$$*\mathbf{R}* = \kappa\mathbf{T}$$

Physicalization of Curvature

Tobias Wowereit

The Einstein equations are completed.
The electromagnetic nature of energetic phenomena is derived.
Alternative units are introduced.

Contents

1. I,II,III

1.1. I

The Einstein equations of General Relativity

$$\begin{aligned}\kappa T_{ik} &= R_{ik} - \frac{1}{2} g_{ik} R \\ &= [*R*]_{ik} \\ &= g^{jl}[*R*]_{ijkl} \\ &= g^{jl}\frac{1}{4}\eta_{ij}{}^{ab}\eta_{kl}{}^{cd}R_{abcd}\end{aligned}$$

can be written as

$$g^{jl}\frac{[*R*]_{ijkl}}{\kappa} = T_{ik}$$

Thus T_{ik} is the trace of $\frac{[*R*]_{ijkl}}{\kappa}$ and if T_{ik} is an energy momentum tensor so is $\frac{[*R*]_{ijkl}}{\kappa}$ because the metric bears no physical or geometrical unit. Emphasizing this by setting

$$\frac{[*R*]_{ijkl}}{\kappa} =: T_{ijkl}$$

gives

$$g^{jl}T_{ijkl} = T_{ik}$$

More details in part I.

1.2. II

The energy momentum tensor of the electromagnetic field in connection with the Einstein equations reads

$$\begin{aligned}R_{ik} - \frac{1}{4}Rg_{ik} &= \kappa[T_{ik} - \frac{1}{4}Tg_{ik}] \\ &= \kappa[F_i{}^a F_{ka} - \frac{1}{4}F_{ab}F^{ab}]\end{aligned}$$

This expression stems from

$$\begin{aligned}[*R*]_{ik} &= \kappa T_{ik} \\ &= \kappa F_i{}^a F_{ka}\end{aligned}$$

which in turn stems from

$$\begin{aligned}[*R*]_{ijkl} &= \kappa T_{ijkl} \\ &= \kappa F_{ij}F_{kl}\end{aligned}$$

Details in part II.

1.3. III

Usually T_{ik} is called 'energy-momentum tensor' and bears the unit

$$[T_{ik}] = \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \cdot \frac{1}{\mathrm{m}^3}$$

Regarding T_{ik} as a tensorfield defined on spacetime, a canonical unit for T_{ik} should contain a factor $\frac{1}{\mathrm{m}^4}$. This means

$$[T_{ik}] = \frac{\mathrm{X}}{\mathrm{m}^4}$$

The unit X has electromagnetic 'roots' because

$$\begin{aligned} \mathrm{X} &= K_{[1]}^2 \mathrm{esu}^2 \qquad\qquad K_{[1]} \in \mathbb{R} \\ &\approx 4,817 \cdot 10^{51} \cdot \mathrm{esu}^2 \end{aligned}$$

'esu' being the 'electrostatic unit of charge'. Details in part III.

2. π, ω, τ, ρ, $*$, ∇, C, R

Further literature

$\nabla, g, \mathbf{C}, \mathbf{R}$	[13], pp. 23, 30 – 40
	[11], III, §1 – 5
π	[2], pp. 291 – 298
	[3]
τ	[2], pp. 298 – 299
	[9]
ω	[1], pp. 25 – 28
	[4], pp. 374 – 379, 552 – 568
$\omega \circ \alpha$	[10], pp. 120
	[11], I, §7

Let M denote a differentiable manifold, $\dim[M] = n$, and let M^L denote a Lorentzian manifold, $\dim[M^L] = 4$, with metric signature $1, 1, 1, -1$.

2.1. π

Let M be given. The *Skew* or *Anti Symmetrizer* $\alpha^{a_1 \dots a_l}_{i_1 \dots i_l}$ and the *Symmetrizer* $\sigma^{a_1 \dots a_l}_{i_1 \dots i_l}$ are defined by

$$\begin{aligned}
\alpha^{a_1 \dots a_l}_{i_1 \dots i_l} &:= \sum_p \mathrm{sign}[p] \delta^{a_{p[1]}}_{i_1} \dots \delta^{a_{p[l]}}_{i_l} \\
\sigma^{a_1 \dots a_l}_{i_1 \dots i_l} &:= \sum_p \delta^{a_{p[1]}}_{i_1} \dots \delta^{a_{p[l]}}_{i_l}
\end{aligned}$$

$p[1] \dots p[l]$ denotes an array of $1, \dots, l$ which is obtained from $1, \dots, l$ by permutations. The sum is carried out over all possible arrays of $1, \dots, l$. $\mathrm{sign}[p]$, the sign of an array, is positive if $p[1] \dots p[l]$ can be obtained from $1, 2, \dots, l-1, l$ by an even number of permutations, otherwise negative. So the *Permutation Operator* π can be defined as

$$\begin{aligned}
\pi^{ab}_{ij} X_{ab} &:= \frac{1}{2}\sigma^{ab}_{ij} X_{ab} + \frac{1}{2}\alpha^{ab}_{ij} X_{ab} \\
&=: X^{\boxed{i}\boxed{j}}_{ij} + X^{\begin{smallmatrix}\boxed{i}\\ \boxed{j}\end{smallmatrix}}_{ij} \\
&=: X^{\square\square}_{ij} + X^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_{ij} \\
&=: [\pi^{\square\square}]^{ab}_{ij} X_{ab} + [\pi^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}]^{ab}_{ij} X_{ab}
\end{aligned}$$

$\pi^{ab}_{ij} X_{ab}$ leads to

$$\begin{aligned}
\pi^{abc}_{ijk} X_{abc} &= X^{\boxed{i}\boxed{j}\boxed{k}}_{ijk} + X^{\begin{smallmatrix}\boxed{i}\boxed{j}\\ \boxed{k}\end{smallmatrix}}_{ijk} + X^{\begin{smallmatrix}\boxed{i}\boxed{k}\\ \boxed{j}\end{smallmatrix}}_{ijk} + X^{\begin{smallmatrix}\boxed{i}\\ \boxed{j}\\ \boxed{k}\end{smallmatrix}}_{ijk} \\
&= X^{\square\square\square}_{ijk} + X^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}_{ijk} + X^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}_{ijk} \\
&=: [\pi^{\square\square\square}]^{abc}_{ijk} X_{abc} + [\pi^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}]^{abc}_{ijk} X_{abc} + [\pi^{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}]^{abc}_{ijk} X_{abc}
\end{aligned}$$

which leads to

$$\pi^{abcd}_{ijkl} X_{abcd} = X^{\boxed{i}\boxed{j}\boxed{k}\boxed{l}}_{ijkl} + X^{\begin{smallmatrix}\boxed{i}\boxed{j}\boxed{k}\\ \boxed{l}\end{smallmatrix}}_{ijkl} + X^{\begin{smallmatrix}\boxed{i}\boxed{j}\boxed{l}\\ \boxed{k}\end{smallmatrix}}_{ijkl} + X^{\begin{smallmatrix}\boxed{i}\boxed{j}\\ \boxed{k}\boxed{l}\end{smallmatrix}}_{ijkl} + X^{\begin{smallmatrix}\boxed{i}\boxed{j}\\ \boxed{k}\\ \boxed{l}\end{smallmatrix}}_{ijkl}$$

$$
\begin{aligned}
&+X_{ijk}^{\ytableaushort{ikl,j}} + X_{ijkl}^{\ytableaushort{ik,jl}} + X_{ijkl}^{\ytableaushort{ik,j,l}} + X_{ijk}^{\ytableaushort{il,j,k}} + X_{ijkl}^{\ytableaushort{i,j,k,l}} \\
= \quad & X_{ijkl}^{\ydiagram{4}} + X_{ijkl}^{\ydiagram{3,1}} + X_{ijkl}^{\ydiagram{2,2}} + X_{ijkl}^{\ydiagram{2,1,1}} + X_{ijkl}^{\ydiagram{1,1,1,1}} \\
=: \quad & [\pi^{\ydiagram{4}}]_{ij}^{ab} X_{abcd} + [\pi^{\ydiagram{3,1}}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\ydiagram{2,2}}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\ydiagram{2,1,1}}]_{ijkl}^{abcd} X_{abcd} + [\pi^{\ydiagram{1,1,1,1}}]_{ijkl}^{abcd} X_{abcd}
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\pi_{ijklm}^{abcde} X_{abcde} \quad = \quad & X_{ijklm}^{\ytableaushort{ijklm}} + X_{ijklm}^{\ytableaushort{ijkl,m}} \\
&+X_{ijklm}^{\ytableaushort{ijkm,l}} + X_{ijklm}^{\ytableaushort{ijk,lm}} + X_{ijklm}^{\ytableaushort{ijk,l,m}} + X_{ijklm}^{\ytableaushort{ijlm,k}} + X_{ijklm}^{\ytableaushort{ijl,km}} + X_{ijklm}^{\ytableaushort{ijl,k,m}} \\
&+X_{ijklm}^{\ytableaushort{ijm,kl}} + X_{ijklm}^{\ytableaushort{ij,kl,m}} + X_{ijklm}^{\ytableaushort{ijm,k,l}} + X_{ijklm}^{\ytableaushort{ij,km,l}} + X_{ijklm}^{\ytableaushort{ij,k,l,m}} \\
&+X_{ijkm}^{\ytableaushort{iklm,j}} + X_{ijkm}^{\ytableaushort{ikl,jm}} + X_{ijkm}^{\ytableaushort{ikl,j,m}} + X_{ijklm}^{\ytableaushort{ikm,jl}} + X_{ijklm}^{\ytableaushort{ik,jl,m}} \\
&+X_{ijklm}^{\ytableaushort{ikm,j,l}} + X_{ijklm}^{\ytableaushort{ik,jm,l}} + X_{ijklm}^{\ytableaushort{ik,j,l,m}} + X_{ijklm}^{\ytableaushort{ilm,j,k}} + X_{ijklm}^{\ytableaushort{il,jm,k}} + X_{ijklm}^{\ytableaushort{il,j,k,m}} \\
&+X_{ijklm}^{\ytableaushort{im,j,k,l}} + X_{ijklm}^{\ytableaushort{i,j,k,l,m}} \\
= \quad & X_{ijklm}^{\ydiagram{5}} + X_{ijkm}^{\ydiagram{4,1}} + X_{ijklm}^{\ydiagram{3,2}} + X_{ijklm}^{\ydiagram{3,1,1}} + X_{ijklm}^{\ydiagram{2,2,1}} + X_{ijklm}^{\ydiagram{2,1,1,1}} + X_{ijklm}^{\ydiagram{1,1,1,1,1}} \\
=: \quad & [\pi^{\ydiagram{5}}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\ydiagram{4,1}}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\ydiagram{3,2}}]_{ijklm}^{abcde} X_{abcde} \\
&+[\pi^{\ydiagram{3,1,1}}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\ydiagram{2,2,1}}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\ydiagram{2,1,1,1}}]_{ijklm}^{abcde} X_{abcde} + [\pi^{\ydiagram{1,1,1,1,1}}]_{ijklm}^{abcde} X_{abcde}
\end{aligned}
$$

The generalization of the factor $\frac{1}{2}$ in $\pi_{ij}^{ab} X_{ab}$ reads

$$
\frac{A}{B!} \tag{2.1}
$$

where A denotes the number of different configurations of a box-array by indices, and B denotes the order of the tensor. Hence

$$
\begin{aligned}
\pi_{ij}^{ab} \quad &= \quad \frac{1}{2}\sigma_{ij}^{ab} + \frac{1}{2}\alpha_{ij}^{ab} \\
&= \quad [\pi^{\ydiagram{2}}]_{ij}^{ab} + [\pi^{\ydiagram{1,1}}]_{ij}^{ab} \\
\pi_{ijk}^{abc} \quad &= \quad \frac{1}{6}\sigma_{ijk}^{abc} + \frac{1}{3}[\alpha_{ik}^{pc}\sigma_{pj}^{ab} + \alpha_{ij}^{pb}\sigma_{pk}^{ac}] + \frac{1}{6}\alpha_{ijk}^{abc} \\
&= \quad [\pi^{\ydiagram{3}}]_{ijk}^{abc} + [\pi^{\ydiagram{2,1}}]_{ijk}^{abc} + [\pi^{\ydiagram{1,1,1}}]_{ijk}^{abc} \\
\pi_{ijkl}^{abcd} \quad &= \quad \frac{1}{24}\sigma_{ijkl}^{abcd} + \frac{1}{8}[\alpha_{il}^{pd}\sigma_{pjk}^{abc} + \alpha_{ik}^{pc}\sigma_{pjl}^{abd} + \alpha_{ij}^{pb}\sigma_{pkl}^{acd}] + \frac{1}{12}[\alpha_{ik}^{pr}\alpha_{jl}^{qs}\sigma_{pq}^{ab}\sigma_{rs}^{cd} + \alpha_{ij}^{pr}\alpha_{kl}^{qs}\sigma_{pq}^{ac}\sigma_{rs}^{bd}] \\
&\quad +\frac{1}{8}[\alpha_{ikl}^{pcd}\sigma_{pj}^{ab} + \alpha_{ijl}^{pbd}\sigma_{pk}^{ac} + \alpha_{ijk}^{pbc}\sigma_{pl}^{ad}] + \frac{1}{24}\alpha_{ijkl}^{abcd} \\
&= \quad [\pi^{\ydiagram{4}}]_{ij}^{ab} + [\pi^{\ydiagram{3,1}}]_{ijkl}^{abcd} + [\pi^{\ydiagram{2,2}}]_{ijkl}^{abcd} + [\pi^{\ydiagram{2,1,1}}]_{ijkl}^{abcd} + [\pi^{\ydiagram{1,1,1,1}}]_{ijkl}^{abcd}
\end{aligned}
$$

$$
\begin{aligned}
\pi^{abcde}_{ijklm} \quad = \quad & \frac{1}{120}\sigma^{abcde}_{ijklm} + \frac{1}{30}[\alpha^{pe}_{im}\sigma^{abcd}_{pjkl} + \alpha^{pd}_{il}\sigma^{abce}_{pjkm} + \alpha^{pc}_{ik}\sigma^{abde}_{pjlm} + \alpha^{pb}_{ij}\sigma^{acde}_{pklm}] \\
& + \frac{1}{24}[\alpha^{pr}_{il}\alpha^{qs}_{jm}\sigma^{de}_{rs}\sigma^{abc}_{pqk} + \alpha^{pr}_{ik}\alpha^{qs}_{jm}\sigma^{ce}_{rs}\sigma^{abd}_{pql}] \\
& + \frac{1}{24}[\alpha^{pr}_{ik}\alpha^{qs}_{jl}\sigma^{cd}_{rs}\sigma^{abe}_{pqm} + \alpha^{pr}_{ij}\alpha^{qs}_{km}\sigma^{be}_{rs}\sigma^{acd}_{pql} + \alpha^{pr}_{ij}\alpha^{qs}_{kl}\sigma^{bd}_{rs}\sigma^{ace}_{pqm}] \\
& + \frac{1}{20}[\alpha^{pde}_{ilm}\sigma^{abc}_{pjk} + \alpha^{pce}_{ikm}\sigma^{abd}_{pjl} + \alpha^{pcd}_{ikl}\sigma^{abe}_{pjm} + \alpha^{pbe}_{ijm}\sigma^{acd}_{pkl} + \alpha^{pbd}_{ijl}\sigma^{ace}_{pkm} + \alpha^{pbc}_{ijk}\sigma^{ade}_{plm}] \\
& + \frac{1}{24}[\alpha^{pre}_{ikm}\alpha^{qs}_{jl}\sigma^{cd}_{rs}\sigma^{ab}_{pq} + \alpha^{prd}_{ikl}\alpha^{qs}_{jm}\sigma^{ce}_{rs}\sigma^{ab}_{pq} + \alpha^{pre}_{ijm}\alpha^{qs}_{kl}\sigma^{bd}_{rs}\sigma^{ac}_{pq}] \\
& + \frac{1}{24}[\alpha^{prd}_{ijl}\alpha^{qs}_{km}\sigma^{be}_{rs}\sigma^{ac}_{pq} + \alpha^{prc}_{ijk}\alpha^{qs}_{lm}\sigma^{be}_{rs}\sigma^{ad}_{pq}] \\
& + \frac{1}{30}[\alpha^{pcde}_{iklm}\sigma^{ab}_{pj} + \alpha^{pbde}_{ijlm}\sigma^{ac}_{pk} + \alpha^{pbce}_{ijkm}\sigma^{ad}_{pl} + \alpha^{pbcd}_{ijkl}\sigma^{ae}_{pm}] + \frac{1}{120}\alpha^{abcde}_{ijklm} \\
= \quad & [\pi^{(5)}]^{abcde}_{ijklm} + [\pi^{(4,1)}]^{abcde}_{ijklm} + [\pi^{(3,2)}]^{abcde}_{ijklm} + [\pi^{(3,1,1)}]^{abcde}_{ijklm} + [\pi^{(2,2,1)}]^{abcde}_{ijklm} + [\pi^{(2,1,1,1)}]^{abcde}_{ijklm} + [\pi^{(1,1,1,1,1)}]^{abcde}_{ijklm}
\end{aligned}
$$

The number of independent components per term can be obtained from the following procedure. The dimension of the space, on which the tensor is defined, is inserted into the box in the upper left-hand corner of the array of indices. Going on from this, the other boxes are filled. For every step to the right, the number is raised by 1, and for every step down, the number is decreased by 1. The product of all those numbers of an array is finally multiplied by (2.1).

2.2. τ

Let g be a metric on M.

$$
\begin{aligned}
g_{ij} &= g_{ji} \\
\det[g_{ij}] &=: \mathfrak{g} \neq 0
\end{aligned}
$$

With the help of the metric g the *Trace Decomposition Operator* τ and the *Trace Reduction Operator* γ can be defined.

$$
\tau^{a_1\ldots a_l}_{i_1\ldots i_l} X_{a_1\ldots a_l} \quad := \quad [X_{i_1\ldots i_l} - \gamma^{a_1\ldots a_n}_{i_1\ldots i_l} X_{a_1\ldots a_l}] + \gamma^{a_1\ldots a_l}_{i_1\ldots i_l} X_{a_1\ldots a_l}
$$

with

$$
\gamma^{a_1\ldots a_l}_{i_1\ldots i_l} X_{a_1\ldots a_l} \quad := \quad g_{i_1 i_2}\tau^{a_3\ldots a_l}_{i_3\ldots i_l} Y^{[1,2]}_{a_3\ldots a_l} + g_{i_1 i_3}\tau^{a_2 a_4\ldots a_l}_{i_2 i_4\ldots i_l} Y^{[1,3]}_{a_2 a_4\ldots a_l} + \ldots + g_{i_{l-1} i_l}\tau^{a_1\ldots a_{l-2}}_{i_1\ldots i_{l-2}} Y^{[l-1,l]}_{a_1\ldots a_{l-2}}
$$

and

$$
g^{p_1 p_2}\gamma^{a_1\ldots a_l}_{i_1\ldots p_1\ldots p_2\ldots i_l} X_{a_1\ldots a_l} \quad = \quad g^{p_1 p_2} X_{i_1\ldots p_1\ldots p_2\ldots i_l}
$$

2.3. $*$

Let $\mathfrak{g} < 0$.

$$
\begin{aligned}
\alpha^{i_1\ldots i_n}_{1\ldots n} &= g^{i_1 a_1}\ldots g^{i_n a_n} g_{1 b_1}\ldots g_{n b_n}\alpha^{b_1\ldots b_n}_{a_1\ldots a_n} \\
&= g^{i_1 a_1}\ldots g^{i_n a_n} g_{1 b_1}\ldots g_{n b_n}\alpha^{1\ldots n}_{a_1\ldots a_n}\alpha^{b_1\ldots b_n}_{1\ldots n} \\
&= g^{i_1 a_1}\ldots g^{i_n a_n}\alpha^{1\ldots n}_{a_1\ldots a_n}\mathfrak{g} \\
\alpha^{1\ldots n}_{i_1\ldots i_n} &= g_{i_1 a_1}\ldots g_{i_n a_n}\alpha^{a_1\ldots a_n}_{1\ldots n}\frac{1}{\mathfrak{g}}
\end{aligned}
$$

That results in

$$
\begin{aligned}
\alpha^{1\dots n}_{i_1\dots i_n}\alpha^{1\dots n}_{a_1\dots a_n} &= \alpha^{1\dots n}_{i_1\dots i_n} g_{a_1b_1}\dots g_{a_nb_n}\,\alpha^{b_1\dots b_n}_{1\dots n}\frac{1}{\mathfrak{g}} \\
&= \alpha^{b_1\dots b_n}_{i_1\dots i_n} g_{a_1b_1}\dots g_{a_nb_n}\frac{1}{\mathfrak{g}} \\
\alpha^{i_1\dots i_n}_{1\dots n}\alpha^{a_1\dots a_n}_{1\dots n} &= \alpha^{i_1\dots i_n}_{b_1\dots b_n} g^{a_1b_1}\dots g^{a_nb_n}\,\mathfrak{g}
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{-\mathfrak{g}}\alpha^{1\dots n}_{i_1\dots i_n}\sqrt{-\mathfrak{g}}\alpha^{1\dots n}_{a_1\dots a_n} &= -\,g_{a_1b_1}\dots g_{a_nb_n}\alpha^{b_1\dots b_n}_{i_1\dots i_n} = -\det\begin{bmatrix} g_{i_1a_1} & \cdots & g_{i_1a_n} \\ \vdots & \ddots & \vdots \\ g_{i_na_1} & \cdots & g_{i_na_n}\end{bmatrix} \\
\frac{1}{\sqrt{-\mathfrak{g}}}\alpha^{i_1\dots i_n}_{1\dots n}\frac{1}{\sqrt{-\mathfrak{g}}}\alpha^{a_1\dots a_n}_{1\dots n} &= -\,g^{a_1b_1}\dots g^{a_nb_n}\alpha^{i_1\dots i_n}_{b_1\dots b_n} = -\det\begin{bmatrix} g^{i_1a_1} & \cdots & g^{i_1a_n} \\ \vdots & \ddots & \vdots \\ g^{i_na_1} & \cdots & g^{i_na_n}\end{bmatrix}
\end{aligned}
$$

respectively. Set

$$
\begin{aligned}
\eta_{i_1\dots i_n} &:= \sqrt{-\mathfrak{g}}\alpha^{1\dots n}_{i_1\dots i_n} \\
\eta^{i_1\dots i_n} &:= \frac{1}{\sqrt{-\mathfrak{g}}}\alpha^{i_1\dots i_n}_{1\dots n}
\end{aligned}
$$

These definitions make sense because of

$$
\eta_{a_1\dots a_n} g^{i_1a_1}\dots g^{i_na_n} = \eta^{i_1\dots i_n}
$$

so

$$
\eta_{i_1\dots i_n}\eta^{a_1\dots a_n} = -\,\alpha^{a_1\dots a_n}_{i_1\dots i_n} = -\det\begin{bmatrix} \delta^{a_1}_{i_1} & \vdots & \delta^{a_n}_{i_1} \\ \vdots & \ddots & \vdots \\ \delta^{a_1}_{i_n} & \vdots & \delta^{a_n}_{i_n}\end{bmatrix}
$$

