Riemann Hypothesis is incorrect (second proof)

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Abstract. A few years ago, I wrote my paper [4]. In the paper [4], I use Nevanlinna's Second Main Theorem of the value distribution theory, denied the Riemann Hypothesis. In this paper, I use the analytic methods, I once again denied the Riemann Hypothesis.

Keyword. Riemann Hypothesis, Disavowal.

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1.Introduction

Riemann Hypothesis was posed by Riemann in early 50's of the 19 th century in his thesis titled "The Number of Primes less than a Given Number". It is one of the unsolved "Supper" problems of mathematics. The Riemann Hypothesis is closely related to the well-known Prime Number Theorem. The Riemann Hypothesis states that all the nontrivial zeros of the zeta-function lie on the 'critical line'$\{s : Res = \frac{1}{2}\}$. In this paper, we use the analytical methods, refute the Riemann Hypothesis. For convenience, We will below the Riemann Hypothesis abbreviated to RH.

2. Some theorems in the classic theory

In this paper, $\Gamma(s)$ is the Euler gamma function, $\zeta(s)$ is the Riemann zeta function.

Lemma 2.1. If $Re\, w > 0$, then

$$\frac{1}{2\pi i}\int_{(2)}\Gamma(s)w^{-s}ds = e^{-w}$$

where Rew is the real part of complex number w.

Let $\eta > 0$ be given, when $|s| \geq \eta$ and $|\arg s| \leq \pi - \eta$, then

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right)$$

If $-4 \leq \sigma \leq 4, |t| \geq 1$, then

$$\Gamma(\sigma + it) = \sqrt{2\pi}\, t^{\sigma-\frac{1}{2}} \exp\left(-\frac{\pi}{2}|t| + it(\log|t| - 1)) + i\lambda\frac{\pi}{2}\left(\sigma - \frac{1}{2}\right)\right)$$

$$+O\left(t^{\sigma-\frac{3}{2}} \exp\left(-\frac{\pi}{2}|t|\right)\right)$$

where $\exp(x) = e^x$, and $\lambda = 1$ if $t \geq 1$, $\lambda = -1$ if $t \leq -1$.

See[1], page 523, page 525.

Lemma 2.2. If $Res > 1$, then

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ is the Mangoldt function.

Let s is any complex number, we have

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + c_1 + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)}$$

where ρ be the nontrivial zeros of $\zeta(s)$, c_1 be the positive constant.

We write $s = \sigma + it$. If $-1 \leq \sigma \leq 2$, $-\pi < Im\{\log(s-1) \leq \pi$, $-\pi < Im\{\log(s-\rho)\} \leq \pi$, then

$$\log\zeta(s) = -\log(s-1) + \sum_{|\gamma - t| \leq 1} \log(s-\rho) + O\left(\log(|t|+2)\right)$$

where Ims is the imaginary part of complex number s.

See [2], page 4, page 31, page 218.

Lemma 2.3. Let $N(T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1, 0 < t < T$, then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

where $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + iT)$.

See [3], page 98.

Lemma 2.4. Assume that RH, If $x \geq 2$, then

$$\psi(x) = \sum_{2 \leq n \leq x} \Lambda(n) = x + R(x)$$

where $R(x) \ll x^{\frac{1}{2}} \log^2 x$.

See [3], page 113.

3. Some preparation work

Lemma 3.1. Assume that RH, and $0 < \delta \le \frac{1}{50}$, then

$$\int_{\frac{1}{2}+\delta}^{2} |\log \zeta(\sigma + i\gamma_0)| d\sigma \ll 1$$

and

$$\int_{-1}^{\frac{1}{2}-\delta} |\log \zeta(\sigma + i\gamma_0)| d\sigma \ll 1$$

where γ_0 is the ordinate of nontrivial first zero of $\zeta(s)$, $\gamma_0 \approx 14.134725$.

Proof. By lemma 2.2 and RH, we have

$$\log \zeta(\sigma + i\gamma_0) \ll \sum_{|\gamma-\gamma_0|\le 1} \left|\log \left(\sigma - \frac{1}{2} + i\gamma_0 - i\gamma\right)\right| + O(\log \gamma_0)$$

because

$$\log \left(\sigma - \frac{1}{2} + i\gamma_0 - i\gamma\right) = \frac{1}{2} \log \left(\left(\sigma - \frac{1}{2}\right)^2 + (\gamma_0 - \gamma)^2\right)$$

$$+ i \arg \left(\sigma - \frac{1}{2} + i\gamma_0 - i\gamma\right)$$

therefore

$$\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right| \ll \left|\log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right)\right|+1$$

and

$$\log\left(\sigma-\frac{1}{2}\right)^2 \leq \log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right) \leq \log 4$$

$$\log\left((\sigma-\frac{1}{2})^2+(\gamma_0-\gamma)^2\right) \ll \left|\log\left(\sigma-\frac{1}{2}\right)^2\right|+1$$

therefore

$$\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right| \ll \left|\log\left(\sigma-\frac{1}{2}\right)^2\right|+1$$

