A PROOF OF THE FOUR COLOR THEOREM BY INDUCTION

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Abstract. We choose *one of four colors as a temporary color* for all regions that have not been colored, and made *the initial conditions* corresponding to The Four Color Theorem. If these conditions hold for any n regions figure, then they will hold for n + 1 regions figure- formed n regions figure by adding next region. By induction, step by step we have proved The Four Color Theorem successfully on paper.

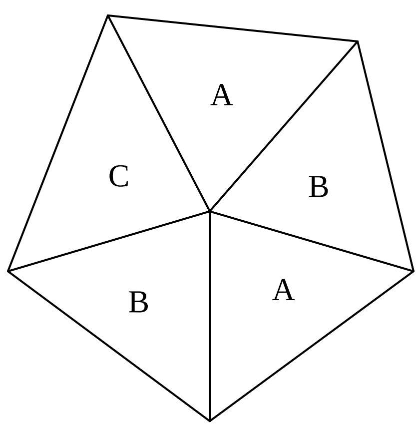
Introduce. The Four Color Theorem has been known since 1852[1,2], and has just proved with the help of computer. The format proof [3] was achieved by Kenneth Appel and Wolfgang Haken and was published in 1976. It was the first major theorem to be proved using a computer.

The Four Color Theorem. *The regions of any simple planar map can be colored with only four colors, in such away that any two adjacent regions have different colors.*

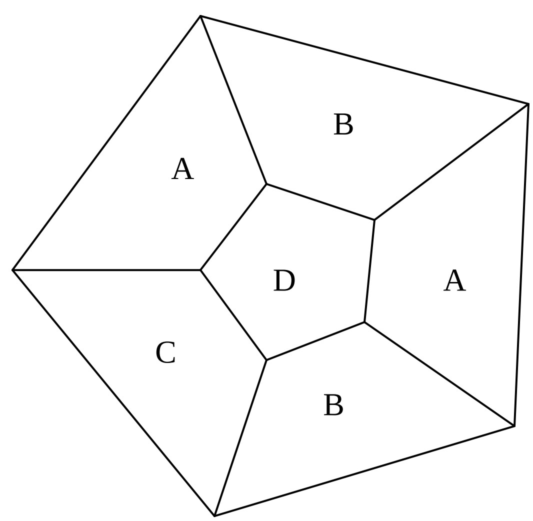
Two regions are called adjacent regions if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.

Proof. Let A, B, C, D be the four colors to color a simple planar map in such a way that any adjacent regions have different colors.

Generally, for any simple planar map, we can take one point and its shared regions or take one region and its adjacent closed regions. The first figures are follows or similar. (f.1 a, f.1 b).



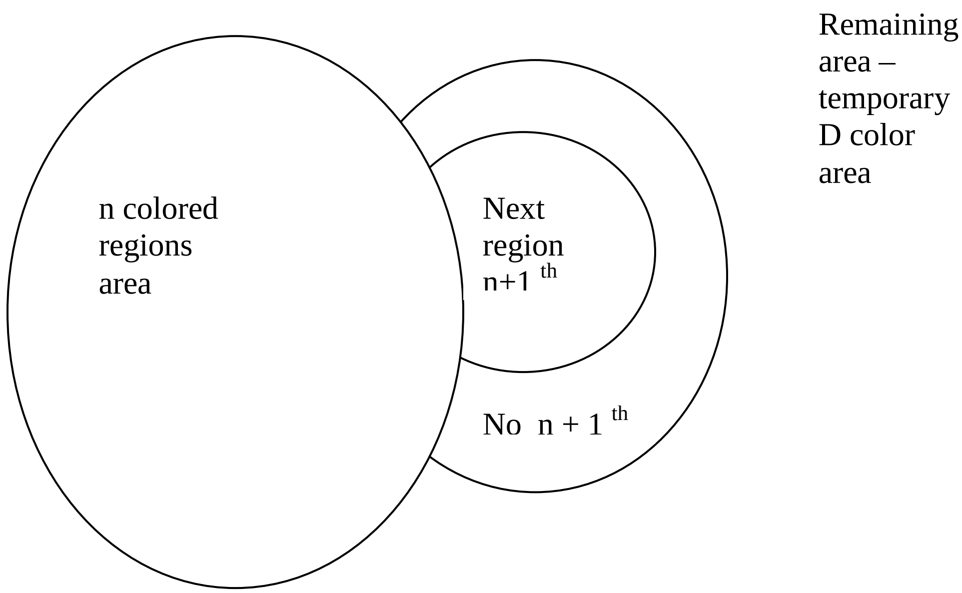
f.1a



f.1b

Note that the first figures with any ring –size can be used 3 colors for f.1a or similar, 4 colors for f.1b or similar maximum. Both of them can only be used 3 colors for ring – regions.

For coloring a simple planar map, we have to divide the map into two areas. The first contains n colored regions, the second is the remaining. We choose D color as a temporary color for the remaining area. The next region (n+1th region), which we will color in order, has a common boundary with both areas, and the common boundary with the colored area must be uninterrupted as following figure (f.2).



f.2

A. By following algorithm:

Clearly, the first figures as f.1 a, f.1 b or similar are true. Assume it is true for any n regions-figure, we prove that, it must be true for n +1 regions figure. That means for any next region (n+1th region), n+1 regions figure- formed n colored regions by adding n+1th region can be used four colors A, B, C, and D maximum, in which three colors A, B, and C for ring-regions including the n+1th region (assign the label N. r).

Let X_1, X_2, X_3, X_i., X_k be the regions of colored area, taken from k regions adjacent to N. r

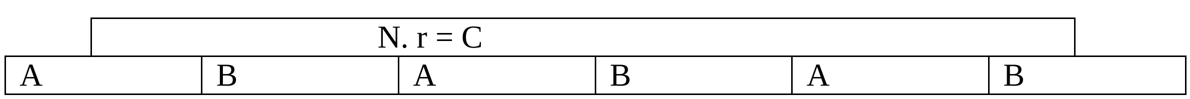
Since three colors A, B, C are the same role, assume that X_1 = A or B color, use C color for N. r, so we must change color so that C color will not exist in the regions from X_1 to X_k .