Thus, the following construction is possible

$$
\begin{aligned}
\alpha^{a_1\dots a_k}_{i_1\dots i_k} X_{a_1\dots a_k} &= \frac{1}{[n-k]!}\alpha^{a_1\dots a_k i_{k+1}\dots i_n}_{i_1\dots i_k i_{k+1}\dots i_n} X_{a_1\dots a_k} \\
&= -\frac{1}{[n-k]!}\eta_{i_1\dots i_k i_{k+1}\dots i_n}\eta^{a_1\dots a_k i_{k+1}\dots i_n} X_{a_1\dots a_k} \\
&=: -\frac{k!}{[n-k]!}\eta_{i_1\dots i_k i_{k+1}\dots i_n}[*X]^{i_{k+1}\dots i_n}
\end{aligned}
$$

with the *Dual Operator* $*$ and the dual tensor $*X$. Particulary for M^L

$$
\begin{aligned}
\alpha X &= \alpha^{abcd}_{abcd}\frac{1}{24}X = -\,\eta_{abcd}\eta^{abcd}\frac{1}{24}X =: -\,\eta_{abcd}\frac{1}{24}[*X]^{abcd} \\
\alpha^{a}_{i} X_a &= \alpha^{abcd}_{ibcd}\frac{1}{6}X_a = -\,\eta_{ibcd}\eta^{abcd}\frac{1}{6}X_a =: -\,\eta_{ibcd}\frac{1}{6}[*X]^{bcd} \\
\alpha^{ab}_{ij} X_{ab} &= \alpha^{abcd}_{ijcd}\frac{1}{2}X_{ab} = -\,\eta_{ijcd}\eta^{abcd}\frac{1}{2}X_{ab} =: -\,\eta_{ijcd}[*X]^{cd} \\
\alpha^{abc}_{ijk} X_{abc} &= \alpha^{abcd}_{ijkd} X_{abc} = -\,\eta_{ijkd}\eta^{abcd} X_{abc} =: -\,\eta_{ijkd}6[*X]^{d}
\end{aligned}
$$

$$\alpha_{ijkl}^{abcd} X_{abcd} \quad = \quad \alpha_{ijkl}^{abcd} X_{abcd} \quad = \quad -\eta_{ijkl}\eta^{abcd} X_{abcd} \quad =: \quad -\eta_{ijkl} 24[*X]$$

and

$$\begin{aligned} \pi_{ijkl}^{abcd}\eta_{abcd} &= \eta_{ijkl}^{\Box} \\ \tau_{ijkl}^{abcd}\eta_{abcd} &= \eta_{ijkl} - \gamma_{ijkl}^{abcd}\eta_{abcd} \end{aligned}$$

2.4. ω

Returning to M.

$$X_{i_1 \ldots i_l} \quad = \quad \delta_{i_1}^{a_1}\delta_{i_2}^{a_2} \ldots \delta_{i_l}^{a_l} X_{a_1 \ldots a_l}$$

Because of the property

$$\delta_i^a X_a \quad = \quad X_i$$

with an arbitrary vector X_i, the Kronecker Delta has the eigenvalue 1 and every vector is an eigenvector. If one chooses an orthonormal basis $\{x_i^\alpha\}_{\alpha=1}^n$ then a spectral factorization of the Kronecker Delta is given by

$$\delta_i^a \quad = \quad \sum_{\alpha=1}^n x_i^\alpha x^{\alpha a}$$

It follows that

$$\begin{aligned} X_{i_1 \ldots i_l} &= \sum_{\alpha_1=1}^n x_{i_1}^{\alpha_1} x^{\alpha_1 a_1} \sum_{\alpha_2=1}^n x_{i_2}^{\alpha_2} x^{\alpha_2 a_2} \ldots \sum_{\alpha_l=1}^n x_{i_l}^{\alpha_l} x^{\alpha_l a_l} X_{a_1 \ldots a_l} \\ &= \sum_{\alpha_1=1}^n \sum_{\alpha_2=1}^n \ldots \sum_{\alpha_l=1}^n x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_l}^{\alpha_l} [x^{\alpha_1 a_1} x^{\alpha_2 a_2} \ldots x^{\alpha_l a_l} X_{a_1 \ldots a_l}] \end{aligned}$$

which is a decomposition of a tensor into a sum of products of vectors. For given $X_{i_1 \ldots i_l}$ only $l-1$ sums are linear independent, the final sum can be obtained by solving a system of linear equations. This gives a *Factorization Operator* ω for a tensor $X_{i_1 \ldots i_l}$ that reads

$$\omega_{i_1 \ldots i_l}^{a_1 \ldots a_l} X_{a_1 \ldots a_l} \quad := \quad \sum_{\alpha=1}^{n^{l-1}} Y_{i_1}^{[1,\alpha]} \ldots Y_{i_l}^{[l,\alpha]}$$

For $l = 2$ this becomes

$$\omega_{ij}^{ab} X_{ab} \quad = \quad \sum_{\alpha=1}^n Y_i^\alpha Z_j^\alpha$$

If there is a positive definite scalar product available, a factorization of X_{ij} can be obtained by using the *Singular Value Decomposition*

$$X_{ij} \quad = \quad Y_i{}^a S_{ab} Z^b{}_j$$

with two orthogonal matrices Y and Z and a diagonal matrix S whose non zero diagonal elements s^α are positive and called the singular values of X. It follows that

$$\omega_{ij}^{ab} X_{ab} \quad = \quad \sum_{\alpha=1}^n s^\alpha y_i^\alpha z_j^\alpha$$

$$
\begin{aligned}
&= \sum_{\alpha=1}^{n} [\sqrt{s^\alpha} y_i^\alpha][\sqrt{s^\alpha} z_j^\alpha] \\
&= \sum_{\alpha=1}^{n} Y_i^\alpha Z_j^\alpha
\end{aligned}
$$

2.4.1. $\omega \circ \sigma$

Again, if there is a positive definite scalar product available, one can use the the *Eigenvalue Decomposition*

$$X_{ij} = Y_i{}^a \Lambda_{ab} Y^b{}_j$$

for a square, symmetric matrix X with an orthogonal matrix Y and a diagonal matrix Λ whose diagonal elements λ^α are the eigenvalues of X. It follows that

$$
\begin{aligned}
[\omega \circ \sigma]_{ij}^{ab} X_{ab} &:= \sum_{\alpha=1}^{n} \lambda^\alpha y_i^\alpha y_j^\alpha \\
&= \sum_{\alpha=1}^{n} [\sqrt{\lambda^\alpha} y_i^\alpha][\sqrt{\lambda^\alpha} y_j^\alpha] \\
&= \sum_{\alpha=1}^{n} Y_i^\alpha Y_j^\alpha
\end{aligned}
$$

or

$$
\begin{aligned}
[\omega \circ \sigma]_{ij}^{ab} X_{ab} &= \sum_{\alpha=1}^{n} \text{sign}[\lambda^\alpha] \sqrt{|\lambda^\alpha|} y_i^\alpha][\sqrt{|\lambda^\alpha|} y_j^\alpha] \\
&= \sum_{\alpha=1}^{n} \text{sign}[\lambda^\alpha] Z_i^\alpha Z_j^\alpha
\end{aligned}
$$

2.4.2. $\omega \circ \alpha$

$$
\begin{aligned}
[\omega \circ \alpha]_{i_1 \ldots i_n}^{a_1 \ldots a_n} X_{a_1 \ldots a_n} &= [\omega \circ \alpha]_{i_1 \ldots i_n}^{a_1 \ldots a_n} g_{a_1 b_1} \ldots g_{a_n b_n} X^{b_1 \ldots b_n} \\
&= \alpha_{i_1 \ldots i_n}^{a_1 \ldots a_n} \sum_{\alpha_1}^{n} x_{a_1}^{\alpha_1} x_{b_1}^{\alpha_1} \ldots \sum_{\alpha_n}^{n} x_{a_n}^{\alpha_n} x_{b_n}^{\alpha_n} X^{b_1 \ldots b_n} \\
&= \alpha_{i_1 \ldots i_n}^{a_1 \ldots a_n} \sum_{p} x_{a_1}^{p[1]} x_{b_1}^{p[1]} \ldots x_{a_n}^{p[n]} x_{b_n}^{p[n]} X^{b_1 \ldots b_n} \\
&= \alpha_{i_1 \ldots i_n}^{a_1 \ldots a_n} x_{a_1}^{[1]} \ldots x_{a_n}^{[n]} \sum_{p} \text{sign}[p] x_{b_1}^{p[1]} \ldots x_{b_n}^{p[n]} X^{b_1 \ldots b_n} \\
&=: \alpha_{i_1 \ldots i_n}^{a_1 \ldots a_n} Y_{a_1}^{[1]} \ldots Y_{a_n}^{[n]}
\end{aligned}
$$

which implies

$$[\omega \circ \alpha]_{i_1 \ldots i_n}^{a_1 \ldots a_n} X_{a_1 \ldots a_{n-1}} * [X]_{a_n} = \alpha_{i_1 \ldots i_n}^{a_1 \ldots a_n} Y_{a_1}^{[1]} \ldots Y_{a_{n-1}}^{[n-1]} * Y_{a_n}^{[n]}$$

Here $*Y_{a_k}^{[k]}$ can be chosen

$$[*Y]_{a_k}^{[k]} = [*X]_{a_k}$$

which gives

$$[\omega \circ \alpha]^{a_1 \ldots a_{n-1}}_{i_1 \ldots i_{n-1}} X_{a_1 \ldots a_{n-1}} \quad = \quad \alpha^{a_1 \ldots a_{n-1}}_{i_1 \ldots i_{n-1}} Y^{[1]}_{a_1} \ldots Y^{[k-1]}_{i_{k-1}} Y^{[k+1]}_{i_{k+1}} \ldots Y^{[n]}_{a_n}$$

So for $n = 3$

$$[\omega \circ \alpha]^{ab}_{ij} X_{ab} \quad = \quad \alpha^{ab}_{ij} Y_a Z_b$$

2.5. ρ

On M^L there is a possible decomposition

$$\omega^{ac}_{ik} g_{ac} \quad = \quad \sum_{\alpha=1}^{3} x_{\alpha i} x_{\alpha k} - u_i u_k$$

for the metric, the $x_{\alpha i}$ being normed and spacelike, and u_i being normed and timelike.

$$u_i u^i \quad = \quad -1$$

Alternatively,

$$v_i = c\, u_i$$

will be used, $c \in \mathbb{R}$, defined in (9.1). Hence

$$\omega^{ka}_{ic} \delta^c_a \quad = \quad \sum_{\alpha=1}^{3} x^{\alpha}_i x^{\alpha k} - u_i u^k \quad =: \quad n^k_i - u_i u^k$$

This gives rise to the *Spatial Temporal Decomposition Operator* ρ

$$\rho^{k_1 \ldots k_l}_{i_1 \ldots i_l} \quad := \quad [n^{k_1}_{i_1} - u_{i_1} u^{k_1}] \ldots [n^{k_l}_{i_l} - u_{i_l} u^{k_l}]$$

and

$$\rho^{abcd}_{ijkl} \eta_{abcd} \quad = \quad \eta_{jkl} u_i - \eta_{ikl} u_j + \eta_{ijl} u_k - \eta_{ijk} u_l$$

with

$$\eta_{ijk} \quad := \quad \eta_{ijkd} u^d$$

2.6. ∇

Covariant Derivative ∇ on M: tensorial, linear, product rule, commutes with contractions. For two covariant derivatives, ∇ and $\widetilde{\nabla}$,

$$\nabla_i X_j - \widetilde{\nabla}_i X_j \quad = \quad -\, G_{ij}{}^a X_a$$

holds. G is a tensorfield. There are special covariant derivatives Y_{ij} of X_j: Y is a tensor which has the components $Y_{ij} = \partial_i X_j$ in one basis. Set

$$\nabla_i X_j \quad = \quad \partial_i X_j - \Gamma_{ij}{}^a X_a$$

for the difference $\nabla_i X_j - \partial_i X_j$. One has

$$Y_{ij} \quad = \quad \frac{\partial x'^a}{\partial x^i} \frac{\partial x'^b}{\partial x^j} Y'_{ab} \qquad (2.2)$$

$$
\begin{aligned}
\partial_i X_j &= \frac{\partial}{\partial x^i}[X'_b \frac{\partial x'^b}{\partial x^j}] && (2.3)\\
&= \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}\frac{\partial X'_b}{\partial x'^a} + X'_b \frac{\partial^2 x'^b}{\partial x^i \partial x^j} && (2.4)\\
\Gamma^k_{ij} &= \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}\frac{\partial x^k}{\partial x'^c}\Gamma'^c_{ab} + \frac{\partial^2 x'^a}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial x'^a} && (2.5)
\end{aligned}
$$

Furthermore, one has

$$
\begin{aligned}
\nabla_i [X^a Y_a] &= \partial_i [X^a Y_a]\\
&= \partial_i X^a Y_a + X^a \partial_i Y_a
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_i [X^a Y_a] &= [\partial_i Y_a - \Gamma^b_{ia} Y_b] X^a + \nabla_i X^a Y_a\\
&= \partial_i Y_a X^a - \Gamma^a_{ib} X^b Y_a + \nabla_i X^a Y_a
\end{aligned}
$$

hence

$$
\nabla_i X^j = \frac{\partial X^j}{\partial x^i} + \Gamma^j_{ia} X^a
$$

By the same reasoning one gets from

$$
\nabla_k [X^{i_1 \ldots i_n}_{j_1 \ldots j_m} Y^{[1]}_{i_1} \ldots Y^{[n]}_{i_n} Z^{[1]j_1} \ldots Z^{[m]j_m}] = \partial_k [X^{i_1 \ldots i_n}_{j_1 \ldots j_m} Y^{[1]}_{i_1} \ldots Y^{[n]}_{i_n} Z^{[1]j_1} \ldots Z^{[m]j_m}]
$$

for the covariant derivative of a tensorfield the formula

$$
\nabla_k X^{i_1 \ldots i_n}_{j_1 \ldots j_m} := \partial_k X^{i_1 \ldots i_n}_{j_1 \ldots j_m} + \sum_{\alpha=1}^{n} X^{i_1 \ldots i_{\alpha-1} a i_{\alpha+1} \ldots i_n}_{j_1 \ldots j_m} \Gamma^{i_\alpha}_{ka} - \sum_{\beta=1}^{m} X^{i_1 \ldots i_n}_{j_1 \ldots j_{\beta-1} b j_{\beta+1} \ldots j_m} \Gamma^b_{k j_\beta}
$$

2.7. C, R

$$
\begin{aligned}
\nabla_k \partial_l &= -\Gamma^a_{kl}\partial_a\\
\nabla_j \nabla_k \partial_l &= -\nabla_j [\Gamma^a_{kl}\partial_a]\\
&= -\partial_j \Gamma^a_{kl}\partial_a + \Gamma^b_{jk}\Gamma^a_{bl}\partial_a + \Gamma^b_{jl}\Gamma^a_{kb}\partial_a\\
\nabla_i \nabla_j \nabla_k \partial_l &= -\nabla_i \partial_j \Gamma^a_{kl}\partial_a - \partial_j \Gamma^a_{kl}\nabla_i \partial_a\\
&\quad + \nabla_i \Gamma^b_{jk}\Gamma^a_{bl}\partial_a + \Gamma^b_{jk}\nabla_i \Gamma^a_{bl}\partial_a + \Gamma^b_{jk}\Gamma^a_{bl}\nabla_i \partial_a\\
&\quad + \nabla_i \Gamma^b_{jl}\Gamma^a_{kb}\partial_a + \Gamma^b_{jl}\nabla_i \Gamma^a_{kb}\partial_a + \Gamma^b_{jl}\Gamma^a_{kb}\nabla_i \partial_a\\
&= -\partial_i \partial_j \Gamma^a_{kl}\partial_a\\
&\quad + [\partial_b \Gamma^a_{kl}\Gamma^b_{ij} + \partial_j \Gamma^a_{bl}\Gamma^b_{ik} + \partial_j \Gamma^a_{kb}\Gamma^b_{il} + \partial_i [\Gamma^b_{jk}\Gamma^a_{bl}] + \partial_i [\Gamma^b_{jl}\Gamma^a_{kb}]]\partial_a\\
&\quad - \Gamma^c_{ij}[\Gamma^b_{ck}\Gamma^a_{bl} + \Gamma^b_{cl}\Gamma^a_{kb}]\partial_a - \Gamma^c_{ik}[\Gamma^b_{jc}\Gamma^a_{bl} + \Gamma^b_{jl}\Gamma^a_{cb}]\partial_a - \Gamma^c_{il}[\Gamma^b_{jk}\Gamma^a_{bc} + \Gamma^b_{jc}\Gamma^a_{kb}]\partial_a
\end{aligned}
$$

Antisymmetric parts

$$
\begin{aligned}
\alpha^{cd}_{kl}\nabla_c \partial_d &= -\alpha^{cd}_{kl}\Gamma_{cd}{}^p \partial_p\\
&=: -S_{kl}{}^p \partial_p\\
\alpha^{cd}_{kl}\nabla_j \nabla_c \partial_d &= -\nabla_j S_{kl}{}^p \partial_p - S_{kl}{}^p \nabla_j \partial_p\\
\alpha^{bc}_{jk}\nabla_b \nabla_c \partial_l &= -\alpha^{bc}_{jk}[\partial_b \Gamma^p_{cl} + \Gamma^q_{cl}\Gamma^p_{bq}]\partial_p - S_{jk}{}^p \nabla_p \partial_l
\end{aligned}
$$

$$
\begin{aligned}
&=: && -K_{jkl}{}^p\partial_p - S_{jk}{}^p\nabla_p\partial_l\\
\alpha_{ij}^{ab}\nabla_a\nabla_b\nabla_k\partial_l &= && -K_{ijk}{}^p\nabla_p\partial_l - K_{ijl}{}^p\nabla_k\partial_p - S_{ij}{}^p\nabla_p\nabla_k\partial_l\\
\alpha_{jk}^{bc}\nabla_i\nabla_b\nabla_c\partial_l &= && -\nabla_i K_{jkl}{}^p\partial_p - K_{jkl}{}^p\nabla_i\partial_p - \nabla_i S_{jk}{}^p\nabla_p\partial_l - S_{jk}{}^p\nabla_i\nabla_p\partial_l
\end{aligned}
$$

with the *Torsion Tensor* $S_{kl}{}^p$ and the *Curvature Tensor* $K_{jkl}{}^p$. Equating

$$\alpha_{jkl}^{bcd}\alpha_{cd}^{pq}\nabla_b\nabla_p\partial_q = \alpha_{jkl}^{bcd}\alpha_{bc}^{pq}\nabla_p\nabla_q\partial_l$$

leads to

$$-\alpha_{jkl}^{bcd}[\nabla_b S_{cd}{}^p\partial_p + S_{cd}{}^q\nabla_b\partial_q] = -\alpha_{jkl}^{bcd}[K_{bcd}{}^p\partial_p - S_{bc}{}^q\nabla_q\partial_d]$$

or

$$\alpha_{jkl}^{bcd}K_{bcd}{}^p = \alpha_{jkl}^{bcd}[\nabla_b S_{cd}{}^p + S_{cd}{}^q S_{qb}{}^p]$$

and equating

$$\alpha_{ijk}^{abc}\alpha_{ab}^{pq}\nabla_p\nabla_q\nabla_c\partial_l = \alpha_{ijk}^{abc}\alpha_{bc}^{pq}\nabla_i\nabla_p\nabla_q\partial_l$$

leads to

$$\alpha_{ijk}^{abc}[K_{abc}{}^p\nabla_p\partial_l + S_{ab}{}^p\nabla_p\nabla_c\partial_l] = \alpha_{ijk}^{abc}[\nabla_a K_{bcl}{}^p\partial_p + \nabla_a S_{bc}{}^p\nabla_p\partial_l + S_{bc}{}^p\nabla_a\nabla_p\partial_l]$$

or

$$0 = \alpha_{ijk}^{abc}[\nabla_a K_{bcl}{}^p - S_{bc}{}^q K_{aql}{}^p]$$

One has

$$\alpha_{ij}^{ab}\nabla_a\nabla_b X_{k_1\ldots k_n} = -K_{ijk_1}{}^p X_{pk_2\ldots k_n} - \ldots - K_{ijk_n}{}^p X_{k_1\ldots k_{n-1}p} - S_{ij}{}^p\nabla_p X_{k_1\ldots k_n}$$

and, therefore

$$
\begin{aligned}
\alpha_{ij}^{ab}\nabla_a\nabla_b g_{kl} &= -K_{ijk}{}^p g_{pl} - K_{ijl}{}^p g_{pk} - S_{ij}{}^p\nabla_p g_{kl}\\
&= -\sigma_{kl}^{cd}K_{ijcd} - S_{ij}{}^p\nabla_p g_{kl}
\end{aligned}
$$

Furthermore, one has

$$
\begin{aligned}
\nabla_i g_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^a g_{ak} - \Gamma_{ik}^a g_{aj}\\
&=: \partial_i g_{jk} - \Gamma_{ijk} - \Gamma_{ikj}
\end{aligned}
$$

and, therefore

$$-\nabla_i g_{jk} + \nabla_k g_{ij} + \nabla_j g_{ik} = -\partial_i g_{jk} + \partial_k g_{ij} + \partial_j g_{ik} + S_{ikj} + S_{ijk} - \sigma_{jk}^{bc}\Gamma_{bci}$$

and so

$$2\Gamma_{ijk} = \sigma_{ij}^{ab}\Gamma_{abk} + S_{ijk}$$

becomes

$$\Gamma_{ijk} = \frac{1}{2}[-\partial_k g_{ij} + \partial_j g_{ki} + \partial_i g_{kj} + S_{kij} + S_{kji} + S_{ijk} + \nabla_k g_{ij} - \nabla_j g_{ki} - \nabla_i g_{kj}]$$

If

$$\Gamma_{ijk} = \frac{1}{2}[-\partial_k g_{ij} + \partial_j g_{ki} + \partial_i g_{kj} + S_{kij} + S_{kji} + S_{ijk}]$$

then

$$C_{ijk}{}^{l} := \alpha_{ij}^{ab}[\partial_a \Gamma_{bk}^{l} + \Gamma_{bk}^{p}\Gamma_{ap}^{l}]$$

and

$$0 = \sigma_{kl}^{cd} C_{ijcd}$$

If

$$\Gamma_{ijk} = \frac{1}{2}[-\partial_k g_{ij} + \partial_j g_{ki} + \partial_i g_{kj}]$$

then

$$R_{ijk}{}^{l} := \alpha_{ij}^{ab}[\partial_a \Gamma_{bk}^{l} + \Gamma_{bk}^{p}\Gamma_{ap}^{l}]$$
$$R_{ijkl} = \frac{1}{2}\alpha_{ij}^{ab}\alpha_{kl}^{cd}[\partial_a \partial_c g_{bd} + g_{pq}\Gamma_{ac}^{p}\Gamma_{bd}^{q}]$$

and

$$0 = \alpha_{jkl}^{bcd} R_{bcd}{}^{p}$$
$$0 = \alpha_{ijk}^{abc} \nabla_a R_{bcl}{}^{p}$$

3. C

3.1. C

3.1.1. π

From the definition of $\mathbf{C}$ it follows that

$$0 = \sigma_{ij}^{ab} C_{abkl} \tag{3.1}$$

$$0 = \sigma_{kl}^{cd} C_{ijcd} \tag{3.2}$$

holds, hence

$$\begin{aligned} \pi_{ijkl}^{abcd} C_{abcd} &= \frac{1}{12} \alpha_{ij}^{pr} \alpha_{kl}^{qs} \sigma_{pq}^{ac} \sigma_{rs}^{bd} C_{abcd} + \frac{1}{8} \alpha_{kl}^{xy} \alpha_{ijx}^{pbc} \sigma_{py}^{ad} C_{abcd} + \frac{1}{24} \alpha_{ijkl}^{abcd} C_{abcd} \\ &= C_{ijkl}^{\boxplus} + C_{ijkl}^{[2,1,1]} + C_{ijkl}^{[1,1,1,1]} \end{aligned}$$

with

$$C_{ijkl}^{\boxplus} = \frac{1}{2}[C_{ijkl} + C_{klij}] - \frac{1}{6}[[C_{ijkl} + C_{klij}] + [C_{iljk} + C_{jkil}] + [C_{iklj} + C_{ljik}]] \tag{3.3}$$

$$C_{ijkl}^{[2,1,1]} = \frac{1}{2}[C_{ijkl} - C_{klij}] \tag{3.4}$$

$$C_{ijkl}^{[1,1,1,1]} = \frac{1}{6}[[C_{ijkl} + C_{klij}] + [C_{iljk} + C_{jkil}] + [C_{iklj} + C_{ljik}]] \tag{3.5}$$

The distribution of components reads $36 = 20 + 15 + 1$. From (3.3) and (3.5) one has

$$C_{ijkl}^{\boxplus} + C_{ijkl}^{[1,1,1,1]} = \frac{1}{2}[C_{ijkl} + C_{klij}] \tag{3.6}$$

3.1.2. τ

$$\begin{aligned} \tau_{ijkl}^{abcd} C_{abcd} &= [\tau^{III}]_{ijkl}^{abcd} C_{abcd} + [\tau^{II}]_{ijkl}^{abcd} C_{abcd} + [\tau^{I}]_{ijkl}^{abcd} C_{abcd} \\ &=: C_{ijkl}^{III} + C_{ijkl}^{II} + C_{ijkl}^{I} \end{aligned}$$

with

$$\begin{aligned} C_{ijkl}^{III} &:= C_{ijkl} - \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} X_{bd} \\ C_{ijkl}^{II} &:= \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} [X_{bd} - \gamma_{bd}^{pq} X_{pq}] \\ C_{ijkl}^{I} &:= \alpha_{ij}^{ab} \alpha_{kl}^{cd} g_{ac} g_{bd} X \end{aligned}$$

and

$$\begin{aligned} g^{ik} C_{ijkl} &=: C_{jl} = 2X_{bd} \\ g^{ik} g^{jl} C_{ijkl} &=: C = 24X \end{aligned}$$

hence

$$\begin{aligned} X_{bd}^{II} &:= [X_{bd} - \gamma_{bd}^{pq} X_{pq}] = \frac{1}{2}[C_{jl} - \frac{1}{4} g_{jl} C] =: \frac{1}{2} C_{bd}^{II} \\ X &= \frac{1}{24} C \end{aligned}$$

The distribution of components reads $36 = 20 + 15 + 1$.

