Trapezoidal Method for Solving the First Order Stiff Systems on a Piecewise Uniform Mesh

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Abstract

In this paper, we introduce a method based on the modification of the Trapezoidal Method with a Piecewise Uniform Mesh proposed in Sumithra and Tamilselvan [1] for a numerical solution of stiff Ordinary Differential Equations (ODEs) of the first order system. Using this modification, the stiff ODEs were successfully solved and it resulted in good solutions.

Keywords: Stiff Systems, Trapezoidal Method, One Step Method, Initial Value Problem, Piecewise Uniform Mesh, Singular Perturbation Problem.

1 Introduction

Blenski [2] states that, in a system of equations the time constant characterizing the system differs from each other by orders of magnitude. Such a system of ODEs is known as a stiff system of equations. For this stiff system, at the initial moment (close to origin) the standard perturbation approach breaks down for t. Therefore, for the small t the standard expansion is supplemented by an inner solution. This approach of stiff system of ODEs leads to the singular perturbation method. The Singular Perturbation Problem (SPP) works well in a piecewise uniform mesh rather than in a uniform mesh which is discussed in Miller et al. [3]. Hence, for the stiff system also we can implement the piecewise uniform mesh logic.

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Consider the stiff linear system of first order equations

$$\begin{cases} u'(t) + a_{11}(t)u(t) + a_{12}(t)v(t) = f_1(t), \\ v'(t) + a_{21}(t)u(t) + a_{22}(t)v(t) = f_2(t) \end{cases} \tag{1.1}$$

$$u(0) = A \ \ and \ \ v(0) = B, \quad t \in (0, 1]. \tag{1.2}$$

In Sumithra and Tamilselvan [1], the Trapezoidal Method with a piecewise uniform mesh was implemented for the stiff system of ODEs for all $a_{ij} \gg 1$. In this paper we extend the work for another class of problems which satisfies the following conditions:

$$\begin{cases} (i) For\ all\ |a_{1j}(t)| \ \gg \ |a_{2j}(t)|,\ \text{for}\ j = 1, 2. \quad where\ t \in (0, 1] \\ \qquad (or) \\ (ii)\ |a_{i1}(t)| \ < \ |a_{i2}(t)|,\ \text{for}\ i = 1, 2\ where\ t \in (0, 1]. \end{cases} \tag{1.3}$$

The system (1.1) - (1.2) is said to be stiff if the eigen values λ_i of the matrix (a_{ij}) have negative real parts and for which

$$max_i|\lambda_i| \gg min_i|\lambda_i|.$$

In a physical system the components for which $|\lambda_i|$ is large and the real part of $|\lambda_i|$ is negative they decay rapidly and are not seen in the solution beside some initial transients. Thus it is fairly paradoxical that the exact components which lead to instabilities in the numerical schemewhich could also be said that the solution has the components which evolve at very different rates.

The initial value problem of stiff differential equations occurs in the fields of:
1. Chemical Reactions: A famous chemical reaction is the originator reaction among $HBrO_2$, Br^- and Ce(IV).
2. Reaction-Diffusion systems: Problems in which the diffusion is modeled via the Laplace operator may become stiff as they are discretized in space by finite differences or finite elements.
The well-known example of such systems which appears so often in the mathematical biology. Several further occurrences of the stiffness can be found in electrical circuits, mechanics, meteorology, oceanography and vibrations.

For a detailed discussion on the stiff nature, implicit methods and the Trapezoidal Method with a uniform mesh for stiff ODEs one may refer to Shampine and Gear [4], Shampine and Thompson [5], Idrees et al. [6], Gordon and Shampine [7], Satek [8], to name a few. One can refer Sumithra and Tamilselvan [9] and Sumithra and Tamilselvan [1] for numerical methods for stiff ODEs on the piecewise uniform mesh.

The rest of the paper is organized as follows: in section 2, we briefly summarize the Trapezoidal

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Method and the description of the piecewise uniform mesh. In section 3, we discuss the local truncation error of the method. In section 4, we show the accuracy of our method. Finally, in section 5, we present some concluding remarks.

