Comparison of two finite- difference methods for solving the damped wave equation

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October 23, 2014

Communicated by *Prof. S.A. Edalatpanah*

Abstract

In this work we present two finite-difference schemes for solving this equation with one initial and boundary conditions. We study stability and consistency of these methods. Two methods are explicit and they approximates the solutions of the wave equation with consistency of order $O\left\{(\Delta t)^2, (\Delta x)^2\right\}$, for examining the accuracy of the results , we compare the results with the solution obtained by the methods of separation of variable, also a numerical example for each methods is presented and compared with each other. Finally, the graphs of the error have been plotted to show the methods work with high accuracy.

Keywords : damped wave equation, finite difference, stability, consistency.

1 Introduction

Wave equation is an extremely important evolution model and it is widely used by physicists and engineers in describing the propagation of water waves, sound waves, electromagnetic waves, seismic waves, gravity waves and oscillatory waves, etc (see [1]). To gain an insight to the physical back ground about the damped wave equation we refer to [2]. Where it is stated that when the neural fields are formulated to predict wave equation. Damped wave equation first occurred in the mathematical description of the telegraph, it is generally known as the equation of telegraphy. The telegrapher's equation (or just telegraph equation) is a linear differential equation which describes the voltage and current on an electrical transmission line with distance and time. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction–diffusion

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for such branches of sciences. In the present research various numerical schemes will be developed and compared for solving this equation [3],[4],[5],[6],[7],[8].

We consider the following equation,

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with the initial and boundary conditions,

$$u(x,0) = f(x), u(0,t) = u(l,t) = 0, \tag{2}$$

$$u_t(x,0) = g(x), \tag{3}$$

where h is a small positive constant. The term $2h\frac{\partial u}{\partial t}$, presents a damping force proportional to the element's velocity u_t.

2 The finite–difference schemes

The main idea behind the finite-difference methods for obtaining the solution of a given partial differential equation is to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points. The most usual way to generate these approximations is through of use of Taylor series. The solution domain of the problem is covered by a mesh of grid-lines

$$x_i = i\Delta x \qquad i = 0, 1, ..., m, \tag{4}$$

$$t_n = n\Delta t \qquad n = 0, 1, ..., N. \tag{5}$$

parallel to the space and time coordinate axes, respectively. Approximations u_i^n to $u(i\Delta x, n\Delta t)$ are calculated at the point of intersection of these lines, namely,$(i\Delta x, n\Delta t)$ which is referred to as the (i, n) grid-point. The constant spatial and temporal grid-spacing are $\Delta x = \frac{1}{M}$ and $\Delta t = \frac{T}{N}$, respectively.

3 Two three-level explicit methods

3.1 The first method

Consider the following approximations of the derivatives for solving the equation 1,

$$\frac{\partial^2 u}{\partial t^2}|_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}|_i^n + O(\Delta t^4), \tag{6}$$

$$\frac{\partial u}{\partial t}|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{\partial^3 u}{\partial t^3}|_i^n + O(\Delta t^4), \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2}|_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}|_i^n + O(\Delta t^4), \tag{8}$$

for $i = 1, ..., M-1$, and $n = 1, ..., N-1$,
where

$$S = \frac{c^2 \Delta t^2}{\Delta x^2} \tag{9}$$

By replacing these terms in equation 1 we obtain,

$$\begin{aligned} &\tfrac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - \tfrac{(\Delta t)^2}{12}\tfrac{(\partial^4 u)}{\partial t^4}|_i^n + O(\Delta t^4) \\ &+2h(\tfrac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \tfrac{(\Delta t)^2}{6}\tfrac{(\partial^3 u)}{\partial t^3}|_i^n + O(\Delta t^4) \\ &-c^2(\tfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - \tfrac{\partial^4 u}{\partial x^4}|_i^n + O(\Delta t^4) = 0 \end{aligned} \tag{10}$$

by omitting all terms of $O\{(\Delta t)^2, (\Delta x)^2\}$ we have,

$$u_i^{n+1} = \frac{1}{1 + h\Delta t}[2(1-s)u_i^n - (1 - h\Delta t)u_i^{n-1} + s(u_{i+1}^n + u_{i-1}^n)] \tag{11}$$