3.1.3. $\tau \circ \pi$

$$
\begin{aligned}
\tau \circ \pi \circ \mathbf{C} &= [\tau^{III} + \tau^{II} + \tau^{I}] \circ [\pi^{[2,2]} + \pi^{[2,1,1]} + \pi^{[1,1,1,1]}] \circ \mathbf{C} \\
&= [\pi^{[2,2]} + \pi^{[2,1,1]} + \pi^{[1,1,1,1]}] \circ [\tau^{III} + \tau^{II} + \tau^{I}] \circ \mathbf{C} \quad = \quad \pi \circ \tau \circ \mathbf{C}
\end{aligned}
$$

and in particular

$$
\begin{aligned}
C^{[2,2]\,III}_{ijkl} &= \frac{1}{2}[C^{III}_{ijkl} + C^{III}_{klij}] - C^{[1,1,1,1]\,III}_{ijkl} \\
C^{[2,2]\,II}_{ijkl} &= \frac{1}{2}[C^{II}_{ijkl} + C^{II}_{klij}] \\
&= \frac{1}{4}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac}\sigma^{pq}_{bd} C^{II}_{pq} \\
C^{[2,2]\,I}_{ijkl} &= \frac{1}{2}[C^{I}_{ijkl} + C^{I}_{klij}] \\
&= \frac{1}{24}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac} g_{bd} C \\
C^{[2,1,1]\,III}_{ijkl} &= \frac{1}{2}[C^{III}_{ijkl} - C^{III}_{klij}] \\
C^{[2,1,1]\,II}_{ijkl} &= \frac{1}{2}[C^{II}_{ijkl} - C^{II}_{klij}] \\
&= \frac{1}{4}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac}\alpha^{pq}_{bd} C^{II}_{pq} \\
C^{[2,1,1]\,I}_{ijkl} &= 0 \\
C^{[1,1,1,1]\,III}_{ijkl} &= \frac{1}{6}[[C^{III}_{ijkl} + C^{III}_{klij}] + [C^{III}_{iljk} + C^{III}_{jkil}] + [C^{III}_{iklj} + C^{III}_{ljik}]] \\
C^{[1,1,1,1]\,II}_{ijkl} &= 0 \\
C^{[1,1,1,1]\,I}_{ijkl} &= 0
\end{aligned}
$$

The distribution of components reads $36 = 10 + 9 + 1 + 9 + 6 + 1$.

3.2. $*\mathbf{C}$

Because of (3.1) and (3.2) one has

$$
\begin{aligned}
C_{ijkl} &= -\frac{1}{4}\eta_{ijpq}\eta^{abpq}C_{abkl} &=: \quad -\frac{1}{2}\eta_{ijab}[*C]^{ab}{}_{kl} \\
C_{ijkl} &= -\frac{1}{4}\eta_{ijpq}\eta^{cdpq}C_{ijcd} &=: \quad -\frac{1}{2}\eta_{klcd}[C*]_{ij}{}^{cd}
\end{aligned}
$$

3.2.1. π

From (3.2) and the definition of $*$ it follows that

$$
\begin{aligned}
0 &= \sigma^{ab}_{ij}[*C]_{abkl} \\
0 &= \sigma^{cd}_{kl}[*C]_{ijcd}
\end{aligned}
$$

holds, that means

$$
\pi \circ [*\mathbf{C}] = [*\mathbf{C}]^{[2,2]} + [*\mathbf{C}]^{[2,1,1]} + [*\mathbf{C}]^{[1,1,1,1]}
$$

and in particular

$$
\begin{aligned}
[*C]^{\yng(2,2)}_{ijkl} &= \frac{1}{2}\eta_{ij}{}^{ab}C^{III}_{abkl} &&= *[C^{III}]_{ijkl} \\
[*C]^{\yng(2,1,1)}_{ijkl} &= \frac{1}{2}\eta_{ijc}{}^{p}\alpha^{cd}_{kl}C^{II}_{pd} &&= *[C^{II}]_{ijkl} \\
[*C]^{\yng(1,1,1,1)}_{ijkl} &= \frac{1}{12}\eta_{ijkl}C &&= *[C^{I}]_{ijkl}
\end{aligned}
$$

thus

$$
\pi \circ [*\mathbf{C}] \quad = \quad *[\tau \circ \mathbf{C}]
$$

The substitution $\mathbf{C} \to *\mathbf{C}$ gives

$$
\pi \circ [**\mathbf{C}] \quad = \quad -\pi \circ \mathbf{C} \quad = \quad *[\tau \circ [*\mathbf{C}]] \tag{3.7}
$$

that means

$$
\begin{aligned}
C^{\yng(2,2)}_{ijkl} &= -\frac{1}{2}\eta_{ij}{}^{ab}[*C]^{III}_{abkl} &&= -*[[*C]^{III}]_{ijkl} \\
C^{\yng(2,1,1)}_{ijkl} &= -\frac{1}{2}\eta_{ijc}{}^{p}\alpha^{cd}_{kl}[*C]^{II}_{pd} &&= -*[[*C]^{II}]_{ijkl} \\
C^{\yng(1,1,1,1)}_{ijkl} &= -\frac{1}{12}\eta_{ijkl}*C &&= -*[[*C]^{I}]_{ijkl}
\end{aligned}
$$

The relations (3.4) and (3.6) become

$$
\begin{aligned}
[*C]^{\yng(2,1,1)}_{ijkl} &= \frac{1}{2}[[*C]_{ijkl} - [*C]_{klij}] &&= *[C^{II}]_{ijkl} \\
[*C]^{\yng(2,2)}_{ijkl} + [*C]^{\yng(1,1,1,1)}_{ijkl} &= \frac{1}{2}[[*C]_{ijkl} + [*C]_{klij}] &&= *[C^{III}]_{ijkl} + *[C^{I}]_{ijkl}
\end{aligned}
$$

and

$$
\begin{aligned}
-*[[*C]^{II}]_{ijkl} &= \frac{1}{2}[C_{ijkl} - C_{klij}] \\
-*[[*C]^{III}]_{ijkl} - *[[*C]^{I}]_{ijkl} &= \frac{1}{2}[C_{ijkl} + C_{klij}]
\end{aligned}
$$

3.2.2. τ

Dualization of (3.7) gives

$$
[\pi \circ \mathbf{C}] \quad = \quad \tau \circ [\mathbf{C}]
$$

and in particular

$$
\begin{aligned}
*[C^{\yng(2,2)}]_{ijkl} &= [*C]_{ijkl} - [*C]^{I}_{ijkl} - [*C]^{II}_{ijkl} &&= [*C]^{III}_{ijkl} \\
*[C^{\yng(2,1,1)}]_{ijkl} &= \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}[*C]^{II}_{bd} &&= [*C]^{II}_{ijkl} \\
*[C^{\yng(1,1,1,1)}]_{ijkl} &= \frac{1}{24}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}g_{bd}[*C] &&= [*C]^{I}_{ijkl}
\end{aligned}
$$

with

$$
g^{jl}[*C]_{ijkl} \quad =: \quad [*C]_{ik}
$$

$$g^{jl}g^{ik}[*C]_{ijkl} \quad =: \quad *C$$

The substitution $\mathbf{C} \to *\mathbf{C}$ gives

$$\tau \circ [**\mathbf{C}] \quad = \quad -\tau \circ \mathbf{C} \quad = \quad *[\pi \circ [*\mathbf{C}]]$$

and in particular

$$\begin{aligned}
C^{III}_{ijkl} &= \frac{1}{2}[C_{ijkl} - [*C*]_{klij}] - C^{I}_{ijkl} \\
&= \frac{1}{6}[C_{ijkl} - [*C*]_{klij}] - \frac{1}{3}\eta_{ij}{}^{ab}\alpha^{cd}_{kl}[*C]_{acbd} &&= -*[[*C]^{\yng(2,2)}]_{ijkl} \\
C^{II}_{ijkl} &= \frac{1}{2}[C_{ijkl} + [*C*]_{klij}] &&= -*[[*C]^{\yng(2,1,1)}]_{ijkl} \\
C^{I}_{ijkl} &= \frac{1}{3}[C_{ijkl} - [*C*]_{klij} + \eta_{ij}{}^{ab}\alpha^{cd}_{kl}[*C]_{acbd}] &&= -*[[*C]^{\yng(1,1,1,1)}]_{ijkl}
\end{aligned}$$

The relations (3.4) and (3.6) become

$$\begin{aligned}
[*C]^{II}_{ijkl} &= \frac{1}{2}[[*C]_{ijkl} - [C*]_{klij}] &&= *[C^{\yng(2,1,1)}]_{ijkl} \\
[*C]^{III}_{ijkl} + [*C]^{I}_{ijkl} &= \frac{1}{2}[[*C]_{ijkl} + [C*]_{klij}] &&= *[C^{\yng(2,2)}]_{ijkl} + *[C^{\yng(1,1,1,1)}]_{ijkl}
\end{aligned}$$

and

$$\begin{aligned}
C^{II}_{ijkl} &= \frac{1}{2}[C_{ijkl} + [*C*]_{klij}] &&= -*[[*C]^{\yng(2,1,1)}]_{ijkl} \\
C^{III}_{ijkl} + C^{I}_{ijkl} &= \frac{1}{2}[C_{ijkl} - [*C*]_{klij}] &&= -*[[*C]^{\yng(2,2)}]_{ijkl} - *[[*C]^{\yng(1,1,1,1)}]_{ijkl}
\end{aligned}$$

3.2.3. $\tau \circ \pi$

$$\begin{aligned}
\tau \circ \pi \circ [*\mathbf{C}] &= \tau \circ *[\tau \circ \mathbf{C}] &&= *[\pi \circ \tau \circ \mathbf{C}] \\
\pi \circ \tau \circ [*\mathbf{C}] &= \pi \circ *[\pi \circ \mathbf{C}] &&= *[\tau \circ \pi \circ \mathbf{C}]
\end{aligned}$$

that means

$$\tau \circ *[\tau \circ \mathbf{C}] \quad = \quad \pi \circ *[\pi \circ \mathbf{C}]$$

and in particular one has

$$\begin{aligned}
[*C]^{\yng(2,2)\,III}_{ijkl} &= [*[C^{III}]]^{III}_{ijkl} &&= *[C^{III\,\yng(2,2)}]_{ijkl} &&= [*[C^{\yng(2,2)}]]^{\yng(2,2)}_{ijkl} \\
[*C]^{\yng(2,2)\,II}_{ijkl} &= [*[C^{III}]]^{II}_{ijkl} &&= *[C^{II\,\yng(2,1,1)}]_{ijkl} &&= [*[C^{\yng(2,1,1)}]]^{\yng(2,2)}_{ijkl} \\
[*C]^{\yng(2,2)\,I}_{ijkl} &= [*[C^{III}]]^{I}_{ijkl} &&= *[C^{III\,\yng(1,1,1,1)}]_{ijkl} &&= [*[C^{\yng(1,1,1,1)}]]^{\yng(2,2)}_{ijkl} \\
[*C]^{\yng(2,1,1)\,III}_{ijkl} &= [*[C^{II}]]^{III}_{ijkl} &&= *[C^{II\,\yng(2,2)}]_{ijkl} &&= [*[C^{\yng(2,2)}]]^{\yng(2,1,1)}_{ijkl} \\
[*C]^{\yng(2,1,1)\,II}_{ijkl} &= [*[C^{II}]]^{II}_{ijkl} &&= *[C^{II\,\yng(2,1,1)}]_{ijkl} &&= [*[C^{\yng(2,1,1)}]]^{\yng(2,1,1)}_{ijkl} \\
[*C]^{\yng(1,1,1,1)\,III}_{ijkl} &= [*[C^{I}]]^{III}_{ijkl} &&= *[C^{I\,\yng(2,2)}]_{ijkl} &&= [*[C^{\yng(2,2)}]]^{\yng(1,1,1,1)}_{ijkl}
\end{aligned}$$

3.3. $*\mathbf{C}*$

Because of (3.1) and (3.2)

$$C_{ijkl} \quad = \quad \frac{1}{4}\eta_{ij}{}^{ab}\eta_{kl}{}^{cd}[*C*]_{abcd}$$

holds, with

$$[*C*]_{ijkl} \quad := \quad \frac{1}{4}\eta_{ij}{}^{ab}\eta_{kl}{}^{cd}C_{abcd} \quad = \quad -C_{klij} + \alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}C_{db} - \frac{1}{4}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}g_{bd}C \tag{3.8}$$

3.3.1. π

From the definition of $*$ it follows that

$$\begin{aligned} 0 &= \sigma^{ab}_{ij}[*C*]_{abkl} \\ 0 &= \sigma^{cd}_{kl}[*C*]_{ijcd} \end{aligned}$$

holds, therefore

$$\pi \circ [*\mathbf{C}*] \quad = \quad [*\mathbf{C}*]^{\boxplus} + [*\mathbf{C}*]^{\boxminus} + [*\mathbf{C}*]^{\boxdot} \quad = \quad *[\pi \circ \mathbf{C}]*$$

By means of (3.8) this becomes

$$\begin{aligned} [*C*]^{\boxplus}_{ijkl} &= -C^{\boxplus}_{ijkl} + \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}\sigma^{pq}_{bd}C_{qp} - \frac{1}{4}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}g_{bd}C \\ [*C*]^{\boxminus}_{ijkl} &= C^{\boxminus}_{ijkl} + \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}\alpha^{pq}_{bd}C_{qp} \\ [*C*]^{\boxdot}_{ijkl} &= -C^{\boxdot}_{ijkl} \end{aligned}$$

3.3.2. τ

$$\tau \circ [*\mathbf{C}*] \quad = \quad [*\mathbf{C}*]^{III} + [*\mathbf{C}*]^{II} + [*\mathbf{C}*]^{I} \quad = \quad *[\tau \circ \mathbf{C}]*$$

and by means of (3.8)

$$[*C*]^{III}_{ijkl} \quad = \quad -C^{III}_{klij} \tag{3.9}$$

$$[*C*]^{II}_{ijkl} \quad = \quad C^{II}_{klij} \tag{3.10}$$

$$[*C*]^{I}_{ijkl} \quad = \quad -C^{I}_{klij} \tag{3.11}$$

with

$$\begin{aligned} g^{jl}[*C*]_{ijkl} \quad &=: \quad [*C*]_{ik} \quad = \quad C_{ki} - \frac{1}{2}Cg_{ik} \\ g^{jl}g^{ik}[*C*]_{ijkl} \quad &=: \quad *C* \quad = \quad -C \end{aligned}$$

3.3.3. $\tau \circ \pi$

From (3.9),(3.10), (3.11), (3.4) and (3.6) one has

$$
\begin{aligned}
[*C*]^{\yng(2,2)\,III}_{ijkl} &= -C^{\yng(2,2)\,III}_{klij} &&= -C^{\yng(2,2)\,III}_{ijkl} \\
[*C*]^{\yng(2,2)\,II}_{ijkl} &= C^{\yng(2,2)\,II}_{klij} &&= C^{\yng(2,2)\,II}_{ijkl} \\
[*C*]^{\yng(2,2)\,I}_{ijkl} &= -C^{\yng(2,2)\,I}_{klij} &&= -C^{\yng(2,2)\,I}_{ijkl} \\
[*C*]^{\yng(2,1,1)\,III}_{ijkl} &= -C^{\yng(2,1,1)\,III}_{klij} &&= C^{\yng(2,1,1)\,III}_{ijkl} \\
[*C*]^{\yng(2,1,1)\,II}_{ijkl} &= C^{\yng(2,1,1)\,II}_{klij} &&= -C^{\yng(2,1,1)\,II}_{ijkl} \\
[*C*]^{\yng(1,1,1,1)\,III}_{ijkl} &= -C^{\yng(1,1,1,1)\,III}_{klij} &&= -C^{\yng(1,1,1,1)\,III}_{ijkl}
\end{aligned}
$$

and, equivalently

$$
\begin{aligned}
\frac{1}{2}[C_{ijkl} - [*C*]_{ijkl}] &= C^{\yng(2,2)\,III}_{ijkl} + C^{\yng(2,2)\,I}_{ijkl} + C^{\yng(2,1,1)\,II}_{ijkl} + C^{\yng(1,1,1,1)\,III}_{ijkl} \\
\frac{1}{2}[C_{ijkl} + [*C*]_{ijkl}] &= C^{\yng(2,2)\,II}_{ijkl} + C^{\yng(2,1,1)\,III}_{ijkl}
\end{aligned}
$$

3.4. ρ

3.4.1. $\mathbf{C}$

$$
\begin{aligned}
\rho^{abcd}_{ijkl} C_{abcd} &= C^{abcd} n_{ia} n_{jb} n_{kc} n_{ld} && = [*C*]^{abcd} \eta_{ija} u_b \eta_{klc} u_d \\
&\quad + C^{abcd} n_{pa} u_b n_{rc} u_d [-\alpha^{pq}_{ij} u_q][-\alpha^{rs}_{kl} u_s] && \quad + [*C*]^{abcd} n_{pa} u_b n_{rc} u_d [-\alpha^{pq}_{ij} u_q][-\alpha^{rs}_{kl} u_s] \\
&\quad + C^{abcd} n_{pa} u_b n_{kc} n_{ld} [-\alpha^{pq}_{ij} u_q] && \quad + [C*]^{abcd} n_{pa} u_b \eta_{klc} u_d [-\alpha^{pq}_{ij} u_q] \\
&\quad + C^{abcd} n_{ia} n_{jb} n_{rc} u_d [-\alpha^{rs}_{kl} u_s] && \quad + [*C]^{abcd} \eta_{ija} u_b n_{rc} u_d [-\alpha^{rs}_{kl} u_s]
\end{aligned}
$$

Set

$$
\begin{aligned}
C^{N\!\!N}_{ijkl} &:= [\rho^{N\!\!N}]^{abcd}_{ijkl} C_{abcd} && := C^{abcd} n_{ia} n_{jb} n_{kc} n_{ld} \\
&=: N\!\!N_{ijkl} && = \eta_{ij}{}^{a} \eta_{kl}{}^{c} [*N\!\!N*]_{ac} \\
C^{U\!\!U}_{ijkl} &:= [\rho^{U\!\!U}]^{abcd}_{ijkl} C_{abcd} && := C^{pqrs} [-\alpha^{ab}_{ij} u_b][-\alpha^{cd}_{kl} u_d] n_{ap} u_q n_{cr} u_s \\
&=: [-\alpha^{ab}_{ij} u_b][-\alpha^{cd}_{kl} u_d] U\!\!U_{ac} && \\
C^{U\!\!N}_{ijkl} &:= [\rho^{N\!\!N}]^{abcd}_{ijkl} C_{abcd} && := C^{pqcd} [-\alpha^{ab}_{ij} u_b] n_{ap} u_q n_{kc} n_{ld} \\
&=: [-\alpha^{ab}_{ij} u_b] U\!\!N_{akl} && = [-\alpha^{ab}_{ij} u_b] \eta_{kl}{}^{c} [U\!\!N*]_{ac} \\
C^{N\!\!U}_{ijkl} &:= [\rho^{N\!\!N}]^{abcd}_{ijkl} C_{abcd} && := C^{abrs} n_{ia} n_{jb} n_{cr} u_s [-\alpha^{cd}_{kl} u_d] \\
&=: [-\alpha^{cd}_{kl} u_d] N\!\!U_{ijc} && = [-\alpha^{cd}_{kl} u_d] \eta_{ij}{}^{a} [*N\!\!U]_{ac}
\end{aligned}
$$

That means

$$
\rho \circ \mathbf{C} = \mathbf{C}^{N\!\!N} + \mathbf{C}^{U\!\!U} + \mathbf{C}^{U\!\!N} + \mathbf{C}^{N\!\!U}
$$

The distribution of components reads $36 = 9 + 9 + 9 + 9$.