2 The Trapezoidal Method with a piecewise uniform mesh

Suppose that we approximate equation (1.1)-(1.2) by applying the Trapezoidal rule, then

$$\begin{cases} u_{j+1} = u_j + \frac{h}{2}[f_1(t_j, u_j) + f_1(t_{j+1}, u_{j+1})] \\ v_{j+1} = v_j + \frac{h}{2}[f_2(t_j, v_j) + f_2(t_{j+1}, v_{j+1})], \qquad j \ = \ 0\text{to}(N-1), \end{cases} \tag{2.1}$$

where u_{j+1} and v_{j+1} are determined implicitly, by an explict Euler Method as

$$\begin{cases} u_{j+1} = u_j + hf_1(t_j, u_j) \\ v_{j+1} = v_j + hf_2(t_j, v_j), \qquad j \ = \ 0\text{to}(N-1). \end{cases} \tag{2.2}$$

The piecewise uniform mesh is given by $\{t_j\}_0^N$, where

$$\begin{cases} t_j = jh_1 \quad where \quad h_1 = \frac{4\ \sigma}{N}, \quad j = 0(1)\frac{N}{4}, \\ t_j = \sigma + (j - \frac{N}{4})h_2 \quad where \quad h_2 = \frac{4\ (1-\sigma)}{3\ N}, \quad j = (\frac{N}{4}+1)(1)N. \end{cases} \tag{2.3}$$

The transition point σ is defined as

$$\sigma = \min\Big\{\frac{1}{4}, (\frac{\varepsilon}{\alpha})\ lnN\Big\}, \tag{2.4}$$

where

$$\varepsilon < \frac{1}{M}, \tag{2.5}$$

M is the greatest eigen value and α is the smallest eigen value of (1.1). Here N is the number of mesh point which satisfies the following conditions:

$$\begin{cases} (i)N = 2^m \quad with \quad m \geq 11\ for \quad all\ |a_{1j}(t)| \ \gg \ |a_{2j}(t)|,\ \text{for}\ j = 1, 2. \quad where \ \ t \in (0, 1] \\ (ii)N = 2^m \quad with \quad m \geq 7 \quad for \quad all\ |a_{i1}(t)| \ < \ |a_{i2}(t)|,\ for\ i = 1, 2. \quad where \ \ t \in (0, 1] \end{cases}$$

3 The local truncation error

In general, $u(t_{j+1})$ is the exact value and u_{j+1} is the approximate numerical value.
The local truncation error at the point $t_{(j+1)}$ in the Trapezoidal Method with a piecewise uniform

mesh for $0 \leq j \leq \frac{N}{4} - 1$ is

$$\begin{aligned}
T_{j+1} &= u(t_{j+1}) - u_{j+1} \quad j = 0, 1...N-1 \\
&= u(t_{j+1}) - [u(t_j) + \frac{h_1}{2} f(u(t_{j+1}), t_{j+1}) + \frac{h_1}{2} f(u(t_j), t_j)] \\
&= u(t_{j+1}) - [u(t_{j+1}) - h_1 u^{'}(t_{j+1}) + \frac{h_1^2}{2} u^{''}(t_{j+1}) - \frac{h_1^3}{3} u^{'''}(t_{j+1})] \\
&\quad - \frac{h_1}{2} u^{'}(t_{j+1}) - \frac{h_1}{2} u^{'}(t_j) \\
&= \frac{h_1}{2} u^{'}(t_{j+1}) - \frac{h_1^2}{2} u^{''}(t_{j+1}) + \frac{h_1^3}{3} u^{'''}(t_{j+1}) - \frac{h_1}{2}[u^{'}(t_{j+1}) \\
&\quad - h_1 u^{''}(t_{j+1}) + \frac{h_1^2}{2} u^{'''}(t_{j+1})]
\end{aligned}$$