The initial conditions can be written as follows:

- *Any adjacent regions have different colors.*
- *There is no C color in the regions from X_1 to X_k, and colors of X_1 and X_k are different from both C, D (for X_1 and X_k can be adjacent to D area).*
- *There is no D color from the regions (remaining ring regions) that adjacent to D area.*

Begin by considering regions from X_1 to X_k.

If k regions are alternately colored A and B colors, there is no problem, use C color for N. r. (f.3)



f.3

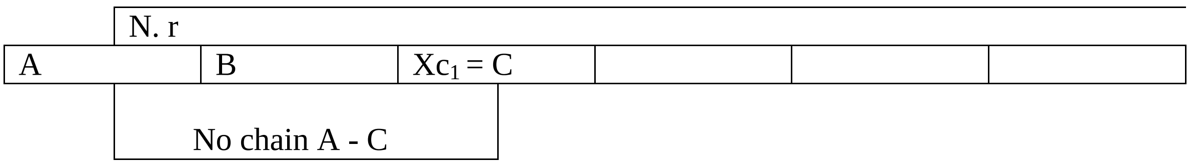
If there are three A, B, C colors in the regions from X_1 to X_k, then do the following steps.

Step1:

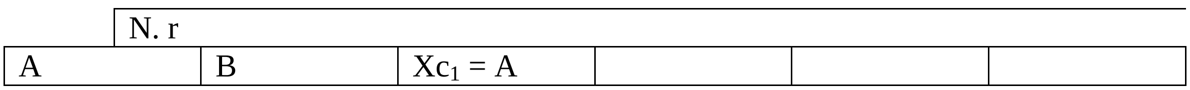
Begin from X_1, take the first C color region (Xc_1= C), and consider the regions from X_1 to Xc_1.

1 a. If there is no chain X_i - Xc_1 = A - C (f.4a), change Xc_1 = C to A, and then interchange the color of the C/A regions in the chain joining Xc_1 (f.4b). Since X_i = A is not in the chain, it remains A color, and there is now no C color region adjacent to Nr from X_1 to Xc_1. If there is no chain X_i - Xc_1 = B - C, change Xc_1 = C to B, and then interchange the color of the C/B regions in the chain joining Xc_1. Since X_i = B is not in the chain, it remains B color, and there is now no C color region adjacent to N. r from X_1 to Xc_1. *Leave blank for the regions after Xc_1, they will be proceeded naturally*.

Chain X_i - Xc_i = A - C, adjacent regions from X_i to Xc_i are alternately colored A and C colors, two adjacent regions are also considered as a chain.

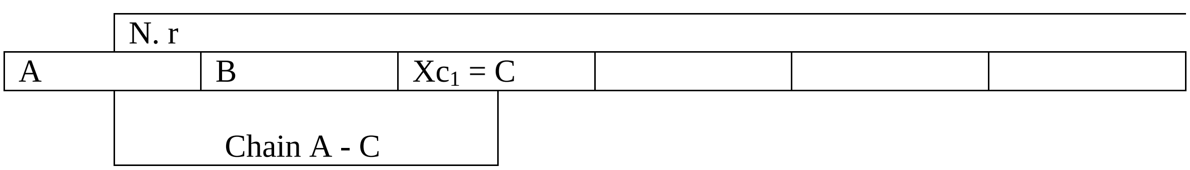


f.4a

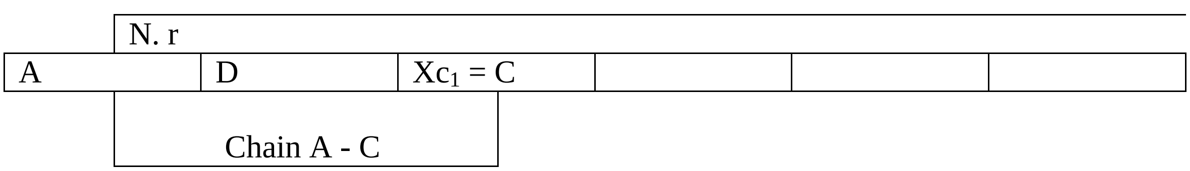


f.4b

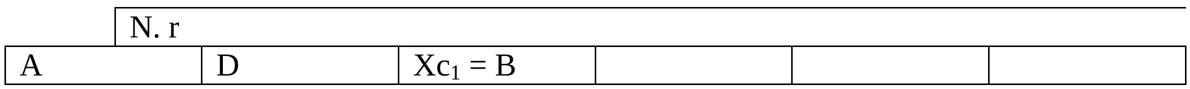
1 b. If there are the chains X_i - Xc_1 = A - C, the chains X_i - Xc_1 = B - C before Xc_1, assume that the first chain of chains X_i- Xc_1= A - C is before the first chain of chains X_i- Xc_1 = B – C (f.5a),(f6.a), take the chain A – C on the basic, change X_i = B to D, and then interchange the color of the B/D regions in the chain joining X_i= B (f.5b), (f6.b). Since there is the chain X_i - Xc_1 = A – C , interchange B/ D colors inside chain X_i- Xc_1 = A - C has no affect to outside chain A-C, outside that it was (By planarity, chains cannot cross). Change Xc_1 = C to B, and then interchange the color of the C/B regions in the chain joining Xc_1, and there is now no C color region adjacent to N. r from X_1 to Xc_1 (f.5c), (f6.c).