3.4.2. $*\mathbf{C}$

$$\rho \circ [*\mathbf{C}] \quad = \quad *[\rho \circ \mathbf{C}]$$

but

$$\begin{aligned}
[*C]^{NN}_{ijkl} &= -[*UN]_{ijkl} &&= -\eta_{ij}{}^{a}\eta_{kl}{}^{c}[UN*]_{ac} &&= *[C^{UN}]_{ijkl} \\
[*C]^{UU}_{ijkl} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d][*NU]_{ac} &&&&= *[C^{NU}]_{ijkl} \\
[*C]^{UN}_{ijkl} &= [-\alpha^{ab}_{ij}u_b][*NN]_{akl} &&= [-\alpha^{ab}_{ij}u_b]\eta_{kl}{}^{c}[*NN*]_{ac} &&= *[C^{NN}]_{ijkl} \\
[*C]^{NU}_{ijkl} &= -[-\alpha^{cd}_{kl}u_d][*UU]_{ijc} &&= -\eta_{ij}{}^{a}[-\alpha^{cd}_{kl}u_d]UU_{ac} &&= *[C^{UU}]_{ijkl}
\end{aligned}$$

3.4.3. $*\mathbf{C}*$

$$\rho \circ [*\mathbf{C}*] \quad = \quad *[\rho \circ \mathbf{C}]*$$

but

$$\begin{aligned}
[*C*]^{NN}_{ijkl} &= [*UU*]_{ijkl} &&= \eta_{ij}{}^{a}\eta_{kl}{}^{c}UU_{ac} &&= [*[C^{UU}]*]_{ijkl} \\
[*C*]^{UU}_{ijkl} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d][*NN*]_{ac} &&&&= [*[C^{NN}]*]_{ijkl} \\
[*C*]^{UN}_{ijkl} &= -[-\alpha^{ab}_{ij}u_b][*NU*]_{akl} &&= -[-\alpha^{ab}_{ij}u_b]\eta_{kl}{}^{c}[*NU]_{ac} &&= [*[C^{NU}]*]_{ijkl} \\
[*C*]^{NU}_{ijkl} &= -[-\alpha^{cd}_{kl}u_d][*UN*]_{ijc} &&= -[-\alpha^{cd}_{kl}u_d]\eta_{ij}{}^{a}[UN*]_{ac} &&= [*[C^{UN}]*]_{ijkl}
\end{aligned}$$

3.5. $\rho \circ \pi$

$$\begin{aligned}
\rho \circ \pi \circ \mathbf{C} &= [\rho^{NN} + \rho^{UU} + \rho^{UN} + \rho^{NU}] \circ [\pi^{\boxplus} + \pi^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}} + \pi^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}] \circ \mathbf{C} \\
&= [\pi^{\boxplus} + \pi^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}} + \pi^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}] \circ [\rho^{NN} + \rho^{UU} + \rho^{UN} + \rho^{NU}] \circ \mathbf{C} \quad = \quad \pi \circ \rho \circ \mathbf{C}
\end{aligned}$$

and

$$\rho \circ \pi \circ \mathbf{C} \quad = \quad -\rho \circ *[\tau \circ [*\mathbf{C}]]$$

in particular

$$\begin{aligned}
[*NN*]^{\boxplus}_{ik} &= \frac{1}{2}\sigma^{ac}_{ik}[*NN*]_{ac} & UU^{\boxplus}_{ik} &= \frac{1}{2}\sigma^{ac}_{ik}UU_{ac} \\
[*NN*]^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}_{ik} &= \frac{1}{2}\alpha^{ac}_{ik}[*NN*]_{ac} & UU^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}_{ik} &= \frac{1}{2}\alpha^{ac}_{ik}UU_{ac} \\
[*NN*]^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ik} &= 0 & UU^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ik} &= 0 \\
[UN*]^{\boxplus}_{ik} &= \frac{1}{2}[[UN*]_{ik} + [*NU]_{ki}] - [UN*]^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ik} & [*NU]^{\boxplus}_{ik} &= [UN*]^{\boxplus}_{ki} \\
[UN*]^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}_{ik} &= \frac{1}{2}[[UN*]_{ik} - [*NU]_{ki}] & [*NU]^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}_{ik} &= -[UN*]^{\begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}}_{ki} \\
[UN*]^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ik} &= \frac{1}{2}\gamma^{ac}_{ik}[[UN*]_{ac} + [*NU]_{ca}] & [*NU]^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ik} &= [UN*]^{\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}}_{ki}
\end{aligned}$$

and

$$[*NN*]^{\boxplus}_{ik} = \frac{1}{2}\eta_i{}^{ab}u^p[*C]^{III}_{kpab} \qquad UU^{\boxplus}_{ik} = -\frac{1}{2}\eta_i{}^{ab}u^p[*C]^{III}_{abkp}$$

$$
\begin{aligned}
[*\mathcal{NN}*]_{ik}^{(2,1,1)} &= -\frac{1}{2}\eta_{ik}{}^{p}u^{q}[*C]^{II}_{qp} & \mathcal{W}_{ik}^{(2,1,1)} &= -\frac{1}{2}\eta_{ik}{}^{p}u^{q}[*C]^{II}_{pq}\\
[\mathcal{UN}*]_{ik}^{\boxplus} &= u^{b}u^{d}[*C]^{III}_{kbid}\\
[\mathcal{UN}*]_{ik}^{(2,1,1)} &= \frac{1}{2}[n^{a}{}_{i}n^{c}{}_{k} - n_{ik}n^{ac}][*C]^{II}_{ca}\\
[\mathcal{UN}*]_{ik}^{(1,1,1,1)} &= -\frac{1}{12}n_{ik}[*C]
\end{aligned}
$$

3.6. $\rho \circ \tau$

$$
\begin{aligned}
\rho\circ\tau\circ\mathbf{C} &= [\rho^{\mathcal{NN}} + \rho^{\mathcal{W}} + \rho^{\mathcal{UN}} + \rho^{\mathcal{NU}}]\circ[\tau^{III} + \tau^{II} + \tau^{I}]\circ\mathbf{C}\\
&= [\tau^{III} + \tau^{II} + \tau^{I}]\circ[\rho^{\mathcal{NN}} + \rho^{\mathcal{W}} + \rho^{\mathcal{UN}} + \rho^{\mathcal{NU}}]\circ\mathbf{C} \quad = \quad \tau\circ\rho\circ\mathbf{C}
\end{aligned}
$$

in particular

$$
\begin{aligned}
[*\mathcal{NN}*]^{III}_{ik} &= \frac{1}{2}[[*\mathcal{NN}*]_{ik} - \mathcal{W}_{ki}] - [*\mathcal{NN}*]^{I}_{ik} & \mathcal{W}^{III}_{ik} &= -[*\mathcal{NN}*]^{III}_{ki}\\
[*\mathcal{NN}*]^{II}_{ik} &= \frac{1}{2}[[*\mathcal{NN}*]_{ik} + \mathcal{W}_{ki}] & \mathcal{W}^{II}_{ik} &= [*\mathcal{NN}*]^{II}_{ki}\\
[*\mathcal{NN}*]^{I}_{ik} &= \frac{1}{2}\gamma^{ac}_{ik}[[*\mathcal{NN}*]_{ac} - \mathcal{W}_{ca}] & \mathcal{W}^{I}_{ik} &= -[*\mathcal{NN}*]^{I}_{ki}\\
[\mathcal{UN}*]^{III}_{ik} &= \frac{1}{2}\sigma^{ac}_{ik}[\mathcal{UN}*]_{ac} & [*\mathcal{NU}]^{III}_{ik} &= \frac{1}{2}\sigma^{ac}_{ik}[*\mathcal{NU}]_{ac}\\
[\mathcal{UN}*]^{II}_{ik} &= \frac{1}{2}\alpha^{ac}_{ik}[\mathcal{UN}*]_{ac} & [*\mathcal{NU}]^{II}_{ik} &= \frac{1}{2}\alpha^{ac}_{ik}[*\mathcal{NU}]_{ac}\\
[\mathcal{UN}*]^{I}_{ik} &= 0 & [*\mathcal{NU}]^{I}_{ik} &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
[*\mathcal{NN}*]^{III}_{ik} &= -u^{p}u^{q}C^{III}_{kpiq}\\
[*\mathcal{NN}*]^{II}_{ik} &= \frac{1}{2}[n_{ik}n^{ac} - n^{c}{}_{i}n^{a}{}_{k}]C^{II}_{ac}\\
[*\mathcal{NN}*]^{I}_{ik} &= \frac{1}{12}n_{ik}C\\
[\mathcal{UN}*]^{III}_{ik} &= \frac{1}{2}\eta_{k}{}^{cd}u^{p}C^{III}_{ipcd} & [*\mathcal{NU}]^{III}_{ik} &= \frac{1}{2}\eta_{i}{}^{ab}u^{q}C^{III}_{abkq}\\
[\mathcal{UN}*]^{II}_{ik} &= -\frac{1}{2}\eta_{ik}{}^{c}u^{a}C^{II}_{ac} & [*\mathcal{NU}]^{II}_{ik} &= \frac{1}{2}\eta_{ik}{}^{a}u^{c}C^{II}_{ac}
\end{aligned}
$$

3.7. $\rho \circ \tau \circ \pi$

$$
\rho\circ\tau\circ\pi\circ\mathbf{C} = [\rho^{\mathcal{NN}} + \rho^{\mathcal{UN}} + \rho^{\mathcal{NU}} + \rho^{\mathcal{W}}]\circ[\tau^{III} + \tau^{II} + \tau^{I}]\circ[\rho^{\boxplus} + \rho^{(2,1,1)} + \rho^{(1,1,1,1)}]\circ\mathbf{C}
$$

and

$$
\begin{aligned}
[*\mathcal{NN}*]^{\boxplus\,III}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}[[*\mathcal{NN}*]_{ac} - \mathcal{W}_{ac}] - [*\mathcal{NN}*]^{\boxplus\,I}_{ik} & \mathcal{W}^{\boxplus\,III}_{ik} &= -[*\mathcal{NN}*]^{\boxplus\,III}_{ik}\\
[*\mathcal{NN}*]^{\boxplus\,II}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}[[*\mathcal{NN}*]_{ac} + \mathcal{W}_{ac}] & \mathcal{W}^{\boxplus\,II}_{ik} &= [*\mathcal{NN}*]^{\boxplus\,II}_{ik}\\
[*\mathcal{NN}*]^{\boxplus\,I}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}\gamma^{pq}_{ac}[[*\mathcal{NN}*]_{pq} - \mathcal{W}_{pq}] & \mathcal{W}^{\boxplus\,I}_{ik} &= -[*\mathcal{NN}*]^{\boxplus\,I}_{ik}
\end{aligned}
$$

$$
\begin{aligned}
[*NN*]_{ik}^{\yng(2,1)\,III} &= \frac{1}{4}\alpha_{ik}^{ac}[[*NN*]_{ac} + W_{ac}] & W_{ik}^{\yng(2,1)\,III} &= [*NN*]_{ik}^{\yng(2,1)\,III} \\
[*NN*]_{ik}^{\yng(2,1)\,II} &= \frac{1}{4}\alpha_{ik}^{ac}[[*NN*]_{ac} - W_{ac}] & W_{ik}^{\yng(2,1)\,II} &= -[*NN*]_{ik}^{\yng(2,1)\,II} \\
[UN*]_{ik}^{III\,\boxplus} &= \frac{1}{4}\sigma_{ik}^{ac}[[UN*]_{ac} + [*NU]_{ac}] - [UN*]_{ik}^{III\,\boxplus} & [*NU]_{ik}^{III\,\boxplus} &= [UN*]_{ik}^{III\,\boxplus} \\
[UN*]_{ik}^{III\,\yng(2,1)} &= \frac{1}{4}\sigma_{ik}^{ac}[[UN*]_{ac} - [*NU]_{ac}] & [*NU]_{ik}^{III\,\yng(2,1)} &= -[UN*]_{ik}^{III\,\yng(2,1)} \\
[UN*]_{ik}^{III\,\yng(1,1,1)} &= \frac{1}{4}\sigma_{ik}^{ac}\gamma_{ac}^{pq}[[UN*]_{pq} + [*NU]_{pq}] & [*NU]_{ik}^{III\,\yng(1,1,1)} &= [UN*]_{ik}^{III\,\yng(1,1,1)} \\
[UN*]_{ik}^{II\,\boxplus} &= \frac{1}{4}\alpha_{ik}^{ac}[[UN*]_{ac} - [*NU]_{ac}] & [*NU]_{ik}^{II\,\boxplus} &= -[UN*]_{ik}^{II\,\boxplus} \\
[UN*]_{ik}^{II\,\yng(2,1)} &= \frac{1}{4}\alpha_{ik}^{ac}[[UN*]_{ac} + [*NU]_{ac}] & [*NU]_{ik}^{II\,\yng(2,1)} &= [UN*]_{ik}^{II\,\yng(2,1)}
\end{aligned}
$$

and

$$\rho\circ\tau\circ\pi\circ\mathbf{C} \quad = \quad -\rho\circ *[\pi\circ\tau\circ[*\mathbf{C}]]$$

therefore

$$
\begin{aligned}
[*NN*]_{ik}^{\boxplus\,III} &= -\frac{1}{2}\sigma_{ik}^{ac}u^b u^d C_{abcd}^{III} \\
&= \frac{1}{4}[\eta_i{}^{ab}n_k^c - \frac{1}{3}n_{ik}\eta^{abc}]u^d[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
[*NN*]_{ik}^{\boxplus\,II} &= \frac{1}{4}\sigma_{ik}^{ac}[n_{ac}u^p u^q - n_a^p n_c^q]C_{pq}^{II} \\
&= -\frac{1}{4}\eta_i{}^{ab}u^d[[*C]_{abkd}^{III} - [*C]_{kdab}^{III}] \\
[*NN*]_{ik}^{\boxplus\,I} &= \frac{1}{12}n_{ik}C \\
&= \frac{1}{12}n_{ik}\eta^{abc}u^d[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
[*NN*]_{ik}^{\yng(2,1)\,III} &= \frac{1}{2}\alpha_{ik}^{ac}u^b u^d C_{abcd}^{III} \\
&= -\frac{1}{4}\eta_{ik}{}^{b}u^d\sigma_{bd}^{pq}[*C]_{pq}^{II} \\
[*NN*]^{\yng(2,1)\,II} &= \frac{1}{4}\alpha_{ik}^{ac}n_a^p n_c^q C_{pq}^{II} \\
&= \frac{1}{4}\eta_{ik}{}^{b}u^d\alpha_{bd}^{pq}[*C]_{pq}^{II} \\
[UN*]_{ik}^{III\,\boxplus} &= \frac{1}{4}[\eta_i{}^{ab}n_k^c - \frac{1}{3}n_{ik}\eta^{abc}]u^d[C_{abcd}^{III} + C_{cdab}^{III}] \\
&= \frac{1}{2}\sigma_{ik}^{ac}u^b u^d[*C]_{abcd}^{III} \\
[UN*]_{ik}^{III\,\yng(2,1)} &= -\frac{1}{4}\eta_i{}^{ab}u^d[C_{abkd}^{III} - C_{kdab}^{III}] \\
&= -\frac{1}{4}\sigma_{ik}^{ac}[n_{ac}u^p u^q - n_a^p n_c^q][*C]_{pq}^{II} \\
[UN*]_{ik}^{III\,\yng(1,1,1)} &= \frac{1}{12}n_{ik}\eta^{abc}u^d[C_{abcd}^{III} + C_{cdab}^{III}] \\
&= -\frac{1}{12}n_{ik}*C
\end{aligned}
$$

$$
\begin{aligned}
[UN*]_{ik}^{II\,\yng(2,2)} &= -\frac{1}{4}\eta_{ik}{}^{b}u^{d}\sigma_{bd}^{pq}C_{pq}^{II} \\
&= -\frac{1}{2}\alpha_{ik}^{ac}u^{b}u^{d}[*C]_{abcd}^{III} \\
[UN*]^{II\,\yng(2,1,1)} &= \frac{1}{4}\eta_{ik}{}^{b}u^{d}\alpha_{bd}^{pq}C_{pq}^{II} \\
&= -\frac{1}{4}\alpha_{ik}^{ac}n_{a}^{p}n_{c}^{q}[*C]_{pq}^{II}
\end{aligned}
$$

and

$$
\begin{aligned}
-\sigma_{ik}^{ac}u^{b}u^{d}C_{abcd}^{III} &= \frac{1}{2}[\eta_{i}{}^{ab}n_{k}^{c} - \frac{1}{3}n_{ik}\eta^{abc}]u^{d}[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
\sigma_{ik}^{ac}[n_{ac}u^{p}u^{q} - n_{a}^{p}n_{c}^{q}]C_{pq}^{II} &= -\eta_{i}{}^{ab}u^{d}[[*C]_{abkd}^{III} - [*C]_{kdab}^{III}] \\
C &= \eta^{abc}u^{d}[[*C]_{abcd}^{III} + [*C]_{cdab}^{III}] \\
\alpha_{ik}^{ac}u^{b}u^{d}C_{abcd}^{III} &= -\frac{1}{2}\eta_{ik}{}^{b}u^{d}\sigma_{bd}^{pq}[*C]_{pq}^{II} \\
\alpha_{ik}^{ac}n_{a}^{p}n_{c}^{q}C_{pq}^{II} &= \eta_{ik}{}^{b}u^{d}\alpha_{bd}^{pq}[*C]_{pq}^{II} \\
\frac{1}{2}[\eta_{i}{}^{ab}n_{k}^{c} - \frac{1}{3}n_{ik}\eta^{abc}]u^{d}[C_{abcd}^{III} + C_{cdab}^{III}] &= \sigma_{ik}^{ac}u^{b}u^{d}[*C]_{abcd}^{III} \\
\eta_{i}{}^{ab}u^{d}[C_{abkd}^{III} - C_{kdab}^{III}] &= \sigma_{ik}^{ac}[n_{ac}u^{p}u^{q} - n_{a}^{p}n_{c}^{q}][*C]_{pq}^{II} \\
\eta^{abc}u^{d}[C_{abcd}^{III} + C_{cdab}^{III}] &= -*C \\
\frac{1}{2}\eta_{ik}{}^{b}u^{d}\sigma_{bd}^{pq}C_{pq}^{II} &= \alpha_{ik}^{ac}u^{b}u^{d}[*C]_{abcd}^{III} \\
\eta_{ik}{}^{b}u^{d}\alpha_{bd}^{pq}C_{pq}^{II} &= -\alpha_{ik}^{ac}n_{a}^{p}n_{c}^{q}[*C]_{pq}^{II}
\end{aligned}
$$

or

$$
\begin{aligned}
u^{b}u^{d}C_{ibkd}^{\yng(2,2)\,III} &= -\frac{1}{2}\eta_{i}{}^{ab}u^{d}[*C]_{abkd}^{III\,\yng(2,2)} \\
[n_{ik}u^{a}u^{c} - n_{i}^{a}n_{k}^{c}]g^{bd}C_{abcd}^{\yng(2,2)\,II} &= -\eta_{i}{}^{ab}u^{d}[*C]_{abkd}^{III\,\yng(2,1,1)} \\
g^{ac}g^{bd}C_{abcd}^{\yng(2,2)\,I} &= 2\eta^{abc}u^{d}[*C]_{abcd}^{III\,\yng(1,1,1,1)} \\
2u^{b}u^{d}C_{ibkd}^{\yng(2,1,1)\,III} &= -\eta_{ik}{}^{a}u^{c}g^{bd}[*C]_{abcd}^{II\,\yng(2,2)} \\
n_{i}^{a}n_{k}^{c}g^{bd}C_{abcd}^{\yng(2,1,1)\,II} &= \eta_{ik}{}^{a}u^{c}g^{bd}[*C]_{abcd}^{II\,\yng(2,1,1)} \\
\eta_{i}{}^{ab}u^{d}C_{abkd}^{III\,\yng(2,2)} &= 2u^{b}u^{d}[*C]_{ibkd}^{\yng(2,2)\,III} \\
\eta_{i}{}^{ab}u^{d}C_{abkd}^{III\,\yng(2,1,1)} &= [n_{ik}u^{a}u^{c} - n_{i}^{a}n_{k}^{c}]g^{bd}[*C]_{abcd}^{\yng(2,2)\,II} \\
2\eta^{abc}u^{d}C_{abcd}^{III\,\yng(1,1,1,1)} &= -g^{ac}g^{bd}[*C]_{abcd}^{\yng(2,2)\,I} \\
\eta_{ik}{}^{a}u^{c}g^{bd}C_{abcd}^{II\,\yng(2,2)} &= 2u^{b}u^{d}[*C]_{ibkd}^{\yng(2,1,1)\,III} \\
\eta_{ik}{}^{a}u^{c}g^{bd}C_{abcd}^{II\,\yng(2,1,1)} &= -n_{i}^{a}n_{k}^{c}g^{bd}[*C]_{abcd}^{\yng(2,1,1)\,II}
\end{aligned}
$$

3.8. $\omega \circ \rho$

Let $u = (0,0,0,1)^{T}$.

$$
\omega_{ac}^{pq}[*NN*]_{pq} = \sum_{\alpha=1}^{3}[*N]_{\alpha a}^{\hat{N}}[*N]_{\alpha c}^{N\hat{N}}
$$

$$
\begin{aligned}
\omega_{ac}^{pq} UU_{pq} &= \sum_{\alpha=1}^{3} U_{\alpha a}^{\hat{U}U} U_{\alpha c}^{U\hat{U}} \\
\omega_{ac}^{pq} [UN*]_{pq} &= \sum_{\alpha=1}^{3} U_{\alpha a}^{UN} [*N]_{\alpha c}^{UN} \\
\omega_{ac}^{pq} [*NU]_{pq} &= \sum_{\alpha=1}^{3} [*N]_{\alpha a}^{NU} U_{\alpha c}^{NU}
\end{aligned}
$$

That gives

$$
\begin{aligned}
\omega \circ C_{ijkl}^{NN} &= \sum_{\alpha=1}^{3} {\eta_{ij}}^{a} [*N]_{\alpha a}^{\hat{N}N} {\eta_{kl}}^{c} [*N]_{\alpha c}^{N\hat{N}} &&= \sum_{\alpha=1}^{3} N_{\alpha ij}^{\hat{N}N} N_{\alpha kl}^{N\hat{N}} \\
\omega \circ C_{ijkl}^{UU} &= \sum_{\alpha=1}^{3} [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{\hat{U}U}][-\alpha_{kl}^{cd} u_d U_{\alpha c}^{U\hat{U}}] &&= \sum_{\alpha=1}^{3} U_{\alpha ij}^{\hat{U}U} U_{\alpha kl}^{U\hat{U}} \\
\omega \circ C_{ijkl}^{UN} &= \sum_{\alpha=1}^{3} [-\alpha_{ij}^{ab} u_b U_{\alpha a}^{UN}] {\eta_{kl}}^{c} [*N]_{\alpha c}^{UN} &&= \sum_{\alpha=1}^{3} U_{\alpha ij}^{UN} N_{\alpha kl}^{UN} \\
\omega \circ C_{ijkl}^{NU} &= \sum_{\alpha=1}^{3} [-\alpha_{kl}^{cd} u_d U_{\alpha c}^{NU}] {\eta_{ij}}^{a} [*N]_{\alpha a}^{NU} &&= \sum_{\alpha=1}^{3} N_{\alpha ij}^{NU} U_{\alpha kl}^{NU}
\end{aligned}
$$

so

$$
\omega \circ \rho \circ C_{ijkl} = \sum_{\alpha=1}^{3} [N_{\alpha ij}^{\hat{N}N} N_{\alpha kl}^{N\hat{N}} + U_{\alpha ij}^{\hat{U}U} U_{\alpha kl}^{U\hat{U}} + U_{\alpha ij}^{UN} N_{\alpha kl}^{UN} + N_{\alpha ij}^{NU} U_{\alpha kl}^{NU}]
$$

N_α and U_α have rank 2. N_α are spacelike and U_α are timelike. Setting

$$
\begin{aligned}
N_{\alpha ij}^{+\hat{N}U} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}N} + N_{\alpha ij}^{NU}] &\qquad U_{\alpha ij}^{+\hat{U}N} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}U} + U_{\alpha ij}^{UN}] \\
N_{\alpha ij}^{-\hat{N}U} &:= \frac{1}{2}[N_{\alpha ij}^{\hat{N}N} - N_{\alpha ij}^{NU}] &\qquad U_{\alpha ij}^{-\hat{U}N} &:= \frac{1}{2}[U_{\alpha ij}^{\hat{U}U} - U_{\alpha ij}^{UN}] \\
N_{\alpha ij}^{+U\hat{N}} &:= \frac{1}{2}[N_{\alpha ij}^{N\hat{N}} + N_{\alpha ij}^{UN}] &\qquad U_{\alpha ij}^{+N\hat{U}} &:= \frac{1}{2}[U_{\alpha ij}^{U\hat{U}} + U_{\alpha ij}^{NU}] \\
N_{\alpha ij}^{-U\hat{N}} &:= \frac{1}{2}[N_{\alpha ij}^{N\hat{N}} - N_{\alpha ij}^{UN}] &\qquad U_{\alpha ij}^{-N\hat{U}} &:= \frac{1}{2}[U_{\alpha ij}^{U\hat{U}} - U_{\alpha ij}^{NU}]
\end{aligned}
$$

the factorization becomes

$$
\begin{aligned}
[\omega \circ \rho \circ C]_{ijkl} = \sum_{\alpha=1}^{3} [& [N_{\alpha ij}^{+\hat{N}U} + U_{\alpha ij}^{+\hat{U}N}][N_{\alpha kl}^{+U\hat{N}} + U_{\alpha kl}^{+N\hat{U}}] + [N_{\alpha ij}^{-\hat{N}U} + U_{\alpha ij}^{-\hat{U}N}][N_{\alpha kl}^{-U\hat{N}} + U_{\alpha kl}^{-N\hat{U}}] \\
& + [N_{\alpha ij}^{+\hat{N}U} - U_{\alpha ij}^{+\hat{U}N}][N_{\alpha kl}^{-U\hat{N}} - U_{\alpha kl}^{-N\hat{U}}] + [N_{\alpha ij}^{-\hat{N}U} - U_{\alpha ij}^{-\hat{U}N}][N_{\alpha kl}^{+U\hat{N}} - U_{\alpha kl}^{+N\hat{U}}]]
\end{aligned}
$$

$[N+U]_{\alpha ij}$ and $[N-U]_{\alpha ij}$ have rank 4.