Clearly, the total of the truncation errors in using the 11 instead of 1 is,

$$E\{u_i^n\} = [-\frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}|_i^n - \frac{h(\Delta t)^2}{3}\frac{\partial^3 u}{\partial t^3}|_i^n + c^2\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\Delta x^4}|_i^n + O\{(\Delta x^4), (\Delta t^4)\}], \tag{12}$$

This may be written

$$E\{u\} = O\{(\Delta t^2), (\Delta x^2)\} \tag{13}$$

In this method the initial condition is $g(x) = 0$. In equation 11, when $n = 0$ we need u_i^{-1} , which is calculated as follows,

$$u_t(x, 0) = g(x) = 0 \Rightarrow \frac{\partial u_i^0}{\partial t} = 0 \Rightarrow \frac{u_i^1 - u_i^{-1}}{\Delta t} = 0 \Rightarrow u_i^1 = u_i^{-1}, \tag{14}$$

Then from equation 11 we have,

$$u_i^1 = \frac{1}{2}[2(1-s)u_i^0 + s(u_{i+1}^0 + u_{i-1}^0)], \tag{15}$$

As en example we let $k = 3, 0 \leq j \leq k+1$ then

$$u_0^n = u(0,t) = 0, \tag{16}$$

$$u_{k+1}^n = u_4^n = u(\pi, t) \tag{17}$$

$$u^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix} \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{pmatrix} + \tfrac{s}{2} \begin{pmatrix} u_0^0 \\ 0 \\ u_4^0 \end{pmatrix} \tag{18}$$

And the solution for time steps $u^{k+1}; k \geq 1$ is,

$$u^{k+1} = Au^k + b - eu^{k-1}, \tag{19}$$

where

$$u^k = \begin{pmatrix} u_1^k \\ u_2^k \\ u_3^k \end{pmatrix}, A = \tfrac{1}{1+h\Delta t} \begin{pmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{pmatrix}, \quad b = \tfrac{s}{1+h\Delta t} \begin{pmatrix} u_0^k \\ 0 \\ u_k^k \end{pmatrix} + \tfrac{1-h\Delta t}{1+h\Delta t}, \tag{20}$$

In the follow the computational molecule of this formula is shown in Fig. 1. In the following the procedure using this formula will be referred to as the (1,3,1) method. This means that the molecule involves one grid point at the $(n+1)$ time level and 3 at the (n) and 1 at the $(n-1)$ time level.

3.1.1 Stability by Von Neumann's method

The Von Neumann method is the most commonly used method of stability analysis for linear finite difference formula with constant coefficients [9]. A solution of the error equation is now sought in the variables separable form

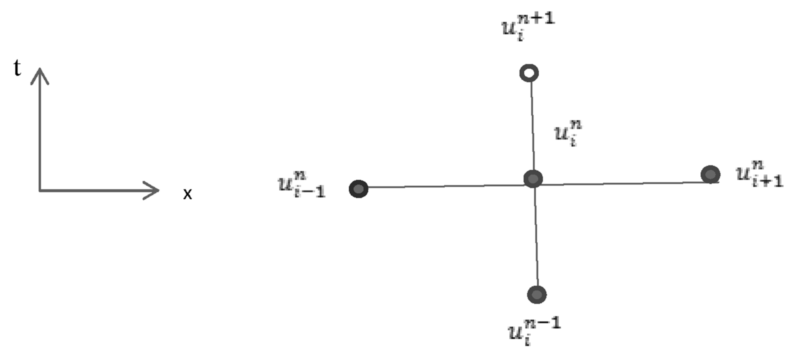


Figure 1: The computational molecule for the (1, 3, 1) scheme.

