$$

Case 2: $k \geq p + 2$

Equation (3) on page 636 of Watson [14] can be rearranged as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_m(nx)}{(nx)^{m-2s}} = 0 \tag{57}$$

where $m > 2s$ with m and s both positive integers, and $0 \le x \le \pi$. In particular, one set of values of interest is

$$S(k,1) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_k(n\pi)}{n\pi} = 0 \ \text{ for } k = 3, 5, 7, ... \tag{58}$$

as obtained by substituting $\{m = 3 \ \& \ s = 1\}$, $\{m = 5 \ \& \ s = 2\}$, $\{m = 7 \ \& \ s = 3\}$, ... into Eq. (57). The other needed set of values is

$$S(k,3) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_k(n\pi)}{(n\pi)^3} = 0 \ \text{ for } k = 5, 7, 9, ... \tag{59}$$

as obtained by substituting $\{m = 5 \ \& \ s = 1\}$, $\{m = 7 \ \& \ s = 2\}$, $\{m = 9 \ \& \ s = 3\}$, ... into Eq. (57).

Case 3: $k = 1 \ \& \ p = 3$

The standard Bessel recursion relation can be written as

$$J_s(z) = \frac{z}{2s} J_{s-1}(z) + \frac{z}{2s} J_{s+1}(z). \tag{60}$$

Multiply this equation through by $(-1)^{n+1} z^{-m}$, let $z \equiv nx$, and then sum over n to get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_s(nx)}{(nx)^m} = \frac{1}{2s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_{s-1}(nx)}{(nx)^{m-1}} + \frac{1}{2s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_{s+1}(nx)}{(nx)^{m-1}}. \tag{61}$$

Next substitute $s = 1$, $m = 3$, and $x = \pi$ so that this result specifically becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_1(n\pi)}{(n\pi)^3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_0(n\pi)}{(n\pi)^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_2(n\pi)}{(n\pi)^2}. \tag{62}$$

According to Eq. (54), the third sum is $S(2,2) = 1/16$. To get the other sum, use the Schlömilch series [15]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_0(nx)}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{8}. \tag{63}$$

Divide through by x^2 and then let $x = \pi$ to obtain

$$S(0,2) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_0(n\pi)}{(n\pi)^2} = \frac{1}{12} - \frac{1}{8} = -\frac{1}{24}. \tag{64}$$

Finally substitute these numerical values of $S(2,2)$ and $S(0,2)$ into the right-hand side of Eq. (62) to get

$$S(1,3) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} J_1(n\pi)}{(n\pi)^3} = \frac{1}{2} \left[\frac{-1}{24} + \frac{1}{16} \right] = \frac{1}{96} \tag{65}$$

which happens to equal $S(3,3)$ from Eq. (56).

Appendix B

The Schlömilch series for x is [15]

$$x = \frac{\pi^2}{4} - 2 \sum_{\text{odd } n=1}^{\infty} \frac{J_0(nx)}{n^2}. \tag{66}$$

Take its derivative with respect to x to get

$$1 = -2 \sum_{\text{odd } n=1}^{\infty} \frac{1}{n} \frac{dJ_0(nx)}{d(nx)}. \tag{67}$$

However

$$\frac{dJ_0(z)}{dz} = -J_1(z) \tag{68}$$

and thus

$$\sum_{\text{odd } n=1}^{\infty} \frac{J_1(nx)}{n} = \frac{1}{2}. \tag{69}$$

Remarkably, the sum is independent of x over the range from 0 to π. In particular, it can be evaluated at $x = \pi$ to get the result needed in Eq. (26).

Appendix C

Expression (i) of Petković [16] with $k \to k+1$ and $m = 1$ becomes

$$\sum_{n=1}^{\infty} \frac{J_{2k+1}(nx)}{n} = \frac{1}{2k+1} \tag{70}$$

for integers $k \geq 1$ and $0 \leq x \leq 2\pi$. Evaluate Eq. (70) at $x = \pi$ to obtain

$$\sum_{n=1}^{\infty} \frac{J_{2k+1}(n\pi)}{n} = \frac{1}{2k+1}. \tag{71}$$

Next put $n = p/2$ into Eq. (70) to get

$$\sum_{\text{even } p=2}^{\infty} \frac{J_{2k+1}(px/2)}{p} = \frac{1}{2(2k+1)}, \tag{72}$$

and substitute $x = 2\pi$ to find

$$\sum_{\text{even } n=2}^{\infty} \frac{J_{2k+1}(n\pi)}{n} = \frac{1}{2(2k+1)}. \tag{73}$$

Finally, subtract Eq. (73) from (71) to establish that

$$\sum_{\text{odd } n=1}^{\infty} \frac{J_{2k+1}(n\pi)}{n} = \frac{1}{2(2k+1)}. \tag{74}$$

Remarkably, Eq. (74) is valid for $k = 0$, as Eq. (69) shows, even though neither Eq. (71) nor (73) is valid for $k = 0$ (as one can verify by numerically approximating the latter two sums for $k = 0$).

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