4. R

4.1. R

4.1.1. π

$$\pi \circ \mathbf{R} \quad = \quad \mathbf{R}^{\boxplus}$$

4.1.2. τ

$$\tau \circ \mathbf{R} \quad = \quad \mathbf{R}^{III} + \mathbf{R}^{II} + \mathbf{R}^{I}$$

with

$$\begin{aligned} R^{III}_{ijkl} &:= R_{ijkl} - R^{II}_{ijkl} - R^{I}_{ijkl} \\ R^{II}_{ijkl} &:= \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}[R_{bd} - \frac{1}{4}g_{bd}R] \quad =: \quad \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}R^{II}_{bd} \\ R^{I}_{ijkl} &:= \frac{1}{24}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}g_{bd}R \end{aligned}$$

and

$$\begin{aligned} g^{jl}R_{ijkl} &=: R_{ik} \\ g^{jl}g^{ik}R_{ijkl} &=: R \end{aligned}$$

4.2. $*\mathbf{R}$

4.2.1. τ

$$\tau \circ [*\mathbf{R}] \quad = \quad [*\mathbf{R}]^{III}$$

4.2.2. π

$$\pi \circ [*\mathbf{R}] \quad = \quad [*R]^{\boxplus}_{ijkl} + [*R]^{(2,1,1)}_{ijkl} + [*R]^{(1,1,1,1)}_{ijkl}$$

with

$$\begin{aligned} [*R]^{\boxplus}_{ijkl} &= \frac{1}{2}\eta_{ij}{}^{ab}R^{III}_{abkl} &&= *[R^{III}]_{ijkl} \\ [*R]^{(2,1,1)}_{ijkl} &= \frac{1}{2}\alpha^{cd}_{kl}\eta_{ijc}{}^{p}R^{II}_{pd} &&= *[R^{II}]_{ijkl} \\ [*R]^{(1,1,1,1)}_{ijkl} &= \frac{1}{12}\eta_{ijkl}R &&= *[R^{I}]_{ijkl} \end{aligned}$$

and

$$\begin{aligned} [*R]^{(2,1,1)}_{ijkl} &= \frac{1}{2}[[*R]_{ijkl} - [R*]_{ijkl}] &&= *[R^{II}]_{ijkl} \\ [*R]^{\boxplus}_{ijkl} + [*R]^{(1,1,1,1)}_{ijkl} &= \frac{1}{2}[[*R]_{ijkl} + [R*]_{ijkl}] &&= *[R^{III}]_{ijkl} + *[R^{I}]_{ijkl} \end{aligned}$$

4.3. $*\mathbf{R}*$

4.3.1. π

$$\pi^{abcd}_{ijkl}[*R*]_{abcd} \quad = \quad [*R*]^{\boxplus}_{ijkl} \quad = \quad -R_{ijkl} + \alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}R_{bd} - \frac{1}{4}\alpha^{ab}_{ij}\alpha^{cd}_{kl}g_{ac}g_{bd}R$$

4.3.2. τ

$$\tau^{abcd}_{ijkl}[*R*]_{abcd} \quad = \quad [*R*]^{III}_{ijkl} + [*R*]^{II}_{ijkl} + [*R*]^{I}_{ijkl} \quad = \quad -R^{III}_{ijkl} + R^{II}_{ijkl} - R^{I}_{ijkl}$$

and

$$\begin{aligned} R^{III}_{ijkl} + R^{I}_{ijkl} &= \frac{1}{2}[R_{ijkl} - [*R*]_{ijkl}] \\ R^{II}_{ijkl} &= \frac{1}{2}[R_{ijkl} + [*R*]_{ijkl}] \end{aligned}$$

with

$$\begin{aligned} g^{jl}[*R*]_{ijkl} &=: [*R*]_{ik} = R_{ik} - \frac{1}{2}Rg_{ik} \\ g^{jl}g^{ik}[*R*]_{ijkl} &=: *R* = -R \end{aligned}$$

4.4. ρ

4.4.1. $\mathbf{R}$

$$\rho \circ \mathbf{R} \quad = \quad \mathbf{R}^{NN} + \mathbf{R}^{UN} + \mathbf{R}^{NU} + \mathbf{R}^{UU}$$

with

$$\begin{aligned} R^{NN}_{ijkl} &:= R_{abcd}[\rho^{NN}]^{abcd}_{ijkl} &&:= R^{abcd}n_{ia}n_{jb}n_{kc}n_{ld} \\ &=: NN^{\boxplus}_{ijkl} &&= [*NN*]^{\boxplus}_{ac}\eta_{ij}{}^{a}\eta_{kl}{}^{c} \\ R^{UU}_{ijkl} &:= R_{abcd}[\rho^{UU}]^{abcd}_{ijkl} &&:= R^{pqrs}n_{ap}u_{q}n_{cr}u_{s}[-\alpha^{ab}_{ij}u_{b}][-\alpha^{cd}_{kl}u_{d}] \\ &=: UU^{\boxplus}_{ac}[-\alpha^{ab}_{ij}u_{b}][-\alpha^{cd}_{kl}u_{d}] && \\ R^{UN}_{ijkl} &:= R_{abcd}[\rho^{UN}]^{abcd}_{ijkl} &&:= R^{pqcd}n_{ap}u_{q}n_{kc}n_{ld}[-\alpha^{ab}_{ij}u_{b}] \\ &=: [-\alpha^{ab}_{ij}u_{b}]UN^{\boxplus}_{akl} &&= [UN*]^{\boxplus}_{ac}[-\alpha^{ab}_{ij}u_{b}]\eta_{kl}{}^{c} \\ R^{NU}_{ijkl} &:= R_{abcd}[\rho^{NU}]^{abcd}_{ijkl} &&:= R^{abrs}n_{ia}n_{jb}n_{cr}u_{s}[-\alpha^{cd}_{kl}u_{d}] \\ &=: [-\alpha^{cd}_{kl}u_{d}]NU^{\boxplus}_{ijc} &&= [*NU]^{\boxplus}_{ac}\eta_{ij}{}^{a}[-\alpha^{cd}_{kl}u_{d}] \end{aligned}$$

4.4.2. $*\mathbf{R}$

$$\begin{aligned} [*R]^{NN}_{ijkl} &= -[*UN]^{\boxplus}_{ijkl} &&= -\eta_{ij}{}^{a}\eta_{kl}{}^{c}[UN*]^{\boxplus}_{ac} &&= *[R^{UN}]_{ijkl} \\ [*R]^{UU}_{ijkl} &= [-\alpha^{ab}_{ij}u_{b}][-\alpha^{cd}_{kl}u_{d}][*NU]^{\boxplus}_{ac} && &&= *[R^{NU}]_{ijkl} \\ [*R]^{UN}_{ijkl} &= [-\alpha^{ab}_{ij}u_{b}][*NN]^{\boxplus}_{akl} &&= [-\alpha^{ab}_{ij}u_{b}]\eta_{kl}{}^{c}[*NN*]^{\boxplus}_{ac} &&= *[R^{NN}]_{ijkl} \\ [*R]^{NU}_{ijkl} &= -[-\alpha^{cd}_{kl}u_{d}][*UU]^{\boxplus}_{ijc} &&= -\eta_{ij}{}^{a}[-\alpha^{cd}_{kl}u_{d}]UU^{\boxplus}_{ac} &&= *[R^{UU}]_{ijkl} \end{aligned}$$

4.4.3. $*\mathbf{R}*$

$$
\begin{aligned}
[*R*]^{NN}_{ijkl} &= [*UU*]^{\boxplus}_{ijkl} &&= \eta_{ij}{}^{a}\eta_{kl}{}^{c}UU^{\boxplus}_{ac} &&= [*[R^{UU}]*]_{ijkl}\\
[*R*]^{UU}_{ijkl} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d][*NN*]^{\boxplus}_{ac} &&&&= [*[R^{NN}]*]_{ijkl}\\
[*R*]^{UN}_{ijkl} &= -[-\alpha^{ab}_{ij}u_b][*NU*]^{\boxplus}_{akl} &&= -[-\alpha^{ab}_{ij}u_b]\eta_{kl}{}^{c}[*NU]^{\boxplus}_{ac} &&= [*[R^{NU}]*]_{ijkl}\\
[*R*]^{NU}_{ijkl} &= -[-\alpha^{cd}_{kl}u_d][*UN*]^{\boxplus}_{ijc} &&= -\eta_{ij}{}^{a}[-\alpha^{cd}_{kl}u_d][UN*]^{\boxplus}_{ac} &&= [*[R^{UN}]*]_{ijkl}
\end{aligned}
$$

4.5. $\rho\circ\tau$

$$\rho\circ\tau\circ\mathbf{R} \quad = \quad [\rho^{NN}+\rho^{UN}+\rho^{NU}+\rho^{UU}]\circ[\tau^{III}+\tau^{II}+I]\circ\mathbf{R}$$

and

$$
\begin{aligned}
[*NN*]^{\boxplus\,III}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}[[*NN*]^{\boxplus}_{ac}-UU^{\boxplus}_{ac}]-[*NN*]^{\boxplus\,I}_{ik} & UU^{\boxplus\,III}_{ik} &= -[*NN*]^{\boxplus\,III}_{ik}\\
[*NN*]^{\boxplus\,II}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}[[*NN*]^{\boxplus}_{ac}+UU^{\boxplus}_{ac}] & UU^{\boxplus\,II}_{ik} &= [*NN*]^{\boxplus\,II}_{ik}\\
[*NN*]^{\boxplus\,I}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}\gamma^{pq}_{ac}[[*NN*]^{\boxplus}_{pq}-UU^{\boxplus}_{pq}] & UU^{\boxplus\,I}_{ik} &= -[*NN*]^{\boxplus\,I}_{ik}\\
[UN*]^{III\,\boxplus}_{ik} &= \frac{1}{4}\sigma^{ac}_{ik}[[UN*]^{\boxplus}_{ac}+[*NU]^{\boxplus}_{ac}] & [*NU]^{III\,\boxplus}_{ik} &= [UN*]^{III\,\boxplus}_{ik}\\
[UN*]^{II\,\boxplus}_{ik} &= \frac{1}{4}\alpha^{ac}_{ik}[[UN*]^{\boxplus}_{ac}-[*NU]^{\boxplus}_{ac}] & [*NU]^{II\,\boxplus}_{ik} &= -[UN*]^{II\,\boxplus}_{ik}
\end{aligned}
$$

and

$$
\begin{aligned}
[*NN*]^{\boxplus\,III}_{ik} &= -u^b u^d R^{III}_{ibkd}\\
&= \frac{1}{4}[\eta_i{}^{ab}n^c{}_k-\frac{1}{3}n_{ik}\eta^{abc}]u^d[[*R]_{abcd}+[*R]_{cdab}]\\
[*NN*]^{\boxplus\,II}_{ik} &= \frac{1}{2}[n_{ik}u^p u^q-n^p_i n^q_k]R^{II}_{pq}\\
&= -\frac{1}{4}\eta_i{}^{ab}u^d[[*R]_{abkd}-[*R]_{kdab}]\\
[*NN*]^{\boxplus\,I}_{ik} &= \frac{1}{12}n_{ik}R\\
&= \frac{1}{12}n_{ik}\eta^{abc}u^d[[*R]_{abcd}+[*R]_{cdab}]\\
[UN*]^{III\,\boxplus}_{ik} &= \frac{1}{2}\eta_i{}^{ab}u^d R^{III}_{kdab}\\
&= \frac{1}{2}\sigma^{ac}_{ik}u^b u^d[*R]_{abcd}\\
[UN*]^{II\,\boxplus}_{ik} &= -\frac{1}{2}\eta_{ik}{}^{b}u^d R^{II}_{bd}\\
&= -\frac{1}{2}\alpha^{ac}_{ik}u^b u^d[*R]_{abcd}
\end{aligned}
$$

or

$$
\begin{aligned}
-u^b u^d R^{III}_{ibkd} &= \frac{1}{2}\eta_i{}^{ab}u^d[*R]^{\boxplus}_{abkd}\\
-[n_{ik}u^p u^q-n^p_i n^q_k]R^{II}_{pq} &= \eta_i{}^{ab}u^d[*R]^{\boxplus}_{abkd}
\end{aligned}
$$

$$
\begin{aligned}
R &= 2\eta^{abc}u^d[*R]^{\yng(1,1,1,1)}_{abcd} \\
\eta_i{}^{ab}u^d R^{III}_{abkd} &= 2u^b u^d [*R]^{\boxplus}_{ibkd} \\
\eta_{ik}{}^{a}u^c R^{II}_{ac} &= 2u^b u^d [*R]^{\yng(2,1,1)}_{ibkd}
\end{aligned}
$$

4.6. $\omega \circ \rho$

$$
\begin{aligned}
[\omega\circ\sigma]^{pq}_{ac}[*NN*]^{\boxplus}_{pq} &= \sum_{\alpha=1}^{3} *N^{\boxplus}_{\alpha a}{}^{NN} *N^{\boxplus}_{\alpha c}{}^{NN} \\
[\omega\circ\sigma]^{pq}_{ac}UU^{\boxplus}_{pq} &= \sum_{\alpha=1}^{3} U^{\boxplus}_{\alpha a}{}^{UU} U^{\boxplus}_{\alpha c}{}^{UU} \\
\omega^{pq}_{ac}[UN*]^{\boxplus}_{pq} &= \sum_{\alpha=1}^{3} U^{\boxplus}_{\alpha a}{}^{UN} *N^{\boxplus}_{\alpha c}{}^{UN} \\
\omega^{pq}_{ac}[*NU]^{\boxplus}_{pq} &= \sum_{\alpha=1}^{3} *N^{\boxplus}_{\alpha a}{}^{UN} U^{\boxplus}_{\alpha c}{}^{UN}
\end{aligned}
$$

leads to

$$
\begin{aligned}
\omega\circ R^{NN}_{ijkl} &= \sum_{\alpha=1}^{3} \eta_{ij}{}^{a} *N^{\boxplus}_{\alpha a}{}^{NN} \eta_{kl}{}^{c} *N^{\boxplus}_{\alpha c}{}^{NN} &&= \sum_{\alpha=1}^{3} N^{\boxplus}_{\alpha ij}{}^{NN} N^{\boxplus}_{\alpha kl}{}^{NN} \\
\omega\circ R^{UU}_{ijkl} &= \sum_{\alpha=1}^{3} [-\alpha^{ab}_{ij}u_b U^{\boxplus}_{\alpha a}{}^{UU}][-\alpha^{cd}_{kl}u_d U^{\boxplus}_{\alpha c}{}^{UU}] &&= \sum_{\alpha=1}^{3} U^{\boxplus}_{\alpha ij}{}^{UU} U^{\boxplus}_{\alpha kl}{}^{UU} \\
\omega\circ R^{UN}_{ijkl} &= \sum_{\alpha=1}^{3} [-\alpha^{ab}_{ij}u_b U^{\boxplus}_{\alpha a}{}^{UN}]\eta_{kl}{}^{c} *N^{\boxplus}_{\alpha c}{}^{UN} &&= \sum_{\alpha=1}^{3} U^{\boxplus}_{\alpha ij}{}^{UN} N^{\boxplus}_{\alpha kl}{}^{UN} \\
\omega\circ R^{NU}_{ijkl} &= \sum_{\alpha=1}^{3} \eta_{ij}{}^{a} *N^{\boxplus}_{\alpha a}{}^{UN}[-\alpha^{cd}_{kl}u_d U^{\boxplus}_{\alpha c}{}^{UN}] &&= \sum_{\alpha=1}^{3} N^{\boxplus}_{\alpha ij}{}^{UN} U^{\boxplus}_{\alpha kl}{}^{UN}
\end{aligned}
$$

so

$$
\begin{aligned}
[\omega\circ\rho\circ R]_{ijkl} &= \sum_{\alpha=1}^{3}[N^{\boxplus}_{\alpha ij}{}^{NN} N^{\boxplus}_{\alpha kl}{}^{NN} + U^{\boxplus}_{\alpha ij}{}^{UU} U^{\boxplus}_{\alpha kl}{}^{UU} + N^{\boxplus}_{\alpha ij}{}^{UN} U^{\boxplus}_{\alpha kl}{}^{UN} + U^{\boxplus}_{\alpha ij}{}^{UN} N^{\boxplus}_{\alpha kl}{}^{UN}] \\
&= \sum_{\alpha=1}^{3}[N^{\boxplus}_{\alpha ij}{}^{NN} N^{\boxplus}_{\alpha kl}{}^{NN} + U^{\boxplus}_{\alpha ij}{}^{UU} U^{\boxplus}_{\alpha kl}{}^{UU} + N^{\boxplus}_{\alpha ij}{}^{UN} N^{\boxplus}_{\alpha kl}{}^{UN} + U^{\boxplus}_{\alpha ij}{}^{UN} U^{\boxplus}_{\alpha kl}{}^{UN} - Z^{-UN}_{\alpha ij} Z^{-UN}_{\alpha kl}] \\
&= \sum_{\alpha=1}^{3}[N^{\boxplus}_{\alpha ij}{}^{NN} N^{\boxplus}_{\alpha kl}{}^{NN} + U^{\boxplus}_{\alpha ij}{}^{UU} U^{\boxplus}_{\alpha kl}{}^{UU} - N^{\boxplus}_{\alpha ij}{}^{UN} N^{\boxplus}_{\alpha kl}{}^{UN} - U^{\boxplus}_{\alpha ij}{}^{UN} U^{\boxplus}_{\alpha kl}{}^{UN} + Z^{+UN}_{\alpha ij} Z^{+UN}_{\alpha kl}]
\end{aligned}
$$

with

$$
\begin{aligned}
Z^{-UN}_{\alpha ij} &:= N^{\boxplus}_{\alpha ij}{}^{UN} - U^{\boxplus}_{\alpha ij}{}^{UN} \\
Z^{+UN}_{\alpha ij} &:= N^{\boxplus}_{\alpha ij}{}^{UN} + U^{\boxplus}_{\alpha ij}{}^{UN}
\end{aligned}
$$

N_α and U_α have rank 2, $Z^{\pm UN}_\alpha$ have rank 4. N_α are spacelike, while U_α are timelike. Setting

$$
N^{+}_{\alpha ij} := \frac{1}{2}[N^{\boxplus}_{\alpha ij}{}^{NN} + N^{\boxplus}_{\alpha ij}{}^{UN}] \qquad U^{+}_{\alpha ij} := \frac{1}{2}[U^{\boxplus}_{\alpha ij}{}^{UU} + U^{\boxplus}_{\alpha ij}{}^{UN}]
$$

$$N^-_{\alpha ij} := \frac{1}{2}[N^{\boxplus NN}_{\alpha ij} - N^{\boxplus UN}_{\alpha ij}] \qquad U^-_{\alpha ij} := \frac{1}{2}[U^{\boxplus UU}_{\alpha ij} - U^{\boxplus UN}_{\alpha ij}]$$

the factorization becomes

$$\begin{aligned}
[\omega \circ \rho \circ R]_{ijkl} &= \sum_{\alpha=1}^{3} [2N^+_{\alpha ij}N^+_{\alpha kl} + 2N^-_{\alpha ij}N^-_{\alpha kl} + 2U^+_{\alpha ij}U^+_{\alpha kl} + 2U^-_{\alpha ij}U^-_{\alpha kl} - Z^{-UN}_{\alpha ij}Z^{-UN}_{\alpha kl}] \\
&= \sum_{\alpha=1}^{3} [2N^+_{\alpha ij}N^-_{\alpha kl} + 2N^-_{\alpha ij}N^+_{\alpha kl} + 2U^+_{\alpha ij}U^-_{\alpha kl} + 2U^-_{\alpha ij}U^+_{\alpha kl} + Z^{+UN}_{\alpha ij}Z^{+UN}_{\alpha kl}]
\end{aligned}$$