$$\xi_j^n = |G|^n e^{i\beta j} \tag{21}$$

in which

$$\beta = m\pi\Delta x, \tag{22}$$

Where the time dependence of this Fourier component of the error at $x = i\Delta x$, and $t = n\Delta t$, is contained in the coefficient (G^n) which is the n^{th} power of the complex number G. By replacing ξ instead of u in equation 11 we obtain,

$$\xi_j^{n+1} = \frac{1}{1+h\Delta t}[2(1-s)\xi_j^n - (1-h\Delta t)\xi_j^{n-1} + s(\xi_{j+1}^n + \xi_{j-1}^n)], \tag{23}$$

Substituting 21 in the above error equation gives:

$$\begin{aligned} |G|^{n+1}e^{i\beta j} = \tfrac{1}{1+h\Delta t}[2(1-s)|G|^n e^{i\beta j} - (1-h\Delta t)|G|^{n-1}e^{i\beta j} \\ +s(|G|^{n+1}e^{i\beta(j+1)} + |G|^{n+1}e^{i\beta j})], \end{aligned} \tag{24}$$

By division by $|G|^{n+1}e^{i\beta j}$, we obtain ,

$$(1+h\Delta t)G^2 - (2-2s+2s\cos\beta)G + (1+h\Delta t)) = 0, \tag{25}$$

For all tree-level methods the Von Neumann amplification factor G may have one of two values which are the roots of a quadratic equation of the form

$$A(G)^2 + BG + D = 0, \tag{26}$$

In which the coefficients A, B, D are functions of β, s, and may be complex numbers. For coefficients of quadratic equation 26 the Von Neumann stability is established by using the following criteria,
(a) If $|A| > |D|$ then the given finite difference formula is stable if and only if

$$|A|^2 - |B|^2 \geq |B\overline{A} - D\overline{B}|,$$

(b) If $|A| = |D|$ and $|B\overline{A} - D\overline{B}| = 0$ then the given finite difference formula is stable if and only if

$$2|A| \geq |B|,$$

(c) If $|A| < |D|$ then the finite difference formula is unstable.
Consider 25, which may be written in the form 26 with

$$A = 1 + h\Delta t, B = -(2 - 2s + 2s\cos\beta) = -(2 - 4s\sin^2\frac{\beta}{2}), D = 1 - h\Delta t$$

since

$$|A| = 1 - h\Delta t >| 1 - h\Delta t |=| D |, \tag{27}$$

It follows from (a), that the equation 1 is stable if and only if

$$\begin{aligned}(1 + h\Delta t)^2 - (1 - h\Delta t)^2 \geq | - (2 - 4s\sin^2\tfrac{\beta}{2})(1 - (1 + h\Delta t)) \\ -(1 - h\Delta t)(2 - 4s\sin^2\tfrac{\beta}{2})|,\end{aligned} \tag{28}$$

By simplifying this relation we can easily show that

$$|2s(\sin^2\frac{\beta}{2})| \leq 1, \tag{29}$$

So the range of stability for this procedure is $0 \leq s \leq 1$.

3.1.2 Consistency

A finite-difference formula is said to be consistent with a partial differential equation if, in the limit as the grid spacing tend to zero, the finitedifference formula is identical to the partial differential equation at each point in the solution domain. To researching the consistency of equation 1 by first method, consider equation 11 written in the form

$$L_\Delta\{u_j^n\} = (1 + h\Delta t)u_j^{n+1} - 2(1 - s)u_j^n + (1 - h\Delta t)u_j^{n-1} - s(u_{j+1}^n + u_{j+1}^n) = 0, \tag{30}$$