$$\Rightarrow T_{j+1} = \frac{h_1^3}{12} u^{'''}(t_{j+1})$$

The truncation error for $\frac{N}{4} \leq j \leq N - 1$ is

$$\begin{aligned}
T_{j+1} &= u(t_{j+1}) - u_{j+1} \quad j = 0, 1...N-1 \\
&= u(t_{j+1}) - [u(t_j) + \frac{h_2}{2} f(u(t_{j+1}), t_{j+1}) + \frac{h_2}{2} f(u(t_j), t_j)] \\
&= u(t_{j+1}) - [u(t_{j+1}) - h_2 u^{'}(t_{j+1}) + \frac{h_2^2}{2} u^{''}(t_{j+1}) - \frac{h_2^3}{3} u^{'''}(t_{j+1})] \\
&\quad - \frac{h_2}{2} u^{'}(t_{j+1}) - \frac{h_2}{2} u^{'}(t_j) \\
&= \frac{h_2}{2} u^{'}(t_{j+1}) - \frac{h_2^2}{2} u^{''}(t_{j+1}) + \frac{h_2^3}{3} u^{'''}(t_{j+1}) - \frac{h_2}{2}[u^{'}(t_{j+1}) \\
&\quad - h_2 u^{''}(t_{j+1}) + \frac{h_2^2}{2} u^{'''}(t_{j+1})]
\end{aligned}$$

$$\Rightarrow T_{j+1} = \frac{h_2^3}{12} u^{'''}(t_{j+1})$$

Therefore, the truncation error for the Trapezoidal Method with the piecewise uniform mesh is

$$T_{j+1} = \begin{cases} \frac{h_1^3}{12}\, u^{'''}(t_{j+1}) & for \quad 0 \leq j \leq \frac{N}{4} - 1 \\ \frac{h_2^3}{12}\, u^{'''}(t_{j+1}) & for \quad \frac{N}{4} \leq j \leq N - 1 \end{cases} \tag{3.1}$$

Similarly, the truncation error for the second component v can be easily derived.

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We define

$$||Y||_s = sup\{|u^{(s)}(t)|, |v^{(s)}(t)|\} \quad for\ all \quad t \in (0,1].$$

Let

$$h^3 = max({h_1}^3, {h_2}^3).$$

Then,

$$T_{j+1}(h) \leq C\ h^3||Y||_3,$$

$$where||Y||_3 = sup\{|u^{'''}|, |v^{'''}|\} \quad for\ all \quad t \in (0,1].$$

Hence, by the definition given as in Jain [10], a one step method has an order of convergence p, if for any sufficiently smooth solution $y(t)$, there exist constant k and h_0 such that $|T_n(h)| \leq kh^{(p+1)}$, $0 < h \leq h_0$. Hence, the order of convergence of the Trapezoidal Method with the piecewise uniform mesh are two.

4 A numerical example

In this section, we present some numerical results to illustrate the performance of our method. The numerical results of the Trapezoidal Method with the piecewise uniform mesh will be compared with the results with the uniform mesh.

The comparison is based in terms of a maximum error and an average error. The numerical results are recorded in terms of the following quantities and tabulated.

As the formula given in Nasir et al. [11] for the uniform mesh we have,
$h = \frac{(b-a)}{N}$, where b is the end value of t and a is the initial value of t.

The calculation of errors (for the piecewise uniform mesh and the uniform mesh) is given as,
$error_j = |u(t_j)_{(exact\quad solution)} - u_{j(approximate)}|$.

For the maximum error (MAXE) (for the piecewise uniform mesh and the uniform mesh), we use the formula,
$MAXE^N = \max(error_j)$

The average error(AVE) for the Trapezoidal Method with the uniform mesh is defined as,

$$AVE = \frac{\sum\limits_{j=1}^{N} error_j}{\frac{(b-a)}{h}},$$

where b is the end value of t and a is the initial value of t.