Setting

$$\begin{aligned}
\overset{++}{Z}{}^{+}_{\alpha ij} &= [N^+_{\alpha ij} + U^+_{\alpha ij}] & \overset{--}{Z}{}^{+}_{\alpha ij} &= [N^-_{\alpha ij} + U^-_{\alpha ij}] \\
\overset{++}{Z}{}^{-}_{\alpha ij} &= [N^+_{\alpha ij} - U^+_{\alpha ij}] & \overset{--}{Z}{}^{-}_{\alpha ij} &= [N^-_{\alpha ij} - U^-_{\alpha ij}] \\
\overset{+-}{Z}{}^{-}_{\alpha ij} &= [N^+_{\alpha ij} - U^-_{\alpha ij}] & \overset{-+}{Z}{}^{-}_{\alpha ij} &= [N^-_{\alpha ij} - U^+_{\alpha ij}] \\
\overset{+-}{Z}{}^{+}_{\alpha ij} &= [N^+_{\alpha ij} + U^-_{\alpha ij}] & \overset{-+}{Z}{}^{+}_{\alpha ij} &= [N^-_{\alpha ij} + U^+_{\alpha ij}]
\end{aligned}$$

the factorization becomes

$$\begin{aligned}
[\omega \circ \rho \circ R]_{ijkl} &= \sum_{\alpha=1}^{3} [\overset{++}{Z}{}^{+}_{\alpha ij}\overset{++}{Z}{}^{+}_{\alpha kl} + \overset{--}{Z}{}^{+}_{\alpha ij}\overset{--}{Z}{}^{+}_{\alpha kl} + \overset{++}{Z}{}^{-}_{\alpha ij}\overset{--}{Z}{}^{-}_{\alpha kl} + \overset{--}{Z}{}^{-}_{\alpha ij}\overset{++}{Z}{}^{-}_{\alpha kl}] \\
&= \sum_{\alpha=1}^{3} [\overset{+-}{Z}{}^{-}_{\alpha ij}\overset{+-}{Z}{}^{-}_{\alpha kl} + \overset{-+}{Z}{}^{-}_{\alpha ij}\overset{-+}{Z}{}^{-}_{\alpha kl} + \overset{+-}{Z}{}^{+}_{\alpha ij}\overset{-+}{Z}{}^{+}_{\alpha kl} + \overset{-+}{Z}{}^{+}_{\alpha ij}\overset{+-}{Z}{}^{+}_{\alpha kl}] \\
&= \sum_{\alpha=1}^{3} [\overset{++}{Z}{}^{+}_{\alpha ij}\overset{++}{Z}{}^{+}_{\alpha kl} + \overset{--}{Z}{}^{+}_{\alpha ij}\overset{--}{Z}{}^{+}_{\alpha kl} + \overset{+-}{Z}{}^{-}_{\alpha ij}\overset{+-}{Z}{}^{-}_{\alpha kl} + \overset{-+}{Z}{}^{-}_{\alpha ij}\overset{-+}{Z}{}^{-}_{\alpha kl} - N^{\boxplus UN}_{\alpha ij}N^{\boxplus UN}_{\alpha kl} - U^{\boxplus UN}_{\alpha ij}U^{\boxplus UN}_{\alpha kl}] \\
&=: \sum_{\alpha} Z_{\alpha ij}Z_{\alpha kl}
\end{aligned}$$

$\overset{++}{Z}{}^{+}_{\alpha ij}, \ldots, \overset{-+}{Z}{}^{-}_{\alpha ij}$ have rank 4. Comparing

$$[\tau \circ R]_{ijkl} = [\tau \circ [\omega \circ \rho \circ R]]_{ijkl}$$

gives

$$\begin{aligned}
R^{III}_{ijkl} &= \sum_{\alpha} [Z_{\alpha ij}Z_{\alpha kl} - \gamma^{abcd}_{ijkl} Z_{\alpha ab}Z_{\alpha cd}] &&=: \sum_{\alpha} [Z_{\alpha ij}Z_{\alpha kl}]^{III} \\
R^{II}_{ijkl} &= \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac} g^{pr} [\delta^q_b \delta^s_d - \frac{1}{4} g_{bd} g^{qs}] \sum_{\alpha} Z_{\alpha pq}Z_{\alpha rs} && \\
&=: \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac} \sum_{\alpha} [Z_{\alpha b}{}^{q} Z_{\alpha dq}]^{II} &&=: \sum_{\alpha} [Z_{\alpha ij}Z_{\alpha kl}]^{II} \\
R^{I}_{ijkl} &= \frac{1}{24}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac} g_{bd} g^{pr} g^{qs} [\sum_{\alpha} Z_{\alpha pq}Z_{\alpha rs}] && \\
&=: \frac{1}{2}\alpha^{ab}_{ij}\alpha^{cd}_{kl} g_{ac} g_{bd} \sum_{\alpha} Z_{\alpha}{}^{pq} Z_{\alpha pq} &&=: \sum_{\alpha} [Z_{\alpha ij}Z_{\alpha kl}]^{I}
\end{aligned}$$

5. $\nabla \mathbf{C}$

5.0.1. $\nabla[\pi \circ \mathbf{C}]$

$$\nabla[\pi \circ \mathbf{C}] \quad = \quad \nabla \mathbf{C}^{\yng(2,2)} + \nabla \mathbf{C}^{\yng(2,1,1)} + \nabla \mathbf{C}^{\yng(1,1,1,1)}$$

The distribution of components reads $144 = 80 + 60 + 4$.

5.0.2. $\nabla[\tau \circ \mathbf{C}]$

$$\nabla[\tau \circ \mathbf{C}] \quad = \quad \nabla \mathbf{C}^{III} + \nabla \mathbf{C}^{II} + \nabla \mathbf{C}^{I}$$

with

$$\begin{aligned}
\nabla_m C^{III}_{ijkl} \quad &= \quad \nabla_m C_{ijkl} \quad - \quad \nabla_m C^{II}_{ijkl} - \nabla_m C^{I}_{ijkl} \\
\nabla_m C^{II}_{ijkl} \quad &= \quad \alpha^{ab}_{ij} \alpha^{cd}_{kl} [g_{ac} \frac{1}{2} \nabla_m C^{II}_{bd}] \\
\nabla_m C^{I}_{ijkl} \quad &= \quad \alpha^{ab}_{ij} \alpha^{cd}_{kl} [g_{ac} g_{bd} \frac{1}{24} \nabla_m C]
\end{aligned}$$

The distribution of components reads $144 = 80 + 60 + 4$.

5.0.3. $\nabla[\tau \circ \pi \circ \mathbf{C}]$

$$\nabla[\tau \circ \pi \circ \mathbf{C}] \quad = \quad \nabla[\mathbf{C}^{\yng(2,2)\,III} + \mathbf{C}^{\yng(2,2)\,II} + \mathbf{C}^{\yng(2,2)\,I}] + \nabla[\mathbf{C}^{\yng(2,1,1)\,III} + \mathbf{C}^{\yng(2,1,1)\,II}] + \nabla \mathbf{C}^{\yng(1,1,1,1)\,III}$$

The distribution of components reads $144 = 40 + 36 + 4 + 36 + 24 + 4$.

5.0.4. $\nabla[\rho \circ \mathbf{C}]$

$$\nabla[\rho \circ \mathbf{C}] \quad = \quad \nabla \mathbf{C}^{NN} + \nabla \mathbf{C}^{UU} + \nabla \mathbf{C}^{UN} + \nabla \mathbf{C}^{NU}$$

The distribution of components reads $144 = 36 + 36 + 36 + 36$.

5.1. π

$$\pi \circ \nabla \mathbf{C} \quad = \quad [\nabla \mathbf{C}]^{\yng(3,2)} + [\nabla \mathbf{C}]^{\yng(3,1,1)} + [\nabla \mathbf{C}]^{\yng(2,2,1)} + [\nabla \mathbf{C}]^{\yng(2,1,1,1)}$$

with

$$\begin{aligned}
[\nabla_m C_{ijkl}]^{\yng(3,2)} \quad &= \quad \frac{1}{24} \alpha^{pr}_{ij} \alpha^{qs}_{kl} \sigma^{bd}_{rs} \sigma^{ace}_{pqm} \nabla_e C_{abcd} \\
&= \quad \frac{1}{24} \alpha^{ab}_{ij} \alpha^{cd}_{kl} \sigma^{rs}_{bd} [\nabla_m [C_{arcs} + C_{csar}] + \nabla_c [C_{mras} + C_{asmr}] + \nabla_a [C_{crms} + C_{mscr}]] \\
[\nabla_m C_{ijkl}]^{\yng(3,1,1)} \quad &= \quad \frac{1}{20} \alpha^{xy}_{kl} \alpha^{pbc}_{ijx} \sigma^{ade}_{pym} \nabla_e C_{abcd} \\
&= \quad \frac{1}{20} \alpha^{xy}_{kl} \alpha^{pbc}_{ijx} [\nabla_m [C_{pbcy} - C_{cypb}] + \nabla_y [C_{mbcp} - C_{cpmb}] + \nabla_p [C_{ybcm} - C_{cmyb}]] \\
[\nabla_m C_{ijkl}]^{\yng(2,2,1)} \quad &= \quad \frac{1}{24} [\alpha^{pre}_{ijm} \alpha^{qs}_{kl} \sigma^{bd}_{rs} \sigma^{ac}_{pq} + \alpha^{xy}_{kl} \alpha^{prc}_{ijx} \alpha^{qs}_{ym} \sigma^{be}_{rs} \sigma^{ad}_{pq}] \nabla_e C_{abcd}
\end{aligned}$$

$$
\begin{aligned}
&= [\nabla_m C_{ijkl}]^{\young(ik,jl,m)} + \alpha^{xy}_{kl}[\nabla_m C_{ijxy}]^{\young(iy,jm,x)} \\
&= \frac{1}{24}\alpha^{qs}_{kl}[\alpha^{pre}_{ijm}\sigma^{bd}_{rs}\nabla_e[C_{pbqd}+C_{qdpb}] + \alpha^{prc}_{ijq}\sigma^{xy}_{mc}\nabla_r[C_{pxys}+C_{yspx}]] \\
&\quad +\frac{1}{24}\alpha^{qs}_{kl}\alpha^{prc}_{ijq}[\alpha^{xy}_{sm}\nabla_y[C_{prcx}-C_{cxpr}] + \nabla_r[C_{smcp}-C_{cpsm}]]
\end{aligned}
$$

$$
\begin{aligned}
[\nabla_m C_{ijkl}]^{\yng(2,1,1,1)} &= \frac{1}{30}[\alpha^{xy}_{kl}\alpha^{pbce}_{ijxm}\sigma^{ad}_{py} + \alpha^{pbcd}_{ijkl}\sigma^{ae}_{pm}]\nabla_e C_{abcd} \\
&= \alpha^{xy}_{kl}[\nabla_m C_{ijxy}]^{\young(iy,j,x,m)} + [\nabla_m C_{ijkl}]^{\young(im,j,k,l)} \\
&= \frac{1}{30}[\alpha^{qs}_{kl}\alpha^{pbce}_{ijqm}\nabla_e[C_{pbcs}-C_{cspb}] + \frac{1}{2}\alpha^{pbcd}_{ijkl}\nabla_p[C_{mbcd}-C_{cdmb}] \\
&\quad +\frac{1}{30}\alpha^{pbcd}_{ijkl}[\nabla_m C_{pbcd} + \frac{1}{2}\nabla_p[C_{mbcd}+C_{cdmb}]]
\end{aligned}
$$

The distribution of components reads $144 = 60 + 36 + 2 \cdot 20 + 2 \cdot 4$.

5.1.1. $\pi \circ \nabla[\pi \circ \mathbf{C}]$

$$
\pi \circ \nabla[\pi \circ \mathbf{C}] = [\pi^{\yng(3,2)} + \pi^{\yng(2,2,1)}] \circ \nabla \mathbf{C}^{\yng(2,2)} + [\pi^{\yng(3,1,1)} + \pi^{\yng(2,2,1)} + \pi^{\yng(2,1,1,1)}] \circ \nabla \mathbf{C}^{\yng(2,1,1)} + \pi^{\yng(2,1,1,1)} \circ \nabla \mathbf{C}^{\yng(1,1,1,1)}
$$

with

$$
\begin{aligned}
[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)} &= \frac{1}{2}\nabla_m C^{\yng(2,2)}_{ijkl} - \frac{1}{8}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} + \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a C^{\yng(2,2)}_{bmcd} \\
[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{2}\nabla_m C^{\yng(2,2)}_{ijkl} + \frac{1}{8}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} + \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a C^{\yng(2,2)}_{bmcd} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} &= \frac{4}{10}\nabla_m C^{\yng(2,1,1)}_{ijkl} - \frac{1}{10}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} - \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a [C^{\yng(2,1,1)}_{bmcd} + C^{\yng(2,1,1)}_{dmcb}] \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{3}\nabla_m C^{\yng(2,1,1)}_{ijkl} - \frac{1}{6}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} - \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a C^{\yng(2,1,1)}_{dmcb} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \frac{4}{15}\nabla_m C^{\yng(2,1,1)}_{ijkl} + \frac{1}{30}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} - \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a [3C^{\yng(2,1,1)}_{bmcd} + 2C^{\yng(2,1,1)}_{dmcb}] \\
[\nabla_m C^{\yng(1,1,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \nabla_m C^{\yng(1,1,1,1)}_{ijkl}
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)} &= \nabla_m C^{\yng(2,2)}_{ijkl} - \frac{1}{16}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]\nabla_e C^{\yng(2,2)}_{abcd} \\
[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{16}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]\nabla_e C^{\yng(2,2)}_{abcd} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} &= \nabla_m C^{\yng(2,1,1)}_{ijkl} - [\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} - [\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= -\frac{2}{5}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} + \frac{1}{20}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} - \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]\nabla_e C^{\yng(2,1,1)}_{abcd} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{12}\alpha^{qs}_{kl}\alpha^{prc}_{ijq}[\alpha^{xy}_{sm}\nabla_y C^{\yng(2,1,1)}_{prcx} + \nabla_r C^{\yng(2,1,1)}_{smcp}]
\end{aligned}
$$

or as

$$
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} = \nabla_m C^{\yng(2,1,1)}_{ijkl} - [\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} - [\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)}
$$

$$
\begin{aligned}
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} &= \frac{3}{2}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} - \frac{1}{4}[\alpha^{ab}_{ij}\alpha^{cd}_{kl} - \alpha^{ab}_{kl}\alpha^{cd}_{ij}]\nabla_a C^{\yng(2,1,1)}_{bmcd} \\
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \frac{1}{30}[2\alpha^{qs}_{kl}\alpha^{abce}_{ijqm}\nabla_e C^{\yng(2,1,1)}_{abcs} + \alpha^{abcd}_{ijkl}\nabla_a C^{\yng(2,1,1)}_{mbcd}]
\end{aligned}
$$

and

$$
\begin{aligned}
[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \frac{1}{30}[-2\alpha^{qs}_{kl}\alpha^{abcd}_{ijqm}g_{as} + \alpha^{abcd}_{ijkl}g_{am}]\nabla^p[*[C^{\yng(2,1,1)}]*]_{pbcd} \\
[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)} &= -\frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]g_{ac}\nabla^p[[*[C^{\yng(2,2)}]*]_{pdbe} + g_{bd}[*[C^{\yng(2,2)}]*]_{pe}] \\
\frac{1}{20}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{ecd}_{mkl}\alpha^{ab}_{ij}]\nabla_e C^{\yng(2,1,1)}_{abcd} &= -\frac{1}{10}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} - \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]g_{ac}\nabla^p[[*[C^{\yng(2,1,1)}]*]_{pdbe} + g_{bd}[*[C^{\yng(2,1,1)}]*]_{pe}]
\end{aligned}
$$

The distribution of components reads $144 = 60 + 20 + 36 + 20 + 4 + 4$.

5.2. τ

$$
\tau \circ \nabla \mathbf{C} = [\nabla \mathbf{C}]^{III} + [\nabla \mathbf{C}]^{II} + [\nabla \mathbf{C}]^{I}
$$

with

$$
\begin{aligned}
[\nabla_m C_{ijkl}]^{III} &= \nabla_m C_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X_{bdm} + g_{am}Y_{bcd} + g_{cm}Z_{dab}] \\
[\nabla_m C_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X_{bdm}]^{II} + g_{am}[Y_{bcd}]^{II} + g_{cm}[Z_{dab}]^{II}] \\
[\nabla_m C_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X_{bdm}]^{I} + g_{am}[Y_{bcd}]^{I} + g_{cm}[Z_{dab}]^{I}] \\
&=: \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X_m + g_{bm}g_{ac}Y_d + g_{dm}g_{ac}Z_b]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \sigma^{cd}_{kl}Y_{bcd} \\
0 &= \sigma^{ab}_{ij}Z_{abd}
\end{aligned}
$$

The equations for the coefficients read

$$
\begin{aligned}
g^{jl}[\nabla_m C_{ijkl}]^{II} &= [\nabla_m C_{ik}]^{II} &&= 2[X_{ikm} - Y_{ikm} - Z_{kim}]^{II} \\
g^{mi}[\nabla_m C_{ijkl}]^{II} &= [\nabla^p C_{pjkl}]^{II} &&= [\alpha^{cd}_{kl}[X_{jdc} + 2Z_{dcj}] + 6Y_{jkl}]^{II} \\
g^{mk}[\nabla_m C_{ijkl}]^{II} &= [\nabla^p C_{ijpl}]^{II} &&= [\alpha^{ab}_{ij}[X_{bla} + 2Y_{bal}] + 6Z_{lij}]^{II} \\
g^{ik}g^{jl}[\nabla_m C_{ijkl}]^{I} &= \nabla_m C &&= 24X_m + 6Y_m + 6Z_m \\
g^{mi}g^{jl}[\nabla_m C_{ijkl}]^{I} &= \nabla^p C_{pk} &&= 6X_k + 9Y_k + 3Z_k \\
g^{mk}g^{jl}[\nabla_m C_{ijkl}]^{I} &= \nabla^p C_{ip} &&= 6X_i + 3Y_i + 9Z_i
\end{aligned}
$$

so with

$$
\begin{aligned}
[X_{ikm}]^{II} &= [X_{ikm}]^{\yng(3)\,II} + [X_{ikm}]^{\yng(2,1)\,II} + [X_{ikm}]^{\yng(1,1,1)} \\
[Y_{ikm}]^{II} &= [Y_{ikm}]^{\yng(2,1)\,II} + [Y_{ikm}]^{\yng(1,1,1)} \\
[Z_{ikm}]^{II} &= [Z_{ikm}]^{\yng(2,1)\,II} + [Z_{ikm}]^{\yng(1,1,1)}
\end{aligned}
$$

the solution reads

$$
[X_{ikm}]^{II\,\yng(3)} = \frac{1}{2}[\nabla_m C_{ik}]^{II\,\yng(3)}
$$

$$
\begin{aligned}
[X_{ikm}]^{[1,1]} &= \frac{1}{6}[5\nabla_m C_{ik} + \nabla^p C_{pikm} - \nabla^p C_{ikpm}]^{[1,1]}\\
[Y_{ikm}]^{[1,1]} &= \frac{1}{6}[\nabla_m C_{ik} + 2\nabla^p C_{pikm} + \nabla^p C_{kmpi}]^{[1,1]}\\
[Z_{ikm}]^{[1,1]} &= \frac{1}{6}[-\nabla_m C_{ik} + \nabla^p C_{pikm} + 2\nabla^p C_{kmpi}]^{[1,1]}\\
[X_{ikm}]^{II\,[2,1]} &= \frac{1}{15}[10\nabla_m C_{ik} + \sigma^{ac}_{ik}\nabla_m C_{ac} + \frac{1}{4}\nabla^p[11C_{pikm} + C_{kmpi} + 11C_{impk} + C_{pkim}]]^{II\,[2,1]}\\
&\quad + \frac{1}{12}\nabla^p[C_{ikpm} - C_{pmik}]^{II\,[2,1]}\\
[Y_{ikm}]^{II\,[2,1]} &= \frac{1}{15}[4\nabla^p C_{pikm} - \nabla^p C_{kmpi} + \alpha^{ce}_{km}[20\nabla_e C_{ic} + 2\sigma^{rs}_{ic}\nabla_e C_{rs} - 10\nabla_i C_{ce}]]^{II\,[2,1]}\\
[Z_{ikm}]^{II\,[2,1]} &= \frac{1}{15}[4\nabla^p C_{kmpi} - \nabla^p C_{pikm} + \alpha^{ce}_{km}[20\nabla_e C_{ci} + 2\sigma^{rs}_{ic}\nabla_e C_{rs} + 10\nabla_i C_{ce}]]^{II\,[2,1]}
\end{aligned}
$$

and

$$
\begin{aligned}
X_k &= \frac{1}{18}[5X^a{}_{ak} - X^a{}_{ka} - X_{ka}{}^a] &&= \frac{1}{36}[2\nabla_k C - \nabla^a C_{ak} - \nabla^k C_{ka}]\\
Y_k &= \frac{1}{18}[-X^a{}_{ak} + 5X^a{}_{ka} - X_{ka}{}^a] + \frac{1}{3}Y^a{}_{ka} &&= \frac{1}{36}[-\nabla_k C + 5\nabla^a C_{ak} - \nabla^k C_{ka}]\\
Z_k &= \frac{1}{18}[-X^a{}_{ak} - X^a{}_{ka} + 5X_{ka}{}^a] + \frac{1}{3}Z_{ka}{}^a &&= \frac{1}{36}[-\nabla_k C - \nabla^a C_{ak} + 5\nabla^k C_{ka}]
\end{aligned}
$$

The distribution of components reads $144 = 32 + 100 + 12$.