Replacing u_i^n in 32 by the exact solution $\hat{u}_i^n$ of equation 1 and after rearranging, we have,

$$L_\Delta\{\hat{u}_i^n\} = (\hat{u}_i^{n+1} + \hat{u}_i^{n-1} - 2\hat{u}_i^n) + h\Delta t(\hat{u}_i^{n+1} - \hat{u}_i^{n-1}) + s(2\hat{u}_i^n - 2\hat{u}_{i+1}^n - 2\hat{u}_{i-1}^n), \tag{31}$$

which may be no longer equal to zero. Since $\hat{u}_i^n$ is continuously differential, the terms of 31 may be replaced by their Taylor expansions about the point $(i\Delta x, n\Delta t)$.
This gives,

$$\begin{aligned} L_\Delta\{\hat{u}_i^n\} = (\Delta t)^2\{(\tfrac{\partial^2 u}{\partial t^2}|_i^n + 2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4 u}{\partial t^4}|_i^n + ...) + 2h(\tfrac{\partial u}{\partial t}|_i^n + \tfrac{\Delta t}{3!}\tfrac{\partial^3 u}{\partial t^3}|_i^n + ...) \\ -c^2(\tfrac{\partial^2 u}{\partial x^2}|_i^n + 2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4 u}{\partial x^4}|_i^n + ...)\} = 0, \end{aligned} \tag{32}$$

or

$$\begin{aligned} L_\Delta\{\hat{u}_i^n\} = (\Delta t)^2\{\tfrac{\partial^2 u}{\partial t^2}|_i^n + 2\tfrac{\partial u}{\partial t}|_i^n - c^2\tfrac{\partial^2 u}{\partial x^2}|_i^n \\ +(2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4 u}{\partial t^4}|_i^n + 2h\tfrac{\Delta t}{3!}\tfrac{\partial^3 u}{\partial t^3}|_i^n - 2c^2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4 u}{\partial x^4}|_i^n + ...)\}, \end{aligned} \tag{33}$$

where

$$E(\hat{u}) = 2\frac{(\Delta t)^2}{4!}\frac{\partial^4 u}{\partial t^4}|_i^n + 2h\frac{\Delta t}{3!}\frac{\partial^3 u}{\partial t^3}|_i^n - 2c^2\frac{(\Delta x)^2}{4!}\frac{\partial^4 u}{\partial x^4}|_i^n + ..., \tag{34}$$

is the truncation error of the first and second- order accurate in time and space respectively. As previously, seen when the grid spacing get smaller and smaller with the first method, the truncation error gets smaller and smaller at a fixed point (x_*, t_*) in the solution domain. In the limit as $\Delta x \to 0$,$\Delta t \to 0$ the finite-difference formula 32 is equivalent to the partial differential equation 1, so the first method is consistent.

3.2 The second method

Consider the following approximations of the derivatives for solving the equation 1

$$\frac{\partial^2 u}{\partial t^2}|_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}|_i^n + O(\Delta t^4), \tag{35}$$

$$\frac{\partial u}{\partial t}|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}|_i^n + O(\Delta t^2), \tag{36}$$

$$\frac{\partial^2 u}{\partial x^2}|_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial x^4}|_i^n + O(\Delta x^4), \tag{37}$$

for $i = 1, ..., M-1$, and $n = 0...1$, where

$$S = \frac{c^2 \Delta t^2}{\Delta x^2}, \tag{38}$$

By replacing these terms in equation 1 we obtain,

$$\begin{aligned} &\left(\tfrac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - \tfrac{(\Delta t)^2}{12} \tfrac{(\partial^4 u)}{\partial t^4}\right)\Big|_i^n + O(\Delta t^4) \\ &+2h\left(\tfrac{u_i^{n+1} - u_i^{n-1}}{\Delta t} - \tfrac{\Delta t}{2} \tfrac{\partial^2 u}{\partial t^2}\right)\Big|_i^n + O(\Delta t^2) \\ &-c^2\left(\tfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} - \tfrac{(\Delta t)^2}{12} \tfrac{\partial^4 u}{\partial x^4}\Big|_i^n + O(\Delta x^2)\right) = 0, \end{aligned} \tag{39}$$

by omitting all terms of $O\{(\Delta t)^2, (\Delta x)^2\}$ we have,

$$u_i^{n+1} = [(2(1-s) + 2h\Delta t)u_i^n + s(u_{i+1}^n + u_{i-1}^n) - u_i^{n-1}] \tag{40}$$