And because

$$\int_{\frac{1}{2}+\delta}^{2}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|d\sigma = \int_{\frac{1}{2}+\delta}^{\frac{3}{2}}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|d\sigma + \int_{\frac{3}{2}}^{2}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|d\sigma$$

$$= -2\int_{\frac{1}{2}+\delta}^{\frac{3}{2}}\log\left(\sigma-\frac{1}{2}\right)d\sigma + 2\int_{\frac{3}{2}}^{2}\log\left(\sigma-\frac{1}{2}\right)d\sigma$$

$$= -2\int_\delta^1 \log\sigma d\sigma + 2\int_1^{\frac{3}{2}} \log\sigma d\sigma = 2\delta\log\delta + 2\int_\delta^1 d\sigma + O(1) = O(1)$$

therefore

$$\int_{\frac{1}{2}+\delta}^2 |\log\zeta(\sigma+i\gamma_0)|\, d\sigma \ll \sum_{|\gamma-\gamma_0|\le 1} \int_{\frac{1}{2}+\delta}^2 \left|\log\left(\sigma - \frac{1}{2} + i\gamma_0 - i\gamma\right)\right| + 1$$

$$\ll \int_{\frac{1}{2}+\delta}^2 \left|\log\left(\sigma - \frac{1}{2}\right)^2\right| + 1 \ll 1$$

Similarly, we have

$$\int_{-1}^{\frac{1}{2}-\delta} |\log\zeta(\sigma+i\gamma_0)|\, d\sigma \ll 1$$

This completes the proof of Lemma 3.1.

Throughout the paper, we write

$$z = a + ib, \qquad a = \frac{1}{T}, \qquad T \ge 50, \qquad b = 2\pi.$$

It is easy to see that

$$\arctan\frac{b}{a}=\frac{\pi}{2}-h,\qquad h=\sum_{k=0}^{\infty}(-1)^k\frac{a^{2k+1}}{(2k+1)b^{2k+1}},\qquad \frac{1}{4\pi T}\le h\le\frac{1}{\pi T}.$$

Lemma 3.2. We calculate the three complex numbers.

Because

$$a+ib=(a^2+b^2)^{\frac{1}{2}}\exp\left(i\arctan\frac{b}{a}\right)=(a^2+b^2)^{\frac{1}{2}}\exp\left(i\frac{\pi}{2}-ih\right)$$

therefore when t is the real number, we have

$$z^{\frac{3}{4}-it}=(a^2+b^2)^{\frac{3}{8}-i\frac{t}{2}}\exp\left(i\frac{3\pi}{8}-i\frac{3h}{4}+\frac{\pi}{2}t-th\right)\ll\exp\left(\frac{\pi}{2}t-th\right)$$

$$z^{-\frac{1}{2}-it}=(a^2+b^2)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}+\frac{\pi}{2}t-th\right)\ll\exp\left(\frac{\pi}{2}t-th\right)$$

$$z^{-\frac{1}{2}+it}=(a^2+b^2)^{-\frac{1}{4}+i\frac{t}{2}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}-\frac{\pi}{2}t+th\right)\ll\exp\left(-\frac{\pi}{2}t+th\right)$$

The three complex numbers required below.

Lemma 3.3.

$$\int\limits_{(-\frac{3}{4})}\Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}ds\ll 1$$

Proof. By lemma 2.1 and lemma 3.2, we have

$$\int\limits_{(-\frac{3}{4})} \Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}ds = i\int\limits_{-\infty}^{+\infty} \Gamma\left(-\frac{3}{4}+it\right)\frac{\zeta'}{\zeta}\left(-\frac{3}{4}+it\right)(a+ib)^{\frac{3}{4}-it}dt$$

$$\ll \int\limits_{-\infty}^{+\infty} \left|\Gamma\left(-\frac{3}{4}+it\right)\frac{\zeta'}{\zeta}\left(-\frac{3}{4}+it\right)\right|\left|(a+ib)^{\frac{3}{4}-it}\right|dt$$

$$\ll \int\limits_{-\infty}^{+\infty} (|t|+2)^{-\frac{5}{4}}e^{-ht}\log(|t|+2)dt \ll \int\limits_{-\infty}^{+\infty} (|t|+2)^{-\frac{5}{4}}\log(|t|+2)dt \ll 1$$

This completes the proof of Lemma 3.3.