5.2.1. $\tau \circ \nabla[\pi \circ \mathbf{C}]$

$$
\tau \circ \nabla[\pi \circ \mathbf{C}] = [\tau^{III} + \tau^{II} + \tau^{I}] \circ [\nabla \mathbf{C}^{[2,2]} + \nabla \mathbf{C}^{[2,1,1]}] + [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{C}^{[1,1,1,1]}
$$

with

$$
\begin{aligned}
[\nabla_m C^{[2,2]}_{ijkl}]^{III} &= \nabla_m C^{[2,2]}_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{[2,2]}_{bdm} + g_{am}Y^{[2,2]}_{bcd} + g_{cm}Y^{[2,2]}_{dab}]\\
[\nabla_m C^{[2,2]}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{[2,2]}_{bdm}]^{II} + g_{am}[Y^{[2,2]}_{bcd}]^{II} + g_{cm}[Y^{[2,2]}_{dab}]^{II}]\\
[\nabla_m C^{[2,2]}_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{[2,2]}_{m} + g_{bm}g_{ac}Y^{[2,2]}_{d} + g_{dm}g_{ac}Y^{[2,2]}_{b}]\\
[\nabla_m C^{[2,1,1]}_{ijkl}]^{III} &= \nabla_m C^{[2,1,1]}_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{[2,1,1]}_{bdm} + g_{am}Y^{[2,1,1]}_{bcd} - g_{cm}Y^{[2,1,1]}_{dab}]\\
[\nabla_m C^{[2,1,1]}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{[2,1,1]}_{bdm}]^{II} + g_{am}[Y^{[2,1,1]}_{bcd}]^{II} - g_{cm}[Y^{[2,1,1]}_{dab}]^{II}]\\
[\nabla_m C^{[2,1,1]}_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{bm}g_{ac}Y^{[2,1,1]}_{d} - g_{dm}g_{ac}Y^{[2,1,1]}_{b}]\\
[\nabla_m C^{[1,1,1,1]}_{ijkl}]^{III} &= \nabla_m C^{[1,1,1,1]}_{abcd} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[2g_{am}Y^{[1,1,1,1]}_{bcd}]\\
[\nabla_m C^{[1,1,1,1]}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[2g_{am}Y^{[1,1,1,1]}_{bcd}]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \sigma^{cd}_{kl}Y^{[2,2]}_{bcd}\\
0 &= \sigma^{cd}_{kl}Y^{[2,1,1]}_{bcd}\\
0 &= \alpha^{bd}_{jl}X^{[2,2]}_{bdm}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \alpha^{bcd}_{jkl} Y^{\yng(2,2)}_{bcd} \\
0 &= \sigma^{bd}_{jl} X^{\yng(2,1,1)}_{bdm}
\end{aligned}
$$

The equations for the coefficients read

$$
\begin{aligned}
g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{II} &= [\nabla_m C^{\yng(2,2)}_{ik}]^{II} &&= 2[X^{\yng(2,2)}_{ikm} - \sigma^{ac}_{ik} Y^{\yng(2,2)}_{acm}]^{II} \\
g^{mi}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{II} &= [\nabla^p C^{\yng(2,2)}_{pjkl}]^{II} &&= [\alpha^{cd}_{kl} X^{\yng(2,2)}_{jdc} + 8Y^{\yng(2,2)}_{jkl}]^{II} \\
g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{II} &= [\nabla_m C^{\yng(2,1,1)}_{ik}]^{II} &&= 2[X^{\yng(2,1,1)}_{ikm} - \alpha^{ac}_{ik} Y^{\yng(2,1,1)}_{acm}]^{II} \\
g^{mi}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{II} &= [\nabla^p C^{\yng(2,1,1)}_{pjkl}]^{II} &&= [\alpha^{cd}_{kl}[X^{\yng(2,1,1)}_{jdc} - Y^{\yng(2,1,1)}_{dcj}] + 6Y^{\yng(2,1,1)}_{jkl}]^{II} \\
g^{mi}\nabla_m C^{\yng(1,1,1,1)}_{ijkl} &= \nabla^p C^{\yng(1,1,1,1)}_{pjkl} &&= 2Y^{\yng(1,1,1,1)}_{jkl} \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{I} &= \nabla_m C &&= 24X^{\yng(2,2)}_{m} + 12Y^{\yng(2,2)}_{m} \\
g^{mk}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{II} &= \nabla^p C^{\yng(2,2)}_{ip} &&= 6X^{\yng(2,2)}_{i} + 12Y^{\yng(2,2)}_{i} \\
g^{mi}g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{II} &= \nabla^p C^{\yng(2,1,1)}_{pk} &&= 6Y^{\yng(2,1,1)}_{k}
\end{aligned}
$$

so with

$$
\begin{aligned}
[X^{\yng(2,2)}_{ikm}]^{II} &= [X^{\yng(2,2)}_{ikm}]^{\yng(3)\,II} + [X^{\yng(2,2)}_{ikm}]^{\yng(2,1)\,II} \\
[Y^{\yng(2,2)}_{ikm}]^{II} &= [Y^{\yng(2,2)}_{ikm}]^{\yng(2,1)\,II} \\
[X^{\yng(2,1,1)}_{ikm}]^{II} &= [X^{\yng(2,1,1)}_{ikm}]^{\yng(2,1)\,II} + [X^{\yng(2,1,1)}_{ikm}]^{\yng(1,1,1)} \\
[Y^{\yng(2,1,1)}_{ikm}]^{II} &= [Y^{\yng(2,1,1)}_{ikm}]^{\yng(2,1)\,II} + [Y^{\yng(2,1,1)}_{ikm}]^{\yng(1,1,1)} \\
Y^{\yng(1,1,1,1)}_{ikm} &= [Y^{\yng(1,1,1,1)}_{ikm}]^{\yng(1,1,1)}
\end{aligned}
$$

the solution reads

$$
\begin{aligned}
[X^{\yng(2,2)}_{ikm}]^{II\,\yng(3)} &= \frac{1}{2}[\nabla_m C^{\yng(2,2)}_{ik}]^{II\,\yng(3)} \\
[X^{\yng(2,1,1)}_{ikm}]^{\yng(1,1,1)} &= \frac{1}{6}[5\nabla_m C^{\yng(2,1,1)}_{ik} + 2\nabla^p C^{\yng(2,1,1)}_{pikm}]^{II} \\
[Y^{\yng(2,1,1)}_{ikm}]^{\yng(1,1,1)} &= \frac{1}{6}[\nabla_m C^{\yng(2,1,1)}_{ik} + \nabla^p C^{\yng(2,1,1)}_{pikm}]^{II} \\
Y^{\yng(1,1,1,1)}_{ikm} &= \frac{1}{2}\nabla^p C^{\yng(1,1,1,1)}_{pikm} \\
[X^{\yng(2,2)}_{ikm}]^{II\,\yng(2,1)} &= \frac{4}{5}[\nabla_m C^{\yng(2,2)}_{ik} + \frac{1}{4}\sigma^{ac}_{ik}\nabla^p C^{\yng(2,2)}_{pacm}]^{II\,\yng(2,1)} \\
[Y^{\yng(2,2)}_{ikm}]^{II\,\yng(2,1)} &= \frac{1}{5}[\nabla^p C^{\yng(2,2)}_{pikm} + \frac{1}{2}\alpha^{ce}_{km}\nabla_e C^{\yng(2,2)}_{ic}]^{II\,\yng(2,1)} \\
[X^{\yng(2,1,1)}_{ikm}]^{II\,\yng(2,1)} &= \frac{2}{3}[\nabla_m C^{\yng(2,1,1)}_{ik} - \frac{1}{2}\nabla^p C^{\yng(2,1,1)}_{pmik}]^{II\,\yng(2,1)} \\
[Y^{\yng(2,1,1)}_{ikm}]^{II\,\yng(2,1)} &= \frac{1}{3}[\nabla^p C^{\yng(2,1,1)}_{pikm} - \frac{1}{2}\nabla_i C^{\yng(2,1,1)}_{km}]^{II\,\yng(2,1)} \\
X^{\yng(2,2)}_{k} &= \frac{1}{18}[\nabla_k C - \nabla^a C^{\yng(2,2)}_{ak}] \\
Y^{\yng(2,2)}_{k} &= \frac{1}{36}[-\nabla_k C + 4\nabla^a C^{\yng(2,2)}_{ak}] \\
Y^{\yng(2,1,1)}_{k} &= \frac{1}{6}\nabla^a C^{\yng(2,1,1)}_{ak}
\end{aligned}
$$

5.2.2. $\tau \circ \nabla[\tau \circ \mathbf{C}]$

$$\tau \circ \nabla[\tau \circ \mathbf{C}] \quad = \quad [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{C}^{III} + [\tau^{II} + \tau^{I}] \circ \nabla \mathbf{C}^{II} + \tau^{I} \circ \nabla \mathbf{C}^{I}$$

with

$$\begin{aligned}
[\nabla_m C^{III}_{ijkl}]^{III} &= \nabla_m C^{III}_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{III}_{bdm} + g_{am}Y^{III}_{bcd} + g_{cm}Z^{III}_{dab}] \\
[\nabla_m C^{III}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{III}_{bdm}]^{II} + g_{am}[Y^{III}_{bcd}]^{II} + g_{cm}[Z^{III}_{dab}]^{II}] \\
[\nabla_m C^{II}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{III}_{bdm}]^{II}] \\
[\nabla_m C^{II}_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{II}_{m} + g_{bm}g_{ac}Y^{II}_{d} + g_{dm}g_{ac}Z^{II}_{b}] \\
[\nabla_m C^{I}_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{I}_{m}]
\end{aligned}$$

and

$$\begin{aligned}
0 &= \sigma^{cd}_{kl}Y^{III}_{bcd} \\
0 &= \sigma^{ab}_{ij}Z^{III}_{abd} \\
0 &= [X^{III}_{bdm}]^{I} \\
0 &= [Y^{III}_{bcd}]^{I} \\
0 &= [Z^{III}_{dab}]^{I}
\end{aligned}$$

The system gives the traces

$$\begin{aligned}
g^{jl}[\nabla_m C^{III}_{ijkl}]^{II} &= 0 &&= 2X^{III}_{ikm} - 2Y^{III}_{ikm} - 2Z^{III}_{kim} \\
g^{mi}[\nabla_m C^{III}_{ijkl}]^{II} &= \nabla^p C^{III}_{pjkl} &&= \alpha^{cd}_{kl}[X^{III}_{jdc} + 2Z^{III}_{dcj}] + 6Y^{III}_{jkl} \\
g^{mk}[\nabla_m C^{III}_{ijkl}]^{II} &= \nabla^p C^{III}_{ijpl} &&= \alpha^{ab}_{ij}[X^{III}_{bla} + 2Y^{III}_{bal}] + 6Z^{III}_{lij} \\
g^{jl}[\nabla_m C^{II}_{ijkl}]^{II} &= [\nabla_m C^{II}_{ik}]^{II} &&= 2[X^{II}_{ikm}]^{II} \\
g^{ik}g^{jl}[\nabla_m C^{II}_{ijkl}]^{I} &= 0 &&= 24X^{II}_{i} + 6Y^{II}_{i} + 6Z^{II}_{i} \\
g^{mk}g^{jl}[\nabla_m C^{II}_{ijkl}]^{I} &= \nabla^p C^{II}_{ip} &&= 6X^{II}_{i} + 3Y^{II}_{i} + 9Z^{II}_{i} \\
g^{mi}g^{jl}[\nabla_m C^{II}_{ijkl}]^{I} &= \nabla^p C^{II}_{pi} &&= 6X^{II}_{i} + 9Y^{II}_{i} + 3Z^{II}_{i} \\
g^{ik}g^{jl}[\nabla_m C^{I}_{ijkl}]^{I} &= \nabla_m C &&= 24X^{I}_{m}
\end{aligned}$$

so with

$$\begin{aligned}
Y^{III}_{ikm} &= [Y^{III}_{ikm}]^{\ydiagram{2,1}} + [Y^{III}_{ikm}]^{\ydiagram{1,1,1}} \\
Z^{III}_{ikm} &= [Z^{III}_{ikm}]^{\ydiagram{2,1}} + [Z^{III}_{ikm}]^{\ydiagram{1,1,1}} \\
[X^{II}_{ikm}]^{II} &= [X^{II}_{ikm}]^{\ydiagram{3}\,II} + [X^{II}_{ikm}]^{\ydiagram{2,1}\,II} + [X^{II}_{ikm}]^{\ydiagram{1,1,1}}
\end{aligned}$$

the solution reads

$$\begin{aligned}
X^{III}_{ikm} &= Y^{III}_{ikm} + Z^{III}_{kim} \\
[Y^{III}_{ikm}]^{\ydiagram{1,1,1}} &= \frac{1}{6}[2\nabla^p C^{III}_{pikm} + \nabla^p C^{III}_{kmpi}]^{\ydiagram{1,1,1}} \\
[Z^{III}_{ikm}]^{\ydiagram{1,1,1}} &= \frac{1}{6}[\nabla^p C^{III}_{pikm} + 2\nabla^p C^{III}_{kmpi}]^{\ydiagram{1,1,1}} \\
[Y^{III}_{ikm}]^{II\,\ydiagram{2,1}} &= \frac{1}{15}[4\nabla^p C^{III}_{pikm} - \nabla^p C^{III}_{kmpi}]^{II\,\ydiagram{2,1}}
\end{aligned}$$

$$
\begin{aligned}
[Z^{III}_{ikm}]^{II\,\yng(2,1)} &= \frac{1}{15}[4\nabla^p C^{III}_{kmpi} - \nabla^p C^{III}_{pikm}]^{II\,\yng(2,1)} \\
[X^{II}_{ikm}]^{II} &= \frac{1}{2}[\nabla_m C^{II}_{ik}]^{II} \\
X^{II}_m &= -\frac{1}{4}[Y^{II}_m + Z^{II}_m] \\
Y^{II}_d &= \frac{1}{36}[5\nabla^a C^{II}_{ak} - \nabla^k C^{II}_{ka}] \\
Z^{II}_b &= \frac{1}{36}[5\nabla^k C^{II}_{ka} - \nabla^a C^{II}_{ak}] \\
\nabla_m C &= \frac{1}{24} X^I_m
\end{aligned}
$$

The distribution of components reads $144 = 32 + 48 + 52 + 8 + 4$.

5.2.3. $\tau \circ \nabla[\tau \circ \pi \circ \mathbf{C}]$

$$
\pi \circ \nabla[\tau \circ \pi \circ \mathbf{C}] = [\tau^{III} + \tau^{II}] \circ [\mathbf{C}^{\yng(2,2)\,III} \mathbf{C}^{\yng(2,1,1)\,III} + \mathbf{C}^{\yng(1,1,1,1)\,III}] + [\tau^{II} + \tau^{I}] \circ [\mathbf{C}^{\yng(2,2)\,II} + \mathbf{C}^{\yng(2,1,1)\,II}] + \tau^{I} \circ \nabla \mathbf{C}^{\yng(2,2)\,I}
$$

with

$$
\begin{aligned}
[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{III} &= \nabla_m C^{\yng(2,2)\,III}_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac} X^{\yng(2,2)\,III}_{bdm} - g_{am} Y^{\yng(2,2)\,III}_{bcd} - g_{cm} Y^{\yng(2,2)\,III}_{dab}] \\
[\nabla_m C^{\yng(2,1,1)\,III}_{ijkl}]^{III} &= \nabla_m C^{\yng(2,1,1)\,III} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac} X^{\yng(2,1,1)\,III}_{bdm} - g_{am} Y^{\yng(2,1,1)\,III}_{bcd} + g_{cm} Y^{\yng(2,1,1)\,III}_{dab}] \\
[\nabla_m C^{\yng(1,1,1,1)\,III}_{ijkl}]^{III} &= [\nabla_m C^{\yng(1,1,1,1)}_{ijkl}]^{III} \\
[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac} X^{\yng(2,2)\,III}_{bdm} + g_{am} Y^{\yng(2,2)\,III}_{bcd} + g_{cm} Y^{\yng(2,2)\,III}_{dab}] \\
[\nabla_m C^{\yng(2,1,1)\,III}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac} X^{\yng(2,1,1)\,III}_{bdm} + g_{am} Y^{\yng(2,1,1)\,III}_{bcd} - g_{cm} Y^{\yng(2,1,1)\,III}_{dab}] \\
[\nabla_m C^{\yng(1,1,1,1)\,III}_{ijkl}]^{II} &= [\nabla_m C^{\yng(1,1,1,1)}_{ijkl}]^{II} \\
[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{\yng(2,2)\,II}_{bdm}]^{II}] \\
[\nabla_m C^{\yng(2,1,1)\,II}_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{\yng(2,1,1)\,II}_{bdm}]^{II}] \\
[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac} g_{bd} X^{\yng(2,2)\,II}_{m} + g_{bm} g_{ac} Y^{\yng(2,2)\,II}_{d} + g_{dm} g_{ac} Y^{\yng(2,2)\,II}_{b}] \\
[\nabla_m C^{\yng(2,1,1)\,II}_{ijkl}]^{I} &= [\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{I} \\
[\nabla_m C^{\yng(2,2)\,I}_{ijkl}]^{I} &= [\nabla_m C^{I}_{ijkl}]^{I}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \sigma^{cd}_{kl} Y^{\yng(2,2)\,III}_{bcd} \\
0 &= \sigma^{cd}_{kl} Y^{\yng(2,1,1)\,III}_{bcd} \\
0 &= \alpha^{bd}_{jl} X^{\yng(2,2)\,III}_{bdm} \\
0 &= \alpha^{bd}_{jl} X^{\yng(2,2)\,II}_{bdm} \\
0 &= \alpha^{bcd}_{jkl} Y^{\yng(2,2)\,III}_{bcd} \\
0 &= \sigma^{bd}_{jl} X^{\yng(2,1,1)\,III}_{bdm}
\end{aligned}
$$

$$
\begin{array}{rcl}
0 & = & \sigma^{bd}_{jl} X^{\yng(2,1)\,II}_{bdm} \\
0 & = & [X^{\yng(2,2)\,III}_{bdm}]^{II} \\
0 & = & [X^{\yng(2,2)\,III}_{bdm}]^{II} \\
0 & = & [Y^{\yng(2,2)\,III}_{bkl}]^{II} \\
0 & = & [Y^{\yng(2,1)\,III}_{bkl}]^{II}
\end{array}
$$

The traces for the coefficients read

$$
\begin{array}{rcccl}
g^{jl}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{II} & = & 0 & = & 2[X^{\yng(2,2)\,III}_{ikm} - \sigma^{ac}_{ik} Y^{\yng(2,2)\,III}_{acm}] \\
g^{mi}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{II} & = & \nabla^p C^{\yng(2,2)\,III}_{pjkl} & = & \alpha^{cd}_{kl} X^{\yng(2,2)\,III}_{jdc} + 8Y^{\yng(2,2)\,III}_{jkl} \\
g^{jl}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{II} & = & 0 & = & 2[X^{\yng(2,1)\,III}_{ikm} - \alpha^{ac}_{ik} Y^{\yng(2,1)\,III}_{acm}] \\
g^{mi}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{II} & = & \nabla^p C^{\yng(2,1)\,III}_{pjkl} & = & \alpha^{cd}_{kl}[X^{\yng(2,1)\,III}_{jdc} - 2Y^{\yng(2,1)\,III}_{dcj}] + 6Y^{\yng(2,1)\,III}_{jkl} \\
g^{mi}[\nabla_m C^{\yng(1,1,1)\,III}_{ijkl}]^{II} & = & [\nabla^p C^{\yng(1,1,1)\,III}_{pjkl}]^{II} & = & 2Y^{\yng(1,1,1)\,III}_{jkl} \\
g^{jl}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{II} & = & [\nabla_m C^{\yng(2,2)\,II}_{ik}]^{II} & = & 2[X^{\yng(2,2)\,II}_{ikm}]^{II} \\
g^{jl}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{II} & = & [\nabla_m C^{\yng(2,1)\,II}_{ik}]^{II} & = & 2[X^{\yng(2,1)\,II}_{ikm}]^{II} \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{I} & = & 0 & = & 24X^{\yng(2,2)\,II}_{m} + 12Y^{\yng(2,2)\,II}_{m} \\
g^{mk}g^{jl}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{I} & = & \nabla^p C^{\yng(2,2)\,II}_{ip} & = & 6X^{\yng(2,2)\,II}_{i} + 12Y^{\yng(2,2)\,II}_{i} \\
g^{mi}g^{jl}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{I} & = & \nabla^p C^{\yng(2,1)\,II}_{pk} & = & 6Y^{\yng(2,1)\,II}_{k} \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,2)\,I}_{ijkl}]^{I} & = & \nabla_m C & = & 24[X^{\yng(2,2)}]^{I}_{m}
\end{array}
$$

so with

$$
\begin{array}{rcl}
Y^{\yng(2,2)\,III}_{ikm} & = & [Y^{\yng(2,2)\,III}_{ikm}]^{\yng(2,1)} \\
Y^{\yng(2,1)\,III}_{ikm} & = & [Y^{\yng(2,1)\,III}_{ikm}]^{\yng(2,1)} + [Y^{\yng(2,1)\,III}_{ikm}]^{\yng(1,1,1)} \\
Y^{\yng(1,1,1)\,III}_{ikm} & = & [Y^{\yng(1,1,1)\,III}_{ikm}]^{\yng(1,1,1)} \\
[X^{\yng(2,2)\,II}_{ikm}]^{II} & = & [X^{\yng(2,2)\,II}_{ikm}]^{\yng(3)\,II} + [X^{\yng(2,2)\,II}_{ikm}]^{\yng(2,1)\,II} \\
[X^{\yng(2,1)\,II}_{ikm}]^{II} & = & [X^{\yng(2,1)\,II}_{ikm}]^{\yng(2,1)\,II} + [X^{\yng(2,1)}_{ikm}]^{\yng(1,1,1)}
\end{array}
$$

the solution becomes

$$
\begin{array}{rcl}
X^{\yng(2,2)\,III}_{ikm} & = & \sigma^{ac}_{ik} Y^{\yng(2,2)\,III}_{acm} \\
Y^{\yng(2,2)\,III}_{ikm} & = & \frac{1}{5}\nabla^p C^{\yng(2,2)\,III}_{pikm} \\
X^{\yng(2,1)\,III}_{ikm} & = & \alpha^{ac}_{ik} Y^{\yng(2,1)\,III}_{acm} \\
[Y^{\yng(2,1)\,III}_{ikm}]^{\yng(1,1,1)} & = & \frac{1}{6}[\nabla^p C^{\yng(2,1)\,III}_{pikm}]^{\yng(1,1,1)} \\
[Y^{\yng(2,1)\,III}_{ikm}]^{\yng(2,1)} & = & \frac{1}{3}[\nabla^p C^{\yng(2,1)\,III}_{pikm}]^{\yng(2,1)} \\
Y^{\yng(1,1,1)\,III}_{ikm} & = & \frac{1}{2}\nabla^p C^{\yng(1,1,1)\,III}_{pikm}
\end{array}
$$

$$
\begin{aligned}
[X^{\yng(2,2)\,II}_{ikm}]^{II} &= \frac{1}{2}[\nabla_m C^{\yng(2,2)\,II}_{ik}]^{II} \\
[X^{\yng(2,1,1)\,II}_{ikm}]^{II} &= \frac{1}{2}[\nabla_m C^{\yng(2,1,1)\,II}_{ik}]^{II} \\
X^{\yng(2,2)\,II}_{m} &= -\frac{1}{2}Y^{\yng(2,2)\,II}_{m} \\
Y^{\yng(2,2)\,II}_{m} &= \frac{1}{9}\nabla^a C^{\yng(2,2)\,II}_{am} \\
Y^{\yng(2,1,1)\,II}_{m} &= \frac{1}{6}\nabla^a C^{\yng(2,1,1)\,II}_{am} \\
X^{\yng(2,2)\,I}_{m} &= \frac{1}{24}\nabla_m C
\end{aligned}
$$

5.3. $\tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}]$

$$
\begin{aligned}
\tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}] = {} & [\tau^{III} + \tau^{II} + \tau^{I}] \circ [[\nabla \mathbf{C}^{\yng(2,2)}]^{\yng(3,2)} + [\nabla \mathbf{C}^{\yng(2,2)}]^{\yng(2,2,1)} + [\nabla \mathbf{C}^{\yng(2,1,1)}]^{\yng(2,2,1)}] \\
+ {} & [\tau^{III} + \tau^{II}] \circ [[\nabla \mathbf{C}^{\yng(2,1,1)}]^{\yng(3,1,1)} + [\nabla \mathbf{C}^{\yng(2,1,1)}]^{\yng(2,1,1,1)} + [\nabla \mathbf{C}^{\yng(1,1,1,1)}]^{\yng(2,1,1,1)}]
\end{aligned}
$$