Clearly, the total of the truncation errors in using the 40 instead of 1 is,

$$E\{u_i^n\} = [-\frac{(\Delta t)^2}{12} \frac{\partial^4 u}{\partial t^4} + h\Delta t \frac{\partial^2 u}{\partial t^2}|_i^n + c^2 \frac{(\Delta x)^2}{12} \frac{\partial^4 u}{\Delta x^4}|_i^n + O\{(\Delta x)^4, (\Delta t)^4\}], \tag{41}$$

This may be written

$$E(u) = O\{(\Delta t)^2, (\Delta x)^2\} \tag{42}$$

In equation 40, when $n = 0$ we need u_i^{-1} , which is calculated as follows,

$$u_i(x, 0) = g(x) = 0 \Rightarrow \frac{\partial u_i^0}{\partial t} = 0 \Rightarrow \frac{u_i^1 - u_i^{-1}}{\Delta t} = 0 \Rightarrow u_i^1 = u_i^{-1}, \tag{43}$$

Then from equation 40 we have,

$$u_i^1 = \frac{1}{1 + h\Delta t}[((1-s) + h\Delta t)u_i^0 + \frac{s}{2(u_{i+1}^0 + u_{i-1}^0)}], \tag{44}$$

As an example wa let then $k = 3, 0 \le j \le k+1$; then

$$u_0^n = u(0, t) = 0, \tag{45}$$

$$u_{k+1}^n = u_4^n = u(\pi, t) \tag{46}$$

$$u^1 = \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \frac{1}{1+h\Delta t} \begin{bmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{bmatrix} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} + \frac{s}{2} \begin{bmatrix} u_0^0 \\ 0 \\ u_4^0 \end{bmatrix}, \tag{47}$$

and for time $n \geq 1, u^{n+1}$ are given by,

$$u^{n+1} = Au^n + b - eu^{n-1}, \tag{48}$$

where

$$u^n = \begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \end{bmatrix}, A = \frac{1}{1+2h\Delta t} \begin{bmatrix} 2(1-s) & s & 0 \\ s & 2(1-s) & s \\ 0 & s & 2(1-s) \end{bmatrix}, \quad b = \frac{s}{1+2h\Delta t} \begin{bmatrix} u_0^0 \\ 0 \\ u_0^0 \end{bmatrix}, e = \frac{1}{1+2h\Delta t}, \tag{49}$$

We see that similar to the first method, this method has (1, 3, 1) molecule scheme too.

3.2.1 Stability by Von Neumann's method

In the equation 40 we set $\xi_j^n = |G|^n e^{i\beta j}$ for u_i^n, then we have the following equation:

$$(1 + 2h\Delta t)G^2 - (2 + 2h\Delta t - 2s + 2s\cos\beta)G + 1 = 0, \tag{50}$$

consider

$$A = 1 + 2h\Delta t, B = -(2 + 2h\Delta t - 2s + 2s\cos\beta), D = 1, \tag{51}$$

then by apply the condition for stability; i.e.,

$$|A| > |D| \Rightarrow |A|^2 - |D|^2 \geq |B\overline{A} - D\overline{B}|,$$

and replacing the values of A, B, D in this relation we will have,

$$(1 + 2h\Delta t)^2 - (1)^2 \geq | (-2 - 2h\Delta t + 2s - 2s)\cos\beta)(1 + 2h\Delta t) + (2 + 2h\Delta t + 2s - 2s\cos\beta) |, \tag{52}$$