Lemma 3.4.

$$\int\limits_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right)(a+ib)^{-\frac{1}{2}-it}\left(\log\frac{t}{2\pi}\right)dt \ll \log^2 T$$

Proof. By lemma 2.1 and lemma 3.2, we have

$$\int\limits_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right)(a+ib)^{-\frac{1}{2}-it}\left(\log\frac{t}{2\pi}\right)dt$$

$$= \sqrt{2\pi}(a^2+b^2)^{-\frac{1}{4}}e^{-i\frac{\pi}{4}+i\frac{h}{2}}\int\limits_{\gamma_0}^{\infty} e^{-th+it(\log t-1)}(a^2+b^2)^{-i\frac{t}{2}}\left(\log\frac{t}{2\pi}\right)dt$$

$$+O\left(\int\limits_{\gamma_0}^{\infty} t^{-1}e^{-th}\log t\,dt\right) = I_1\left(\sqrt{2\pi}(a^2+b^2)^{-\frac{1}{4}}e^{-i\frac{\pi}{4}+i\frac{h}{2}}\right)+I_2$$

We write $r = (a^2+b^2)^{\frac{1}{2}}, \quad 2\pi \le r \le 2\pi + 1.$

$$I_1 = \int_{\gamma_0}^{\infty} e^{-th+it(\log t - \log r - 1)} \left(\log \frac{t}{2\pi} \right) dt$$

$$= \int_{\gamma_0}^{\infty} \frac{e^{-th}}{i \log t - i \log r} \log \left(\frac{t}{2\pi} \right) d\, e^{it(\log t - \log r - 1)}$$

$$= -i \int_{\gamma_0}^{\infty} \frac{e^{-th}}{\log t - \log r} (\log t - \log 2\pi)\, d\, e^{it \log \frac{t}{re}}$$

$$= -i \int_{\gamma_0}^{\infty} \left(e^{-th} + \frac{e^{-th}}{\log t - \log r} (\log r - \log 2\pi) \right) d\, e^{it \log \frac{t}{re}}$$

$$= O(1) - i \int_{\gamma_0}^{\infty} \left(he^{-th} + \left(\frac{he^{-th}}{\log t - \log r} + \frac{e^{-th}}{t(\log t - \log r)^2} \right) \left(\log \frac{r}{2\pi} \right) \right) e^{it \log \frac{t}{re}}\, d\, t$$

$$\ll \int_{\gamma_0}^{\infty} \left(he^{-ht} + \frac{1}{t(\log t - \log r)^2} \right) dt \ll 1$$

$$I_2 = \int_{\gamma_0}^{h^{-2}} t^{-1} e^{-th} \log t dt + \int_{h^{-2}}^{\infty} t^{-1} e^{-th} \log t dt$$

$$\ll \int_{\gamma_0}^{h^{-2}} t^{-1} \log t dt + h^2 \log h^{-2} \int_{h^{-2}}^{\infty} e^{-th} dt \ll (\log h)^2 \ll \log^2 T$$

This completes the proof of Lemma 3.4.

Lemma 3.5.

$$\int\limits_{\gamma_0}^{\infty} \left| \Gamma'\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it} - \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it}\log z \right| t^{-1}dt \ll \log^2 T$$

Proof. When $t \geq \gamma_0$, by lemma 2.1, we have

$$\Gamma'\left(\frac{1}{2}+it\right) \ll \left|\Gamma\left(\frac{1}{2}+it\right)\log\left(\frac{1}{2}+it\right)\right| + \left|\frac{\Gamma\left(\frac{1}{2}+it\right)}{\frac{1}{2}+it}\right|$$

$$\ll e^{-\frac{\pi}{2}t}\log t + t^{-1}e^{-\frac{\pi}{2}t} \ll e^{-\frac{\pi}{2}t}\log t$$

By lemma 2.1 and lemma 3.2, we have

$$\int\limits_{\gamma_0}^{\infty} \left| \Gamma'\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it} - \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it}\log z \right| t^{-1}dt$$

$$\ll \int\limits_{\gamma_0}^{\infty} t^{-1}\exp(-th)\log t dt \ll \int\limits_{\gamma_0}^{h^{-2}} t^{-1}\exp(-th)\log t dt + \int\limits_{h^{-2}}^{\infty} t^{-1}\exp(-th)\log t dt$$

$$\ll \int\limits_{\gamma_0}^{h^{-2}} t^{-1}\log t dt + h\log h^{-2}\int\limits_{h^{-2}}^{\infty}\exp(-th)dt \ll \log^2 T$$

This completes the proof of Lemma 3.5

Lemma 3.6. Assume that RH, then

$$\int_{\gamma_0}^{\infty} \left(\Gamma'\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it} - \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it}\log z\right) S(t)dt \ll 1$$

where $S(t) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$.