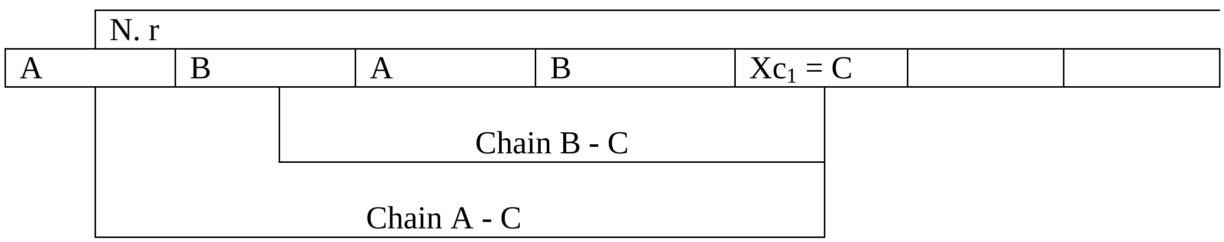
f.5a



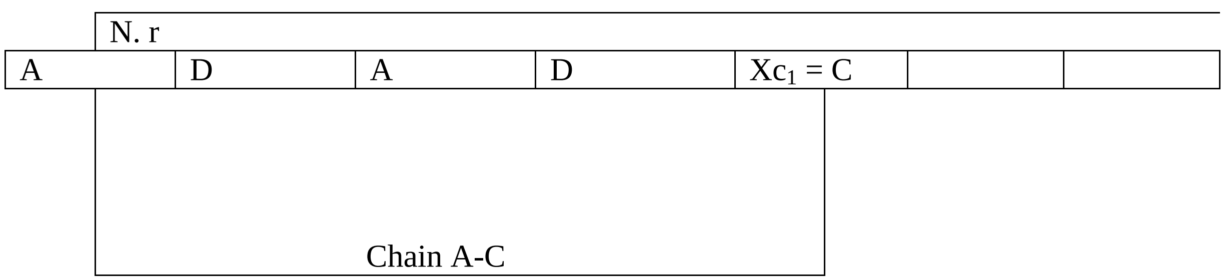
f.5b



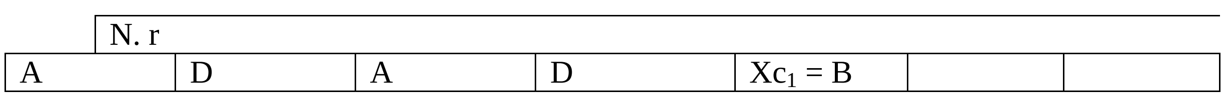
f.5c



f.6a



f.6b



f.6c

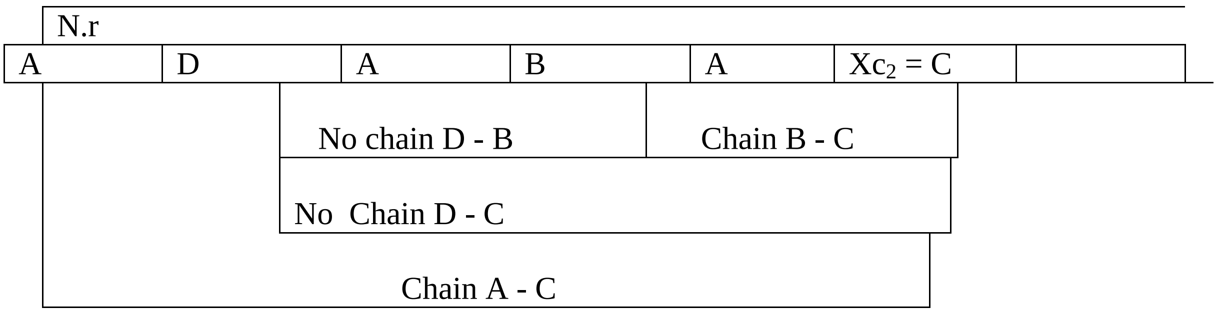
After step1, if there is no more C color region from Xc_1 to $X_{k,}$ then finish by coloring C color for
N. r.

Step 2:

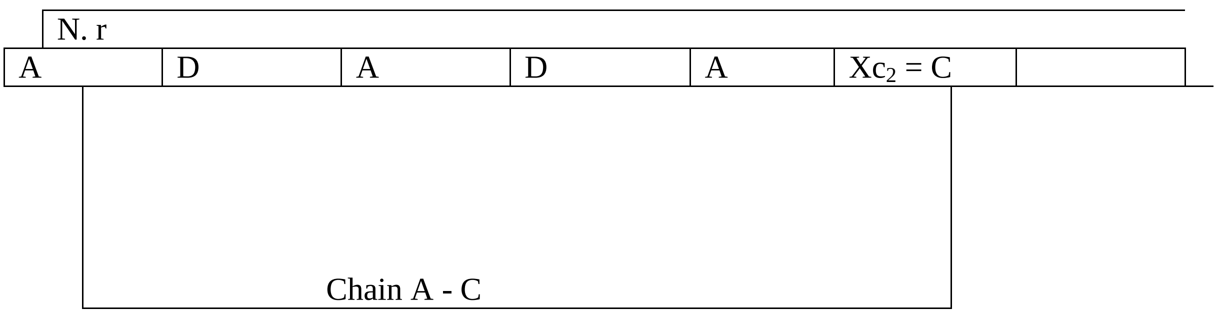
After step1, if there is a C color region left (Xc_2) from Xc_1 to X_k, then consider the regions from X_1 to Xc_2

2a. Repeat step 1 for the following cases:

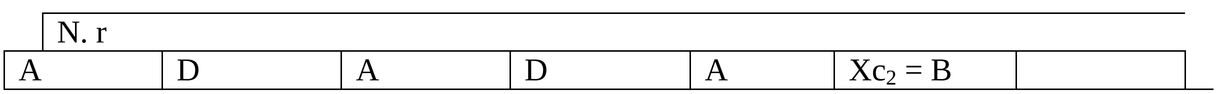
- There is no D color region before Xc_2.
- There are the D color regions, but no chain X_i- Xc_2 = D - C and no chain X_i - X_j = B - D before Xc_2 (f.7a; f.7b; f.7c).