by simplifying this relation we can easily show that

$$| -1 - h\Delta t + 2s\sin^2\frac{\beta}{2}) \,|\leq 1 + h\Delta t, \tag{53}$$

So the range of stability for second method is,

$$0 \leq s \leq 1 \tag{54}$$

3.2.2 Consistency

Consider Equation 40 and replacing u_i^n by the exact solution $\hat{u}_i^n$of equation 1 and rearranging, yields

$$\begin{aligned} L_\Delta\{\hat{u}_i^n\} = (\Delta t)^2\{\tfrac{\partial^2\hat{u}}{\partial t^2}|_i^n + 2h\tfrac{\partial\hat{u}}{\partial t}|_i^n - c^2\tfrac{\partial^2\hat{u}}{\partial x^2}|_i^n + \\ (2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4 u}{\partial t^4}|_i^n + h\Delta t\tfrac{\partial^2\hat{u}}{\partial t^2}|_i^n - 2c^2\tfrac{(\Delta t)^2}{4!}\tfrac{\partial^4\hat{u}}{\partial x^4}|_i^n + ...)\} = 0, \end{aligned} \tag{55}$$

where

$$E(\hat{u}) = 2\frac{(\Delta t)^2}{4!}\frac{\partial^4 u}{\partial t^4}|_i^n + 2h\Delta t\frac{\partial^2\hat{u}}{\partial t^2}|_i^n - 2c^2\frac{(\Delta x)^2}{4!}\frac{\partial^4\hat{u}}{\partial x^4}|_i^n + ..., \tag{56}$$

is the truncation error of the first and second- order accurate in time and space respectively.

4 Numerical tests

In this section we apply the numerical schemes to solve the following example. The accuracy of our proposed numerical methods is measured by computing the difference between the analytic and the numerical solutions at some mesh point. The analytic solution of the equation 1 by initial and boundary conditions 2 and 3 and also $L = \pi, c^2 = 1$, which obtained by method of separation of variables obtained in [3], [10] is given by

$$u(x,t) = \sum \sin(nx)e^{-ht}\lfloor a_n\cos(\sqrt{n^2-h^2 t}) + b_n\sin(\sqrt{n^2-h^2 t})\rfloor, \tag{57}$$

where

$$\begin{aligned} a_n = \tfrac{2}{\pi}\textstyle\int_0^\pi f(x)\sin(nx)dx, \\ b_n = \tfrac{1}{\sqrt{n^2-h^2}}(ha_n + \tfrac{2}{\pi}\textstyle\int_0^\pi g(x)\sin(nx)dx), \end{aligned} \tag{58}$$

Example 4.1:

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad 0 < t \leq 1, \tag{59}$$

with the initial and boundary conditions,

$$u(x,0) = f(x) = \begin{cases} \frac{2}{\pi}x & 0 \leq x \leq \pi/2 \\ \frac{-2}{\pi}(x-\pi) & \pi/2 \leq x \leq \pi \end{cases}, u(0,t) = u(\pi,t) = 0, \tag{60}$$

$$u_t(x,0) = g(x) = 0, \tag{61}$$

with

$$c = 1, h = 0.5$$

By (57) the exact solution for this example is,

$$u(x,t) = \sum \sin(nx)e^{-\frac{t}{2}}[a_n \cos(\sqrt{n^2 - \frac{1}{4}}t) + b_n \sin(\sqrt{n^2 - \frac{1}{4}}t) \tag{62}$$

where

$$a_n = \frac{8}{n^2\pi^2}\sin(n\frac{\pi}{2}), b_n = \frac{a_n}{\sqrt{n^2 - 1/4}}, \ for \ \ n = 1, 2, ... \tag{63}$$

The numerical results for this example are presented in table 1, for $\Delta t = 0.0317$, and $\Delta x = \pi/99$. We can see that the error of the solutions produced by the methods when $\Delta t, \Delta x$ replaced by $\frac{\Delta t}{3}, \frac{\Delta x}{3}$ and $\frac{\Delta t}{5}, \frac{\Delta x}{5}$ respectively reduce.
Where,

$$Error = max_{n,i}|\hat{u}_i^n - u_i^n|,$$

and $\hat{u}_i^n$ is exact solution and u_i^n is numerical.