Proof. We write $\quad G(s) = \Gamma'(s)z^{-s} - \Gamma(s)z^{-s}\log z.$

Assume that RH and $0 < \delta \le \frac{1}{50}$, by the contour integration method, we have

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty} G(s)\log\zeta(s)ds + \int_{-1+i\infty}^{-1+i\gamma_0} G(s)\log\zeta(s)ds + \int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G(s)\log\zeta(s)ds = 0$$

therefore

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty} G(s)\log\zeta(s)ds = -\int_{-1+i\infty}^{-1+i\gamma_0} G(s)\log\zeta(s)ds - \int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G(s)\log\zeta(s)ds = J_1 + J_2$$

By lemma 2.1 and lemma 3.2,

$$J_1 = -\int_{-1+i\infty}^{-1+i\gamma_0} G(s)\log\zeta(s)ds \ll \int_{\gamma_0}^{\infty} |G(-1+it)|\,|\log\zeta(-1+it)|\,dt$$

$$\ll \int_{\gamma_0}^{\infty} \left(\left|\Gamma'(-1+it)z^{1-it}\right| + \left|\Gamma(-1+it)z^{1-it}\log z\right|\right)(\log t)\,dt$$

$$\ll \int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}e^{-th}\log^2 t dt \ll \int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}\log^2 t dt \ll 1$$

By lemma 2.1, lemma 3.1 and lemma 3.2, we have

$$J_2 = -\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G(s)\log\zeta(s)ds \ll \int_{-1}^{\frac{1}{2}-\delta} |G(\sigma+i\gamma_0)|\,|\log\zeta(\sigma+i\gamma_0)|\,d\sigma$$

$$\ll \int_{-1}^{\frac{1}{2}-\delta} \left|\Gamma'(\sigma+i\gamma_0)z^{-\sigma-i\gamma_0} - \Gamma(\sigma+i\gamma_0)z^{-\sigma-i\gamma_0}\log z\right|\,|\log\zeta(\sigma+i\gamma_0)|\,d\sigma$$

$$\ll \int_{-1}^{\frac{1}{2}-\delta} |\log\zeta(\sigma+i\gamma_0)|\,d\sigma \ll 1$$

When $\delta \to 0$, we have

$$\int_{\gamma_0}^{\infty} G\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right) \ll 1$$

Similarly, assume that RH and $0 < \delta \le \frac{1}{50}$, by the contour integration method, we have

$$\int_{2+i\gamma_0}^{2+i\infty} G(1-s)\log\zeta(s)ds+\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G(1-s)\log\zeta(s)ds+\int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G(1-s)\log\zeta(s)ds=0$$

therefore

$$\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G(1-s)\log\zeta(s)ds=-\int_{2+i\gamma_0}^{2+i\infty} G(1-s)\log\zeta(s)ds-\int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G(1-s)\log\zeta(s)ds$$

$$\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G(1-s)\log\zeta(s)ds \ll 1$$

When $\delta \to 0$, we have

$$\int_{\gamma_0}^{\infty} G\left(\frac{1}{2}-it\right)\log\zeta\left(\frac{1}{2}+it\right)dt \ll 1$$

Synthesize the above conclusion, we have

$$\int_{\gamma_0}^{\infty} \left(G\left(\frac{1}{2}+it\right)+G\left(\frac{1}{2}-it\right)\right)\log\zeta\left(\frac{1}{2}+it\right)dt$$

$$= 2\int_{\gamma_0}^{\infty} ReG\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)dt$$

$$= 2\int_{\gamma_0}^{\infty} ReG\left(\frac{1}{2}+it\right)\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right| + i\arg\zeta\left(\frac{1}{2}+it\right)\right)dt$$

therefore

$$\int_{\gamma_0}^{\infty} ReG\left(\frac{1}{2}+it\right)S(t)dt \ll 1$$

where Res is the real part of complex number s

On the other hand, we have

$$\int_{\gamma_0}^{\infty}\left(G\left(\frac{1}{2}+it\right)-G\left(\frac{1}{2}-it\right)\right)\log\zeta\left(\frac{1}{2}+it\right)dt$$

$$= 2i\int_{\gamma_0}^{\infty} ImG\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)dt$$

$$= 2i\int_{\gamma_0}^{\infty} ImG\left(\frac{1}{2}+it\right)\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right| + i\arg\zeta\left(\frac{1}{2}+it\right)\right)dt$$

therefore

$$\int_{\gamma_0}^{\infty} ImG\left(\frac{1}{2}+it\right)S(t)dt \ll 1$$

where Ims is the imaginary part of complex number s

This completes the proof of Lemma 3.6.