f.7a

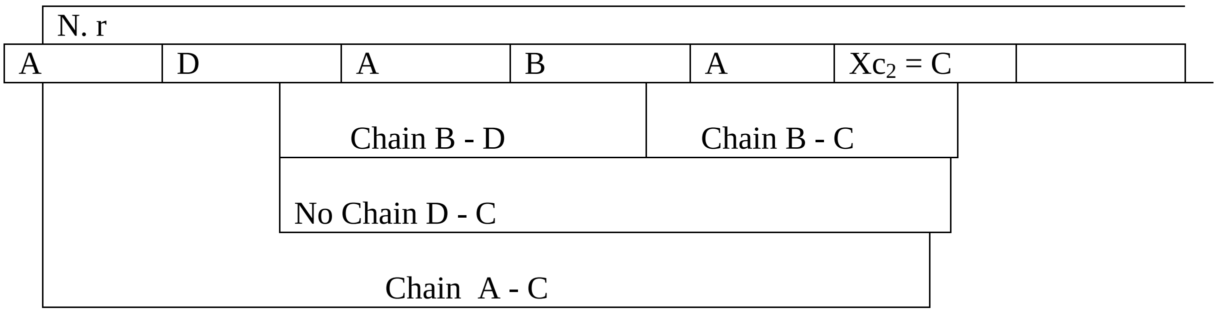


f.7b

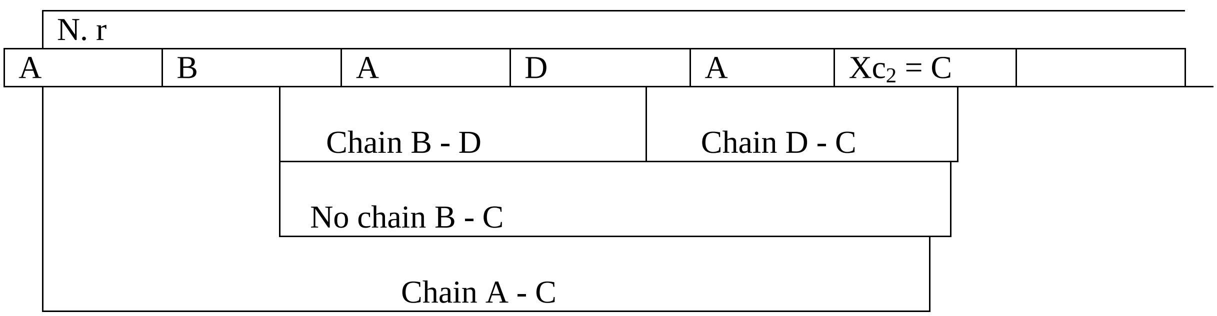


f.7c

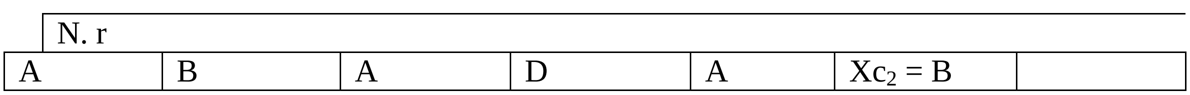
2b. If there are the chains A – C, B – C , no chain X_i- Xc_2 = D - C, and there is a chain X_i – X_j = B - D, (f.8a), a chain A – C is before, take the chain A – C on the basic, then interchange all B/D colors inside chain A-C, the chain $X_i(X_4)$ - Xc_2 = B - C become chain $X_i(X_4)$ - Xc_2 = D - C, since there was no chain $X_i(X_2)$ - Xc_2 = D - C, so after interchange all D/B colors, there will be also no chain $X_i(X_2)$ - Xc_2 = B - C(f.8b). Change Xc_2 = C to B, and then interchange the color of the C/B regions in the chain joining Xc_2. Since X_2= B is not in the chain, it remains B color, and there is now no C color region adjacent to N. r from X_1 to Xc_2 (f.8c).



f.8a



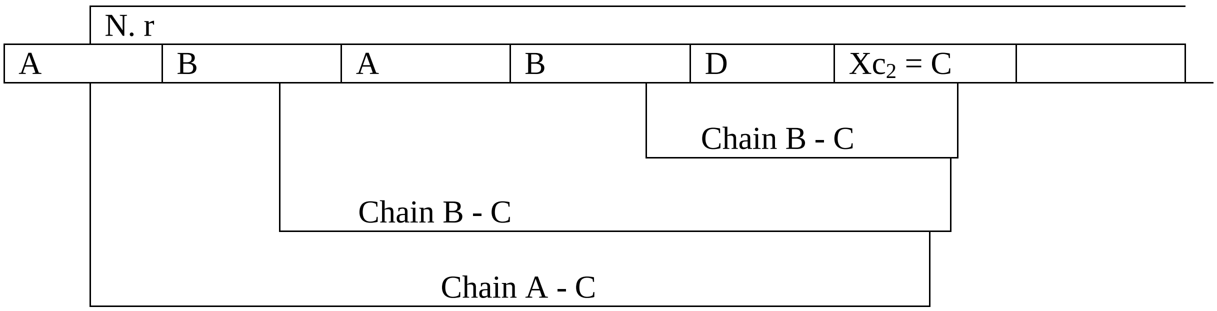
f.8b



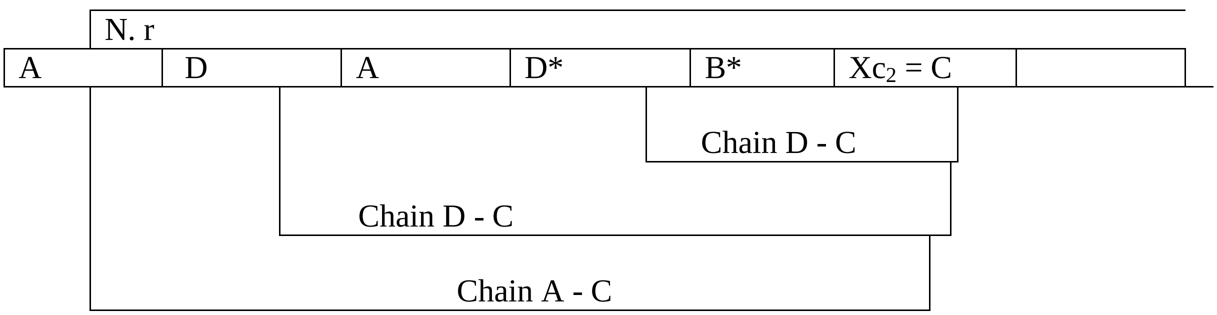
f.8c

2c. If there are the chains X_i - Xc_2 = A - C, B - C, D - C before Xc_2, we apply the following rules:

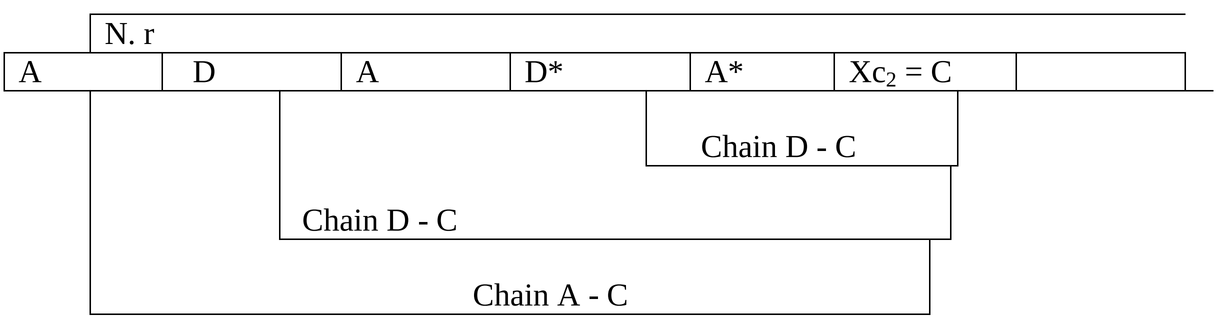
Assume that the first chain of chains A - C is before the first chain of chains B-C, the first chain of chains B – C is before the first chain of chain D - C(f.9a), take the chain A-C on the basic, then interchange all B/D colors inside chain A-C (for the regions between two chains A - C if after chain D - C is a chain A-C left), after interchange, the chains X_i - Xc_2 = B - C become the chains X_i- Xc_2 = D - C, and the chains X_i - Xc_2 = D - C become the chains X_i - Xc_2 = B - C(f.9b). Then take the chain X_i - Xc_2 = D*- C (it was B - C before) which is nearest to the chain B^* - C (It was D - C before) on the basic, interchange all B/A colors inside chain D*- C (for the regions between two chains D – C if after chains A - C is a chain D - C left), the chains X_i - Xc_2 = B* - C become chains X_i - Xc_2 = A*- C(f.9c), then continue considering the regions from this chain A* - C to Xc_2, if there is no more chain X_i – Xc_2 = B – C, change Xc_2 = C to B , and then interchange the color of the C/B regions in the chain joining Xc_2, and there is now no C color region adjacent to N. r from X_1 to Xc_2 (f9d). If there is a chain X_i - Xc_2 = B - C left, take the nearest chain before this chain B - C on the basic, and continue as above until there is no more chain X_i- Xc_2 = B - C before Xc_2, change Xc_2 = C to B , and then interchange the color of the C/B regions in the chain joining Xc_2.



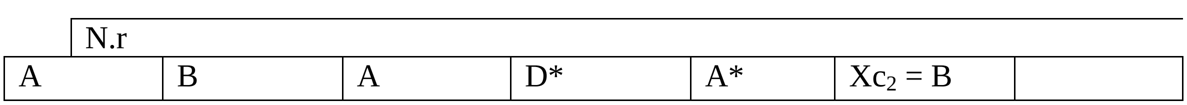
f.9a



f.9b



f.9c

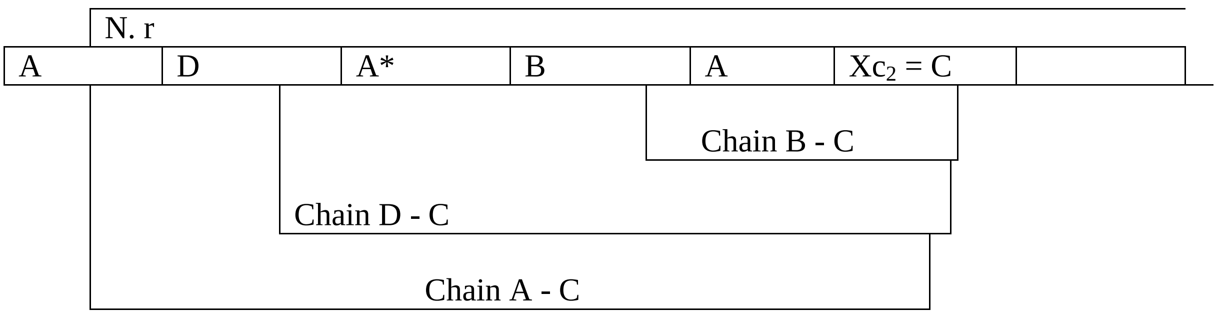


f.9d

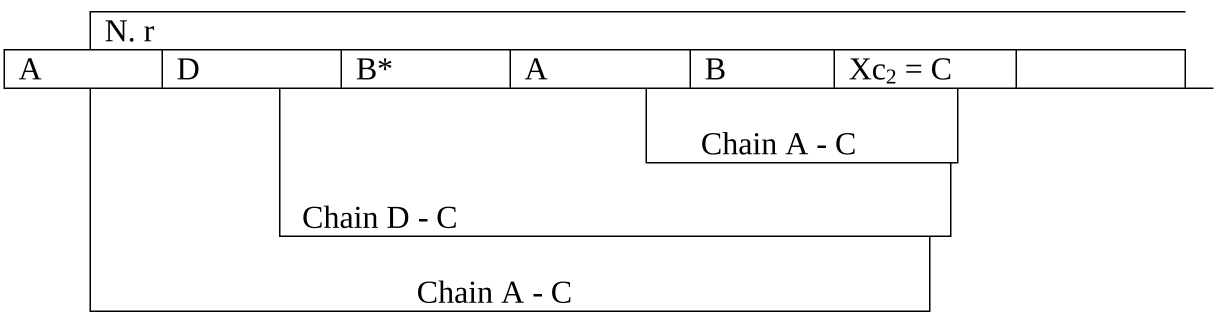
If the first chain of chains A – C is before the first chain of chains D – C, and the first chain D – C is before the first chain of chains B – C (f.10a), then pass over chain A - C, take the chain D - C which is nearest to the first chain of chains B - C on the basic and repeat as above (f.10b,f.10c, f.10d).

If the first chain of chains D – C is before the first chain of chains A – C, and the first chain A – C is before the first chain of chains B – C, then pass over chain D – C, take the chain A - C which is nearest to the chain B - C on the basic and repeat as above.