The average error(AVE) for the Trapezoidal Method with the piecewise uniform mesh is defined as,

$$AVE1 = \frac{\sum_{j=1}^{\frac{N}{4}} (error_j)}{\frac{\sigma}{\frac{h_1}{4}}},$$

$$AVE2 = \frac{\sum_{j=\frac{N}{4}+1}^{N} (error_j)}{\frac{(1-\sigma)}{\frac{3\ h_2}{4}}} \quad and$$

$AVE \ = \ max\{AVE1, AVE2\}$

Example 4.1. $u'(t) = -2000u(t) + 999.75v(t) + 1000.25$
$v'(t) = u(t) - v(t) \quad \forall t \in [0, 1],$
$u(0) = 0, v(0) = -2.$

Exact solution of the above problem is
$u(t) = -1.499875e^{-0.5t} + 0.499875e^{-2000.5t} + 1$
$v(t) = -1.499875e^{-0.5t} + 0.499875e^{-2000.5t} + 1$.
The numerical results obtained by applying the piecewise uniform mesh method (2.3), (2.4), (2.5) to the Example 4.1 *are given in Table 1 .*

Example 4.2. $u'(t) = 32u(t) + 66v(t) + \frac{2}{3}\frac{t}{} + \frac{2}{3} \quad \forall t \in [0, 1],$
$v'(t) = -66u(t) - 133v(t) - \frac{t}{3} - \frac{1}{3}$
$u(0) = v(0) = \frac{1}{3}.$

Exact solution of the above problem is
$u(t) = \frac{2}{3}t + \frac{2}{3}e^{-t} - \frac{1}{3}e^{-100t}$
$v(t) = -\frac{1}{3}t - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-100t}$
The numerical results obtained by applying the piecewise uniform mesh method (2.3), (2.4), (2.5) to the Example 4.2 are given in Table 2 .

Example 4.3. $u'(t) = 9u(t) + 24v(t) + 5cos(t) - \frac{1}{3}sin(t) \quad \forall t \in [0, 1],$
$v'(t) = -24u(t) - 51v(t) - 9cos(t) + \frac{1}{3}sin(t)$
$u(0) = \frac{4}{3} \quad v(0) = \frac{2}{3}.$

Exact solution of the above problem is
$u(t) = 2e^{-3t} - e^{-39t} + \frac{1}{3}cos(t)$
$v(t) = -e^{-3t} + 2e^{-39t} - \frac{1}{3}cos(t)$
The numerical results obtained by applying the piecewise uniform mesh method (2.3), (2.4), (2.5) to the Example 4.3 are given in Table 3.

Table 1: *Values of $MAXE(u)$, $AVE(u)$ for the solution components u for the Example 4.1*

		Number of mesh points N			
		2048	4096	8192	16384
$MAXE(u)$	*piecewise uniform mesh*	0.25289e-008	0.14734e-009	0.25247e-009	0.12173e-009
	uniform mesh	0.35326e-004	0.59767e-005	0.12862e-005	0.30063e-006
$AVE(u)$	*piecewise uniform mesh*	0.13815e-010	0.20260e-011	0.29543e-012	0.42704e-013
	uniform mesh	0.17249e-007	0.14591e-008	0.15701e-009	0.18349e-010

Table 2: *Values of $MAXE(u)$, $AVE(u)$ for the solution components u for the Example 4.2*

		Number of mesh points N			
		128	256	512	1024
$MAXE(u)$	*piecewise uniform mesh*	0.31436e-003	0.63106e-004	0.20212e-004	0.67283e-005
	uniform mesh	0.15276e-001	0.26043e-002	0.56925e-003	0.13432e-003
$AVE(u)$	*piecewise uniform mesh*	0.60919e-005	0.10066e-005	0.15960e-006	0.24573e-007
	uniform mesh	0.11934e-003	0.10173e-004	0.11118e-005	0.13117e-006

Example 4.4. $u'(t) = -u(t) + 95v(t)$
$v'(t) = -u(t) - 97v(t) \quad \forall t \in [0, 1],$
$u(0) = 1, v(0) = 1.$

Exact solution of the above problem is
$u(t) = \frac{1}{47}(95e^{-2t} - 48e^{-96t})$
$v(t) = \frac{1}{47}(48e^{-96t} - e^{-2t}).$
The numerical results obtained by applying the piecewise uniform mesh method (2.3), (2.4), (2.5) to the Example 4.4 are given in Table 4.

5 Conclusion

The potential of the Trapezoidal Method with the piecewise uniform mesh has been exposed by many practical experiments and the results show that the new modified Trapezoidal Method with the piecewise uniform mesh method gives better accuracy.