Lemma 3.7. Assume that RH, we have

$$\sum_{-\infty<\gamma<+\infty} \Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma} \ll \log^2 T$$

Proof.

$$\sum_{-\infty<\gamma<+\infty} \Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma} = \sum_{\gamma_0<\gamma<+\infty} \Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}$$

$$+\sum_{\gamma_0<\gamma<+\infty} \Gamma\left(\frac{1}{2}-i\gamma\right)(a+ib)^{-\frac{1}{2}+i\gamma} = A_1 + A_2$$

$$A_1 = \sum_{\gamma_0<\gamma<+\infty} \Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma} = \int_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it} dN(t)$$

by lemma 2.3, the above formula

$$= \int_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it}\, d\left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(t^{-1})\right)$$

$$= \frac{1}{2\pi}\int_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it} \left(\log\frac{t}{2\pi}\right) dt + \int_{\gamma_0}^{\infty} \Gamma\left(\frac{1}{2}+it\right) z^{-\frac{1}{2}-it}\, d\left(S(t)+O(t^{-1})\right)$$

by lemma 3.4, the above formula

$$= -\int_{\gamma_0}^{\infty}\left(i\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it} - i\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right)S(t)dt$$

$$+O\left(\int_{\gamma_0}^{\infty}\left|i\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it} - i\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right|t^{-1}dt\right)+O(\log^2 T)$$

by lemma 3.5 and lemma 3.6, above formulas $\ll \log^2 T$.

By lemma 2.1 and lemma 3.2, we have

$$A_2 = \sum_{\gamma_0<\gamma<+\infty}\Gamma\left(\frac{1}{2}-i\gamma\right)(a+ib)^{-\frac{1}{2}+i\gamma} \ll \sum_{\gamma_0<\gamma<+\infty} e^{-\pi\gamma+h\gamma} \ll 1$$

This completes the proof of Lemma 3.7

Lemma 3.8. Assume that RH, if $T \geq 2$, then

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-\frac{n}{T}} = T + O\left(T^{\frac{1}{2}}\log^2 T\right)$$

Proof. By lemma 2.4, we have

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-\frac{n}{T}} = \int_2^{\infty} e^{-\frac{x}{T}}\,d\,\psi(x) = \int_2^{\infty} e^{-\frac{x}{T}}\,d\left(x+R(x)\right)$$

$$= \int_2^{\infty} e^{-\frac{x}{T}}\,dx + \frac{1}{T}\int_2^{\infty} e^{-\frac{x}{T}}R(x)\,dx + O(1)$$

$$= Te^{-\frac{2}{T}} + O\left(\frac{1}{T}\int_{2}^{\infty} x^{\frac{1}{2}}\log^2 x\ e^{-\frac{x}{T}}\,dx+\right) + O(1)$$

$$= T + O\left(T^{\frac{1}{2}}\int_{0}^{\infty} x^{\frac{1}{2}}\left(\log x + \log T\right)^2\ e^{-x}\,dx\right) + O(1) = T + O\left(T^{\frac{1}{2}}\log^2 T\right)$$

This completes the proof of Lemma 3.8.

4. Conclusion

when $a = \frac{1}{T}$, $T \geq 50$, $b = 2\pi$, n is the positive integer, by lemma 2.1, we have

$$\frac{1}{2\pi i}\int_{(2)} \Gamma(s)\left(a+ib\right)^{-s} n^{-s} ds = e^{-an-ibn} = e^{-\frac{n}{T}}$$

By lemma 2.2, we have

$$-\sum_{n=2}^{\infty} \Lambda(n)e^{-\frac{n}{T}} = \frac{1}{2\pi i}\int_{(2)} \Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}ds$$

by lemma 2.2 and RH, the above formula

$$= -(a+ib)^{-1} + \sum_{-\infty<\gamma<+\infty} \Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma} + \frac{\zeta'}{\zeta}(0)$$

$$+\frac{1}{2\pi i}\int_{(-\frac{3}{4})} \Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}ds$$

by lemma 3.3 and lemma 3.7, the above formula $\ll \log^2 T$.

By lemma 3.8, we get a contradiction, therefore the RH is incorrect.

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