In figure2, we plot the graph of the solutions at time $t = 0.982700000000000$ for $h = 0.5$ by the two methods with $\Delta x = \frac{\pi}{99}, \Delta t = 0.0317$, and the analytical formula 57. This figure shows that for each method, two plots are coincident together. It means that, two solutions (exact solution and numerical solution) are very close to each other. Now we compare the errors of two new methods by changing the value of h. In figure 3 by drawing the plot of error for these two methods with $h = 0.0001, 0.5, 0.99$ at $t = 0.982700000000000$ and $\Delta t = 0.0317, \Delta x = \pi/99$. The results indicates, that when, h is close to zero, errors in first method and second method are small but when h is

International Journal of Mathematical Engineering and Science (IJMES)
Volume 3 Issue 3 (Year 2014) ISSN : 2277-6982
http://www.ijmes.com/

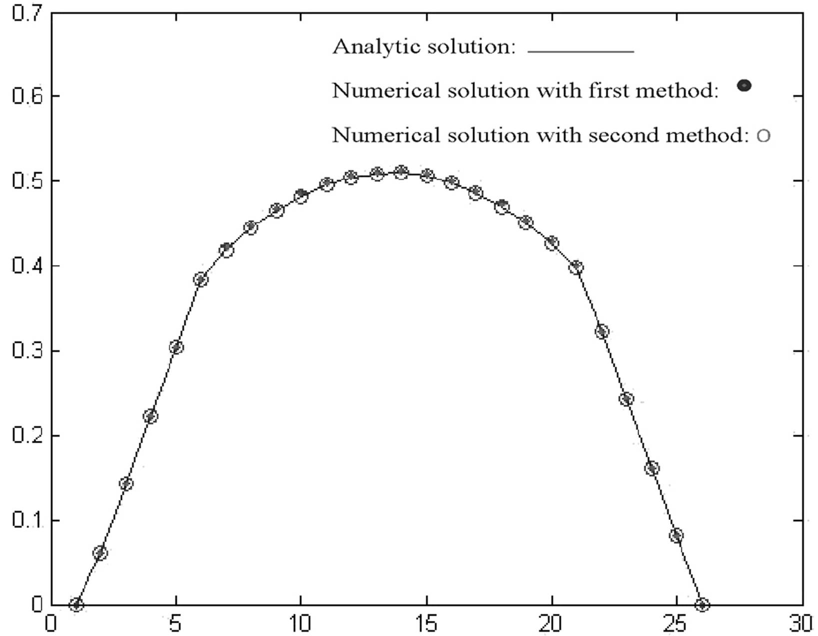


Figure 2: raphs of approximate and exact solutions of the partial differential equation (1) with two methods at $\Delta t = 0.0317, \Delta x = \pi/99, h = 0.5$

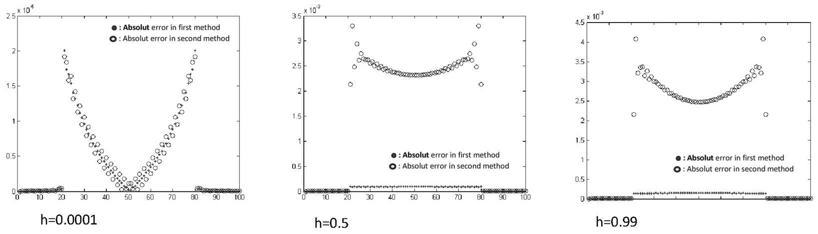
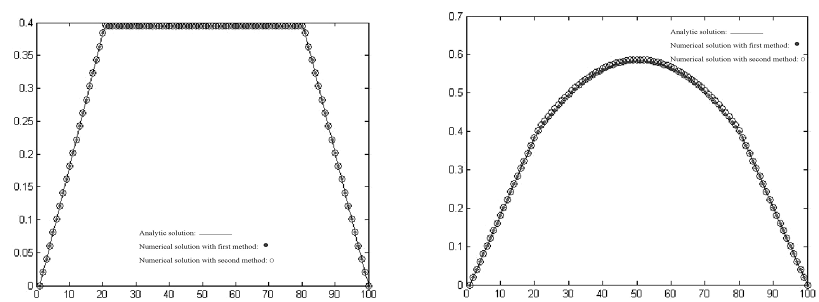