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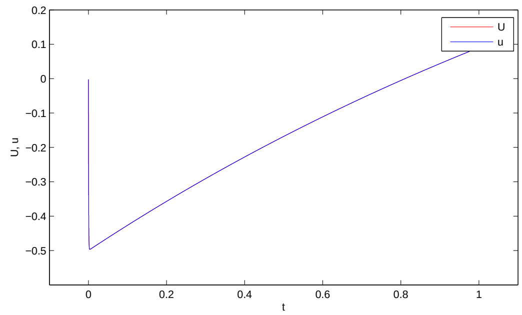
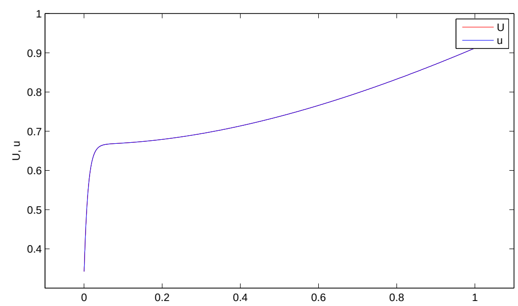
Figure 1:



For the Example 4.1 with $N = 16,384$ and U, u represents the numerical and the exact solution respectively.

Figure 2:

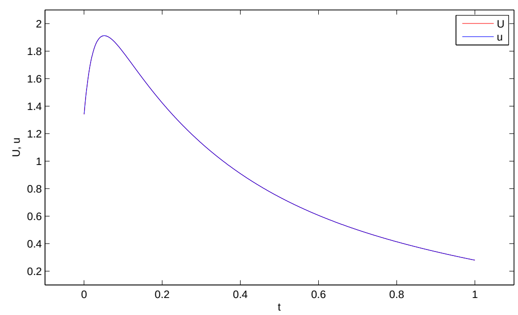


For the Example 4.2 with $N = 1024$ and U, u represents the numerical and the exact solution respectively.

Table 3: *Values of $MAXE(u)$, $AVE(u)$ for the solution component u for the Example 4.3*

		Number of mesh points N			
		128	256	512	1024
$MAXE(u)$	*piecewise uniform mesh*	0.22693e-004	0.20305e-004	0.69571e-005	0.20102e-005
	uniform mesh	0.44536e-003	0.97043e-004	0.22765e-004	0.55192e-005
$AVE(u)$	*piecewise uniform mesh*	0.25224e-006	0.79316e-007	0.13588e-007	0.19631e-008
	uniform mesh	0.34794e-005	0.37907e-006	0.44464e-007	0.53899e-008

Figure 3:

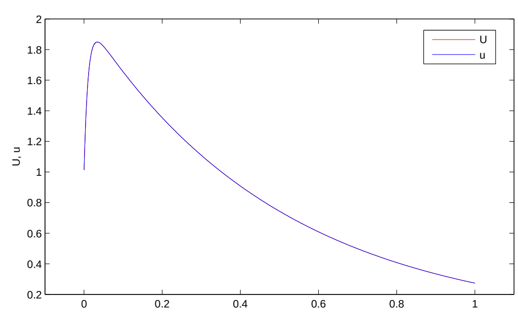


For the Example 4.3 with $N = 1024$ and U, u represents the numerical and the exact solution respectively.

Table 4: *Values of $MAXE(u)$, $AVE(u)$ for the solution component u for the Example 4.4*

		Number of mesh points N			
		128	256	512	1024
$MAXE(u)$	*piecewise uniform mesh*	0.77150e-004	0.10677e-004	0.19461e-005	0.39832e-006
	uniform mesh	0.34506e-003	0.66040e-004	0.14793e-004	0.35165e-005
$AVE(u)$	*piecewise uniform mesh*	0.16543e-005	0.20889e-006	0.25277e-007	0.29406e-008
	uniform mesh	0.26958e-005	0.25797e-006	0.28893e-007	0.34340e-008

Figure 4:



For the Example 4.4 with $N = 1024$ and U, u represents the numerical and the exact solution respectively.

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