If the chains are in order B - C, D - C, A - C or D - C, B - C, A - C, do by the same way, since A and B are the same role (only change roles A and B each other).

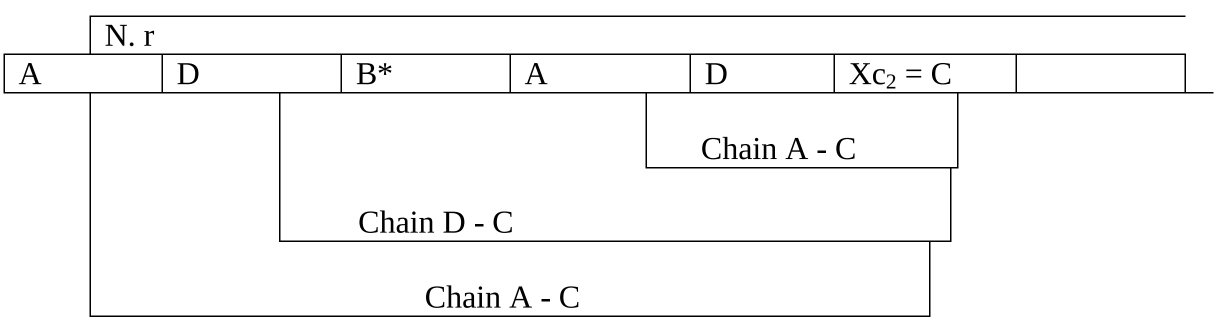


f.10a



f.10b

Since there was no chain A- C, so there is no chain B*- C after interchange all A/B colors inside chain D - C.(if there is a chain A*- C, pass over chain D – C, begin by taking chain A*- C on the basic).*



f.10c

N. r						
A	D	B*	A	D	Xc_2 = B	

f.10d

After steps 2, if there is no more C color region from X_1 to X_k, finish by coloring C color for N.r,

Step3:

If there is a C color region (Xc_3) left from Xc_2 to X_k, then consider the regions from X_1 to Xc_3, repeat as step 2 until there is no C color region before X_k, and finish by coloring C color for N. r.

So, it is true for n+1 regions figure, problem was proved

Note that, when we change the C color of Xc_i to B(A), since there is no chain X_i - Xc_i = B(A)- C respectively before Xc_i, the regions X_i = B(A) color before Xc_i remain their color, so there is no C color region adjacent to Nr from X_1 to Xc_i. they hold from X_1 to Xc_i, a color of other regions can be scramble, but there will be no D color region adjacent to D area because we changed X_i = A(B) to D if only there was its surrounded chain B - C (A - C) respectively, and changed Xc_i only to B(A), not to D. All conditions held.

The cases considered above is general, some specific cases are below, however no problem for all.

- X_1, X_k or both of them no adjacent to D area (f.11), then X_1,X_k can be D color.

N. r				
X_1	X_2			

f.11

- There are the regions from k regions no adjacent each other (X_3 and X_4 adjacent to N. r but no adjacent each other(f.12)), then there can be no chain X_3 - X_4.

N. r					
X_1	X_2	X_3	X_4		

f.12

- N regions figure enclosed by N. r. then use D color for N. r, let n+1 regions figure be as one region (D color region), then continue by using another color as a temporary color for the remaining area.
- There are at most two colors in the regions that adjacent to D area, we can continue as above. Beside, we can use D color for N. r, and take the remaining color as a temporary color for remaining area.

B. Another way of proof:

The initial conditions can be written as follows:

- Any adjacent regions have different colors.

- There is at least one of three A, B, C colors is not in the regions that adjacent to N. r, (this color will be used for N. r) and colors of X_1 and X_k different from D (for X_1, X_k can be adjacent to D area).

- There is no D color from the regions (remaining ring regions) that adjacent to D area.

Let X_1, X_2, X_3, X_i.., X_k regions taken from k regions which adjacent to N. r.

First, consider k=1 to 4:

Since three colors A, B and C are the same role:

k = 1: X_1= A color then B or C color for N. r

k = 2: X_1= A color, X_2=B then C color for N. r

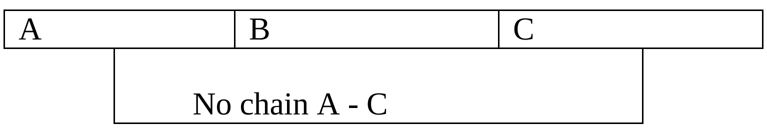
k = 3:

a.X_1 = A color, X_2 = B, X_3 = A then C color for N. r

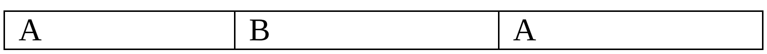
There is no problem for the above cases.

b.X_1 = A, X_2 = B, X_3 = C.

- If there is no chain X_1 - X_3 = A - C (f.13a), change X_3= C to A, and then interchange the color of the C/A regions in the chain joining X_3 = C. Since X_1= A is not in the chain, it remains A color, and there is now no **C** color region, N.r = C (f.13b).

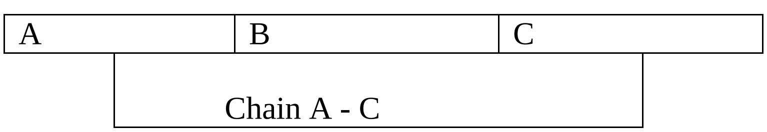


f.13a

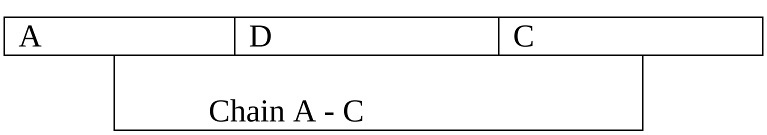


f.13b

- If there is a chain X_1 - X_3 = A - C, a chain X_2 - X_3 =B - C (f.14a), change X_2 = B to D, and then interchange the color of the B/D regions in the chain joining X_2 = B. Since there is the chain X_1 - X_3 =A - C, interchange B and D color inside chain X_1 - X_3 = A - C has no effect to change outside, outside that it was, and there is now no **B** color region, (f14.b), N.r = B.