Figure 3: Plots of error in first method and second method at time $t = 0.982700000000000$ with $\Delta t = 0.0317, \Delta x = \pi/99, h = 0.0001, 0.5, 0.99$

Table 1: Comparison the error of first method and second method in some grid points. First and second-order accurate ($h = 0.5$)

	Δt	Δx	Error
Method 1	0.0317	$\pi/99$	1.1802e-04
	0.0317/3	$\pi/(3 \times 99)$	4.5694e-05
	0.0317/5	$\pi/(5 \times 99)$	6.1146e-05
Method 2	0.0317	$\pi/99$	0.0033
	0.0317/3	$\pi/(3 \times 99)$	0.0012
	0.0317/5	$\pi/(5 \times 99)$	7.6885e-04



h=0.0001 (a) h=0.99 (b)

Figure 4: Comparison of exact solutions with the numerical solutions of the partial differential equation(1) with $h = 0.0001$ at $\Delta t = 0.0317, \Delta x = \pi/99$

near to one, the first method is better than the second method.

And finally in figure (4), we draw the graphs of numerical and exact solutions of these new methods with $h = 0.0001$ and $h = 0.99$, at $\Delta x = \frac{\pi}{99}$, $\Delta t = 0.0317$. In figure 4(b), we have $e_1 = 2.3277e - 0.4$ (error of first method) and $e_2 = 0.00414$ (error of second method) but in figure 4(a), we have $e_1 = 2.012e - 0.5$ and $e_2 = 1.92e - 0.5$, so we can see that when h is close to one the first method works better than the second one .

In this example by using the definition of matrix A from 20 and 49, we have cond(A)= 74.4566(in the first method) and cond(A)= 418.5632 (in the second method), where cond(A) $= \|A\|\|A^{-1}\|$.

5 Conclusions

Wave equation is one of the most important equations in engineering science. In this work, we studied two explicit finite-difference schemes to approximate the solution of this equation. In section 3, the method of separation of variables was used to obtain the analytic solution of the wave equation for comparison with the numerical solutions obtained by finite difference method. In section 4, we presented two explicit finitedifference methods which have (1, 3, 1) molecule schemes. Also we found that those methods are consistent of order $O\{(\Delta t)^2, (\Delta x)^2\}$ and stable. In section 5, we illustrated these two new methods. By an example, the errors of methods compared with each other in table 1. The results show that, the first method is better than the second method. Also in figure (2), we can see that numerical solutions of new methods are very close to the exact solution. Paying attention to the truncation errors that are achieved in the(36) and(58), it is evident that the coefficient of (Δt) in the truncation error of the first method is $\frac{2}{6}h\frac{\partial^3 u}{\partial t^3}|_i^n$, and in the truncation error of the second method is $h\frac{\partial^2 \hat{u}}{\partial t^2}|_i^n$, so the error of the first method is lower than the second method. Finally, we compose the absolute error of these two methods with $h = 0.0001, 0.5$ and 0.99 in figure (3). It is obvious that when h is closed to zero the error of two methods are small but when h is close to 1, the error in the first method is smaller than else. Considering all above results, we get to the conclusion that for solving the wave equation, although two new methods work with high accuracy, but the first method is better than the second method for this equation.

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