f.14a



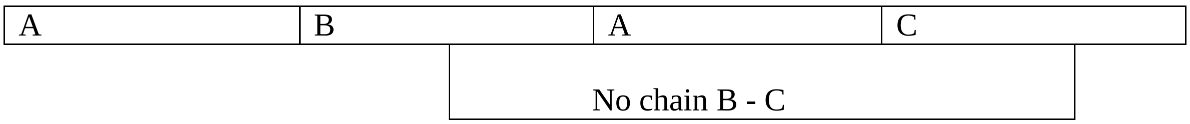
f.14b

k=4:

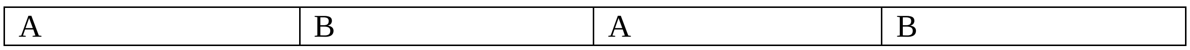
a.$X_1 = A$, $X_2 = B$, $X_3 = A$, $X_4 = B$ then C color for N. r.

b.$X_1 = A$, $X_2 = B$, $X_3 = A$, $X_4 = C$:

If there is no chain X_2 - X_4 = B - C (f.15a), change X_4 = C to B, and then interchange the color of the C/B regions in the chain joining X_4 = C. Since X_2 = B is not in the chain, it remains B color, and there is now no **C** color region, (f.15b), N.r = C.

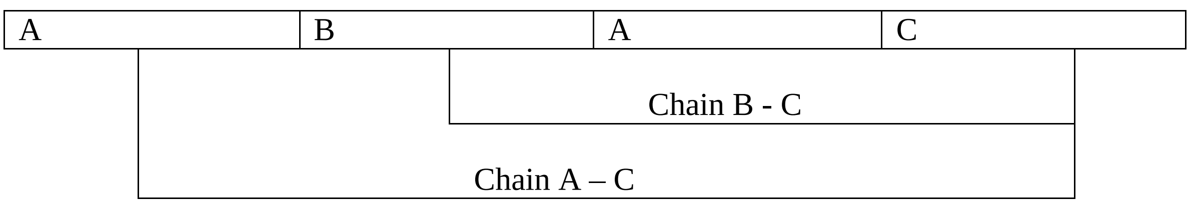


f.15a

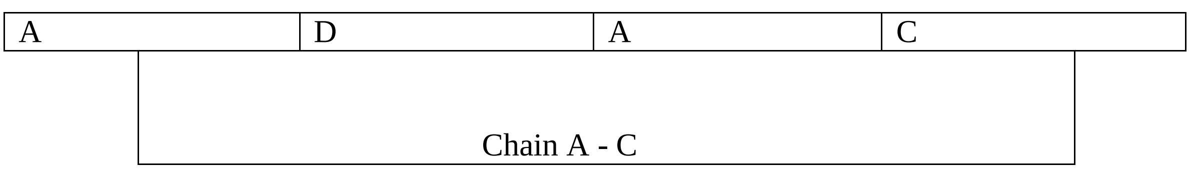


f.15b

If there is a chain X_1 - X_4 = A - C, a chain X_2 - X_4 = B - C (f.16a). Since there is the chain X_1-X_4 = A - C, change X_2 = B to D , and then interchange the color of the B/D regions in the chain joining X_2 = B, and there is now no **B** color region (f.16b) N.r = B,.

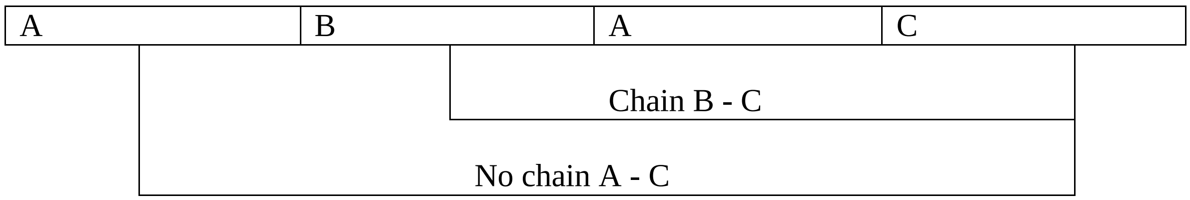


f.16a

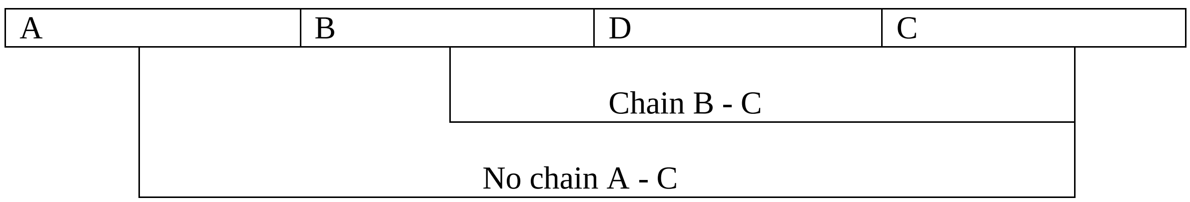


f.16b

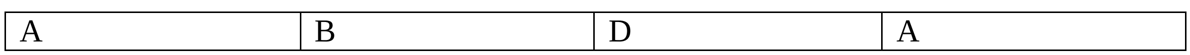
If there is a chain X_2 - X_4 = B - C, no chain X_1 - X_4 = A - C (f.17a). Since there is the chain X_2-X_4 = B - C, change X_3 = A to D, and then interchange the color of the A/D regions in the chain joining X_3 = A (f.17b), change X_4 = C to A, and then interchange the color of the C/A regions in the chain joining X_4 = C. Since X_1 = A is not in the chain, it remains A color, and there is now no **C** color region (f.17c), N.r = C.



f.17a



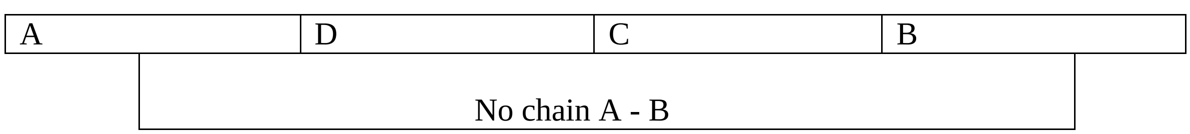
f.17b



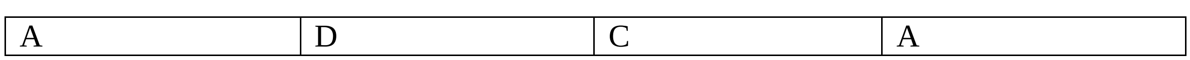
f.17c

c. $X_1 = A$, $X_2 = D$, $X_3 = C$, $X_4 = A$ then B color for N. r.
d. $X_1 = A$, $X_2 = D$, $X_3 = C$, $X_4 = B$:

If there is no chain X_1 - X_4 = A - B (f.18a), change $X_4 = B$ to A, and then interchange the color of the B/A regions in the chain joining $X_4 = B$. Since $X_1 = A$ is not in the chain, it remains A color, and there is now no **B** color region (f.18b), N.r = B.

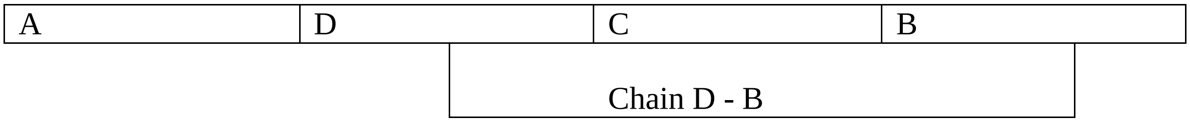


f.18a

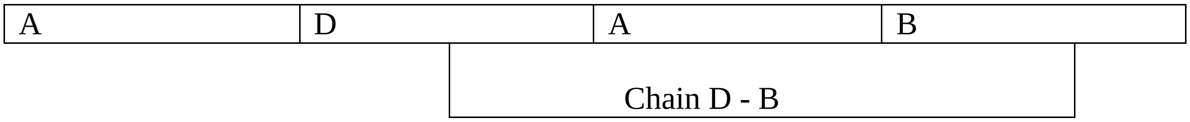


f.18b

If there is a chain X_2 - X_4 = D - B (f.19a), since there is the chain X_2 - X_4 = D - B, change X_3 = C to A , and then interchange the color of the C/A regions in the chain joining $X_3 = C$, and there is now no **C** color region (f.19b), N.r = C.

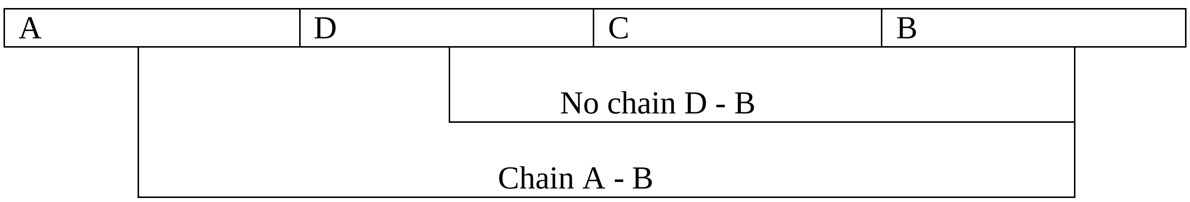


f.19a

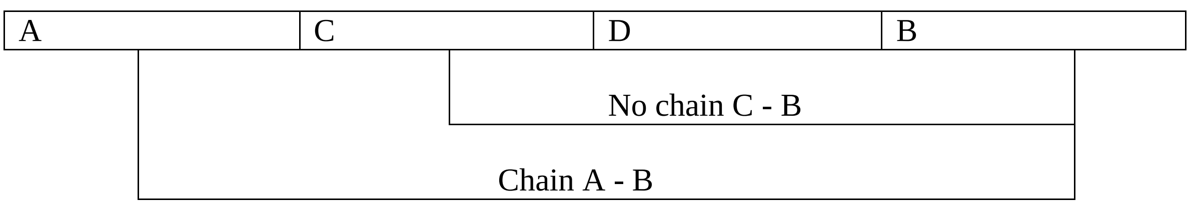


f.19b

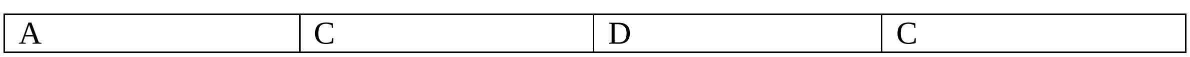
If there is a chain X_1 - X_4 = A - B, no chain X_2 - X_4 = D - B (f.20a), since there is the chain X_1 - X_4 = A - B, interchange all C/D colors inside chain A-B, so X_2 = D become C color, since there is no chain X_2 - X_4 = D - B, interchange all D/C colors does not lead to form new chain X_2 - X_4 = C - B(f.20b), change X_4 = B to C, and then interchange the color of the B/C regions in the chain joining X_4 = B. Since X_2=C is not in the chain, it remains C color, and there is now no **B** color region (f.20c), N.r = B.



f.20a



f.20b



f.20c

So it is true for k=1, k=2, k=3, k=4, assume that if it is true for k regions, we prove that it must be true for k +1 regions.

Since it is true for k regions, so there is at least one of three colors A, B, and C is not in the k regions. Assume that, this color is C color, then X_k = A or B colors.

If X_{k+1} = B or A colors, there is no problem, N.r = C.

If X_{k+1} = C color, then consider regions from X_1 to X_{k+1}, let X_{k+1} be as Xc and do by the same way as step 2 (first proof), given result is also true for k+1 regions, so it holds for any k.
Different points between two proofs above are:
The first: we choose only C color for N.r (X_1 = A)
The second: depending on the cases, we can use B or C color for N.r (X_1 = A)

So, we can conclude that there are many ways to color a simple planar map in such a way that any adjacent regions have different colors.

References:

[1]. Thomas, Robin (1995), The Four - Color Theorem.
[2]. ’O Conner; Robertson (1996), The Four- Color Theorem, Mac Tutor archive.
[3]. Gonthier, Georges (2008),”Formal proof- The Four Color Theorem”,(Notices of the American Mathematical Society).

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