The relations between the corresponding coefficients $X^{\yng(2,2)\yng(2,2,1)}$ and $Y^{\yng(2,2)\yng(2,2,1)}$ and the traces remain unchanged, so $X^{\yng(2,2)\yng(2,2,1)}$ and $Y^{\yng(2,2)\yng(2,2,1)}$ are determined by

$$
\begin{aligned}
[g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)}]^{II} &= [\nabla_m C^{\yng(2,2)}_{ik}]^{II} - \frac{1}{4}\sigma^{ac}_{ik}[\nabla^p C^{\yng(2,2)}_{pacm} - \alpha^{xy}_{cm}\nabla_x C^{\yng(2,2)}_{ay}]^{II} \\
[g^{mi}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)}]^{II} &= [\nabla^p C^{\yng(2,2)}_{pjkl}]^{II} - \frac{1}{4}[\nabla^p C^{\yng(2,2)}_{pjkl} - \alpha^{xy}_{kl}\nabla_x C^{\yng(2,2)}_{jy}]^{II} \\
[g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)}]^{II} &= \frac{1}{4}\sigma^{ac}_{ik}[\nabla^p C^{\yng(2,2)}_{pacm} - \alpha^{xy}_{cm}\nabla_x C^{\yng(2,2)}_{ay}]^{II} \\
[g^{mi}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)}]^{II} &= \frac{1}{4}[\nabla^p C^{\yng(2,2)}_{pjkl} - \alpha^{xy}_{kl}\nabla_x C^{\yng(2,2)}_{jy}]^{II} \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)} &= \nabla_m C + [\nabla^p C^{\yng(2,2)}_{pm} - \frac{1}{2}\nabla_m C] \\
g^{mi}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(3,2)} &= \nabla^p C^{\yng(2,2)}_{pk} - \frac{1}{2}[\nabla^p C^{\yng(2,2)}_{pk} - \frac{1}{2}\nabla_k C] \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)} &= -[\nabla^p C^{\yng(2,2)}_{pm} - \frac{1}{2}\nabla_m C] \\
g^{mi}g^{jl}[\nabla_m C^{\yng(2,2)}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{2}[\nabla^p C^{\yng(2,2)}_{pk} - \frac{1}{2}\nabla_k C] \\
[g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)}]^{II} &= [\nabla_m C^{\yng(2,1,1)}_{ik}]^{II} - \frac{1}{2}[\nabla^p C^{\yng(2,1,1)}_{pmik} + \nabla_m C^{\yng(2,1,1)}_{ik}] + \frac{1}{12}\alpha^{ace}_{ikm}[\nabla^p C^{\yng(2,1,1)}_{pace} - \nabla_a C^{\yng(2,1,1)}_{ce}] \\
[g^{mi}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)}]^{II} &= [\nabla^p C^{\yng(2,1,1)}_{pjkl}]^{II} - \frac{1}{2}[\nabla^p C^{\yng(2,1,1)}_{pjkl} + \nabla_j C^{\yng(2,1,1)}_{kl}] - \frac{1}{12}\alpha^{bcd}_{jkl}[\nabla^p C^{\yng(2,1,1)}_{pbcd} - \nabla_b C^{\yng(2,1,1)}_{cd}] \\
g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} &= \frac{1}{2}[\nabla^p C^{\yng(2,1,1)}_{pmik} + \nabla_m C^{\yng(2,1,1)}_{ik}] - \frac{1}{20}\alpha^{ace}_{ikm}[3\nabla^p C^{\yng(2,1,1)}_{pace} + \nabla_a C^{\yng(2,1,1)}_{ce}] \\
g^{mi}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(3,1,1)} &= \frac{1}{2}[\nabla^p C^{\yng(2,1,1)}_{pjkl} + \nabla_j C^{\yng(2,1,1)}_{kl}] + \frac{1}{20}\alpha^{bcd}_{jkl}[\nabla^p C^{\yng(2,1,1)}_{pbcd} - 3\nabla_b C^{\yng(2,1,1)}_{cd}] \\
g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \frac{1}{15}\alpha^{ace}_{ikm}[\nabla^p C^{\yng(2,1,1)}_{pace} + 2\nabla_a C^{\yng(2,1,1)}_{ce}] \\
g^{mi}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,1,1,1)} &= \frac{1}{30}\alpha^{bcd}_{jkl}[\nabla^p C^{\yng(2,1,1)}_{pbcd} + 2\nabla_b C^{\yng(2,1,1)}_{cd}] \\
g^{ik}g^{jl}[\nabla_m C^{\yng(2,1,1)}_{ijkl}]^{\yng(2,2,1)} &= 0
\end{aligned}
$$

$$g^{mi}g^{jl}[\nabla_m C^{\yng(2,1)}_{ijkl}]^{\yng(2,2,1)} = \nabla^p C^{\yng(2,1)}_{pk}$$

5.3.1. $\tau \circ \pi \circ \nabla[\tau \circ \pi \circ \mathbf{C}]$

$$\begin{aligned}
\tau \circ \pi \circ \nabla[\pi \circ \mathbf{C}] = & \ [\tau^{III} + \tau^{II} + \tau^{I}] \circ [[\nabla[\tau \circ \mathbf{C}^{\yng(2,2)}]]^{\yng(3,2)} + [\nabla[\tau \circ \mathbf{C}^{\yng(2,2)}]]^{\yng(2,2,1)}] \\
+ & \ [\tau^{III} + \tau^{II} + \tau^{I}] \circ [[\nabla[\tau \circ \mathbf{C}^{\yng(2,1)}]]^{\yng(2,2,1)}] \\
+ & \ [\tau^{III} + \tau^{II}] \circ [[\nabla[\tau \circ \mathbf{C}^{\yng(2,1)}]]^{\yng(3,1,1)} + [\nabla[\tau \circ \mathbf{C}^{\yng(2,1)}]]^{\yng(2,1,1,1)}] \\
+ & \ [\tau^{III} + \tau^{II}] \circ [[\nabla \mathbf{C}^{\yng(1,1,1)}]^{\yng(2,1,1)}]
\end{aligned}$$

with

$$g^{jl}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{\yng(3,2)} = -\frac{1}{4}\sigma^{ac}_{ik}\nabla^p C^{\yng(2,2)\,III}_{pacm}$$

$$g^{mi}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{\yng(3,2)} = \nabla^p C^{\yng(2,2)\,III}_{pjkl} - \frac{1}{4}\nabla^p C^{\yng(2,2)\,III}_{pjkl}$$

$$g^{jl}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{\yng(2,2,1)} = \frac{1}{4}\sigma^{ac}_{ik}\nabla^p C^{\yng(2,2)\,III}_{pacm}$$

$$g^{mi}[\nabla_m C^{\yng(2,2)\,III}_{ijkl}]^{\yng(2,2,1)} = \frac{1}{4}\nabla^p C^{\yng(2,2)\,III}_{pjkl}$$

$$g^{jl}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(2,2,1)} = -\frac{1}{2}\nabla^p C^{\yng(2,1)\,III}_{pmik} + \frac{1}{12}\alpha^{ace}_{ikm}\nabla^p C^{\yng(2,1)\,III}_{pace}$$

$$g^{mi}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(2,2,1)} = \nabla^p C^{\yng(2,1)\,III}_{pjkl} - \frac{1}{2}\nabla^p C^{\yng(2,2)\,III}_{pjkl} - \frac{1}{12}\alpha^{bcd}_{jkl}\nabla^p C^{\yng(2,1)\,III}_{pbcd}$$

$$g^{jl}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(3,1,1)} = \frac{1}{2}\nabla^p C^{\yng(2,1)\,III}_{pmik} - \frac{3}{20}\alpha^{ace}_{ikm}\nabla^p C^{\yng(2,1)\,III}_{pace}$$

$$g^{mi}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(3,1,1)} = \frac{1}{2}\nabla^p C^{\yng(2,2)\,III}_{pjkl} + \frac{1}{20}\alpha^{bcd}_{jkl}\nabla^p C^{\yng(2,1)\,III}_{pbcd}$$

$$g^{jl}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(2,1,1,1)} = \frac{1}{15}\alpha^{ace}_{ikm}\nabla^p C^{\yng(2,1)\,III}_{pace}$$

$$g^{mi}[\nabla_m C^{\yng(2,1)\,III}_{ijkl}]^{\yng(2,1,1,1)} = \frac{1}{30}\alpha^{bcd}_{jkl}\nabla^p C^{\yng(2,1)\,III}_{pbcd}$$

$$[g^{jl}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{\yng(3,2)}]^{II} = [\nabla_m C^{\yng(2,2)\,II}_{ik}]^{II} + \frac{1}{8}\sigma^{ac}_{ik}\alpha^{xy}_{cm}[\nabla_x C^{\yng(2,2)\,II}_{ay}]^{II}$$

$$[g^{mi}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{\yng(3,2)}]^{II} = \frac{1}{2}\alpha^{cd}_{kl}[\nabla_c C^{\yng(2,2)\,II}_{jd}]^{II} + \frac{1}{8}\alpha^{xy}_{kl}[\nabla_x C^{\yng(2,2)\,II}_{jy}]^{II}$$

$$[g^{jl}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{\yng(2,2,1)}]^{II} = -\frac{1}{8}\sigma^{ac}_{ik}\alpha^{xy}_{cm}[\nabla_x C^{\yng(2,2)\,II}_{ay}]^{II}$$

$$[g^{mi}[\nabla_m C^{\yng(2,2)\,II}_{ijkl}]^{\yng(2,2,1)}]^{II} = -\frac{1}{8}\alpha^{xy}_{kl}[\nabla_x C^{\yng(2,2)\,II}_{jy}]^{II}$$

$$[g^{jl}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{\yng(2,2,1)}]^{II} = [\nabla_m C^{\yng(2,1)}_{ik}]^{II} - \frac{1}{4}\alpha^{ac}_{ik}\sigma^{ep}_{cm}[\nabla_e C^{\yng(2,1)}_{ap}]^{II} - \frac{1}{6}\alpha^{ace}_{ikm}\nabla_a C^{\yng(2,1)}_{ce}$$

$$[g^{mi}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{\yng(2,2,1)}]^{II} = \frac{1}{2}\alpha^{cd}_{kl}[\nabla_c C^{\yng(2,1)}_{jd}]^{II} - \frac{1}{4}\alpha^{ac}_{kl}\sigma^{ep}_{jc}[\nabla_e C^{\yng(2,1)}_{ap}]^{II} + \frac{1}{6}\alpha^{bcd}_{jkl}\nabla_b C^{\yng(2,1)}_{cd}$$

$$g^{jl}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{\yng(3,1,1)} = \frac{1}{4}\alpha^{ac}_{ik}\sigma^{ep}_{cm}[\nabla_e C^{\yng(2,1)}_{ap}]^{II} + \frac{1}{10}\alpha^{ace}_{ikm}\nabla_a C^{\yng(2,1)}_{ce}$$

$$g^{mi}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{\yng(3,1,1)} = \frac{1}{4}\alpha^{ac}_{kl}\sigma^{ep}_{jc}[\nabla_e C^{\yng(2,1)}_{ap}]^{II} - \frac{1}{5}\alpha^{bcd}_{jkl}\nabla_b C^{\yng(2,1)}_{cd}$$

$$g^{jl}[\nabla_m C^{\yng(2,1)\,II}_{ijkl}]^{\yng(2,1,1,1)} = \frac{1}{15}\alpha^{ace}_{ikm}\nabla_a C^{\yng(2,1)}_{ce}$$

$$
\begin{aligned}
g^{mi}[\nabla_m C^{\ydiagram{2,1,1}\,II}_{ijkl}]^{\ydiagram{2,1,1}} &= && && \frac{1}{30}\alpha^{bcd}_{jkl}\nabla_b C^{\ydiagram{2,1}}_{cd}\\
g^{ik}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,II}_{ijkl}]^{\ydiagram{3,2}} &= && && \nabla^p C^{\ydiagram{2,2}\,II}_{pm}\\
g^{mi}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,II}_{ijkl}]^{\ydiagram{3,2}} &= && \nabla^p C^{\ydiagram{2,2}\,II}_{pk} && - \frac{1}{2}\nabla^p C^{\ydiagram{2,2}\,II}_{pk}\\
g^{ik}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,II}_{ijkl}]^{\ydiagram{2,2,1}} &= && && - \nabla^p C^{\ydiagram{2,2}\,II}_{pm}\\
g^{mi}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,II}_{ijkl}]^{\ydiagram{2,2,1}} &= && && \frac{1}{2}\nabla^p C^{\ydiagram{2,2}\,II}_{pk}\\
g^{ik}g^{jl}[\nabla_m C^{\ydiagram{2,1,1}\,II}_{ijkl}]^{\ydiagram{2,2,1}} &= && 0 &&\\
g^{jl}g^{mi}[\nabla_m C^{\ydiagram{2,1,1}\,II}_{ijkl}]^{\ydiagram{2,2,1}} &= && \nabla^p C^{\ydiagram{2,1,1}}_{pk} &&\\
g^{ik}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,I}_{ijkl}]^{\ydiagram{3,2}} &= && \nabla_m C && - \frac{1}{4}\nabla_m C\\
g^{mi}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,I}_{ijkl}]^{\ydiagram{3,2}} &= && \frac{1}{4}\nabla_k C && + \frac{1}{8}\nabla_k C\\
g^{ik}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,I}_{ijkl}]^{\ydiagram{2,2,1}} &= && && \frac{1}{4}\nabla_m C\\
g^{mi}g^{jl}[\nabla_m C^{\ydiagram{2,2}\,I}_{ijkl}]^{\ydiagram{2,2,1}} &= && && - \frac{1}{8}\nabla_k C
\end{aligned}
$$

5.4. ρ

$$
\begin{aligned}
\rho^{abcde}_{ijklm}\nabla_e C_{abcd} &= \nabla^e C^{abcd}[n_{me} - u_m u_e] n_{ia} n_{jb} n_{kc} n_{ld}\\
&\quad +\nabla^e C^{pqrs}[n_{me} - u_m u_e] n_{ap} u_q n_{cr} u_s [-\alpha^{ab}_{ij} u_b][-\alpha^{cd}_{kl} u_d]\\
&\quad +\nabla^e C^{pqcd}[n_{me} - u_m u_e] n_{ap} u_q n_{kc} n_{ld} [-\alpha^{ab}_{ij} u_b]\\
&\quad +\nabla^e C^{abrs}[n_{me} - u_m u_e] n_{ia} n_{jb} n_{cr} u_s [-\alpha^{cd}_{kl} u_d]\\
&=: [\nabla_m C_{ijkl}]^{NN} + [\nabla_m C_{ijkl}]^{UU} + [\nabla_m C_{ijkl}]^{UN} + [\nabla_m C_{ijkl}]^{NU}\\
= \rho^{abcd}_{ijkl}\delta^e_m \nabla_e C_{abcd} &\\
= \nabla_m[\rho^{abcd}_{ijkl} C_{abcd}] &= \nabla_m C^{NN}_{ijkl} + \nabla_m C^{UU}_{ijkl} + \nabla_m C^{UN}_{ijkl} + \nabla_m C^{NU}_{ijkl}
\end{aligned}
$$

in more detail

$$
\rho^{abcde}_{ijklm}\nabla_e[\rho^{pqrs}_{abcd} C_{pqrs}] = \nabla_m[\rho^{abcd}_{ijkl} C_{abcd}]
$$

but

$$
\begin{aligned}
[\nabla \mathbf{C}^{NN}]^{NN} &= \nabla \mathbf{C}^{NN} - \nabla\rho^{NN} \circ \mathbf{C}^{NN} & [\nabla \mathbf{C}^{UN}]^{UN} &= \nabla \mathbf{C}^{UN} - \nabla\rho^{UN} \circ \mathbf{C}^{UN}\\
[\nabla \mathbf{C}^{UU}]^{NN} &= 0 & [\nabla \mathbf{C}^{NU}]^{UN} &= 0\\
[\nabla \mathbf{C}^{UN}]^{NN} &= -\nabla\rho^{NN} \circ \mathbf{C}^{UN} & [\nabla \mathbf{C}^{NN}]^{UN} &= -\nabla\rho^{UN} \circ \mathbf{C}^{NN}\\
[\nabla \mathbf{C}^{NU}]^{NN} &= -\nabla\rho^{NN} \circ \mathbf{C}^{NU} & [\nabla \mathbf{C}^{UU}]^{UN} &= -\nabla\rho^{UN} \circ \mathbf{C}^{UU}\\
[\nabla \mathbf{C}^{UU}]^{UU} &= \nabla \mathbf{C}^{UU} - \nabla\rho^{UU} \circ \mathbf{C}^{UU} & [\nabla \mathbf{C}^{NU}]^{NU} &= \nabla \mathbf{C}^{NU} - \nabla\rho^{NU} \circ \mathbf{C}^{NU}\\
[\nabla \mathbf{C}^{NN}]^{UU} &= 0 & [\nabla \mathbf{C}^{UN}]^{NU} &= 0\\
[\nabla \mathbf{C}^{UN}]^{UU} &= -\nabla\rho^{UU} \circ \mathbf{C}^{UN} & [\nabla \mathbf{C}^{NN}]^{NU} &= -\nabla\rho^{NU} \circ \mathbf{C}^{NN}\\
[\nabla \mathbf{C}^{NU}]^{UU} &= -\nabla\rho^{UU} \circ \mathbf{C}^{NU} & [\nabla \mathbf{C}^{UU}]^{NU} &= -\nabla\rho^{NU} \circ \mathbf{C}^{UU}
\end{aligned}
$$

which implies

$$\nabla[\rho^{NN} + \rho^{UU} + \rho^{UN} + \rho^{NU}] \circ [[\rho^{NN} + \rho^{UU} + \rho^{UN} + \rho^{NU}] \circ \mathbf{C}] \quad = \quad \nabla\rho \circ \mathbf{C} \quad = \quad 0$$

with

$$\begin{aligned}
-\nabla_m[\rho^{NN}]^{abcd}_{ijkl}C^{NN}_{abcd} &= -\nabla_m u^n[[-\alpha^{ab}_{ij}u_b]NN_{nakl} + [-\alpha^{cd}_{kl}u_d]NN_{ijnc}] \\
-\nabla_m[\rho^{NN}]^{abcd}_{ijkl}C^{UN}_{abcd} &= [-\alpha^{ab}_{ij}\nabla_m u_b]UN_{akl} \\
-\nabla_m[\rho^{NN}]^{abcd}_{ijkl}C^{NU}_{abcd} &= [-\alpha^{cd}_{kl}\nabla_m u_d]NU_{ijc} \\
-\nabla_m[\rho^{UU}]^{abcd}_{ijkl}C^{UU}_{abcd} &= -[[-\alpha^{ab}_{ij}\nabla_m u_b][-\alpha^{cd}_{kl}u_d] + [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}\nabla_m u_d]]UU_{ac} \\
-\nabla_m[\rho^{UU}]^{abcd}_{ijkl}C^{UN}_{abcd} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n UN_{anc} \\
-\nabla_m[\rho^{UU}]^{abcd}_{ijkl}C^{NU}_{abcd} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n NU_{nac} \\
-\nabla_m[\rho^{UN}]^{abcd}_{ijkl}C^{UN}_{abcd} &= -[[-\alpha^{ab}_{ij}\nabla_m u_b]UN_{akl} + [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n UN_{anc}] \\
-\nabla_m[\rho^{UN}]^{abcd}_{ijkl}C^{NN}_{abcd} &= [-\alpha^{ab}_{ij}u_b]\nabla_m u^n NN_{nakl} \\
-\nabla_m[\rho^{UN}]^{abcd}_{ijkl}C^{UU}_{abcd} &= [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}\nabla_m u_d]UU_{ac} \\
-\nabla_m[\rho^{NU}]^{abcd}_{ijkl}C^{NU}_{abcd} &= -[[-\alpha^{cd}_{kl}\nabla_m u_d]NU_{ijc} + [-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n NU_{nac}] \\
-\nabla_m[\rho^{NU}]^{abcd}_{ijkl}C^{NN}_{abcd} &= [-\alpha^{cd}_{kl}u_d]\nabla_m u^n NN_{ijnc} \\
-\nabla_m[\rho^{NU}]^{abcd}_{ijkl}C^{UU}_{abcd} &= [-\alpha^{cd}_{kl}u_d][-\alpha^{ab}_{ij}\nabla_m u_b]UU_{ac}
\end{aligned}$$

and

$$\begin{aligned}
\nabla_m C^{NN}_{ijkl} &= \nabla_m[\rho^{NN}]^{abcd}_{ijkl}C^{NN}_{abcd} + \blacktriangledown_m NN_{ijkl} - u_m \bullet NN_{ijkl} \\
\nabla_m C^{UU}_{ijkl} &= \nabla_m[\rho^{UU}]^{abcd}_{ijkl}C^{UU}_{abcd} + [\blacktriangledown_m UU_{ac} - u_m \bullet UU_{ac}][-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d] \\
\nabla_m C^{UN}_{ijkl} &= \nabla_m[\rho^{UN}]^{abcd}_{ijkl}C^{UN}_{abcd} + [\blacktriangledown_m UN_{akl} - u_m \bullet UN_{akl}][-\alpha^{ab}_{ij}u_b] \\
\nabla_m C^{NU}_{ijkl} &= \nabla_m[\rho^{NU}]^{abcd}_{ijkl}C^{NU}_{abcd} + [\blacktriangledown_m NU_{ijc} - u_m \bullet NU_{ijc}][-\alpha^{cd}_{kl}u_d]
\end{aligned}$$

with the definitions

$$\begin{aligned}
\blacktriangledown_i X_{i_1 \dots i_n} &:= n^a_i n_{i_1}{}^{a_1} \dots n_{i_n}{}^{a_n}\nabla_a X_{a_1 \dots a_n} \\
\bullet X_{i_1 \dots i_n} : &:= u^a n_{i_1}{}^{a_1} \dots n_{i_n}{}^{a_n}\nabla_a X_{a_1 \dots a_n}
\end{aligned}$$

and thus

$$\begin{aligned}
[\nabla_m C^{NN}_{ijkl}]^{NN} &= \blacktriangledown_m NN_{ijkl} - u_m \bullet NN_{ijkl} \\
[\nabla_m C^{UN}_{ijkl}]^{NN} &= UN_{akl}[-\alpha^{ab}_{ij}\nabla_m u_b] \\
[\nabla_m C^{NU}_{ijkl}]^{NN} &= NU_{ijc}[-\alpha^{cd}_{kl}\nabla_m u_d] \\
[\nabla_m C^{UU}_{ijkl}]^{UU} &= [\blacktriangledown_m UU_{ac} - u_m \bullet UU_{ac}][-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d] \\
[\nabla_m C^{UN}_{ijkl}]^{UU} &= UN_{anc}[-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n \\
[\nabla_m C^{NU}_{ijkl}]^{UU} &= NU_{nac}[-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d]\nabla_m u^n \\
[\nabla_m C^{UN}_{ijkl}]^{UN} &= [\blacktriangledown_m UN_{akl} - u_m \bullet UN_{akl}][-\alpha^{ab}_{ij}u_b] \\
[\nabla_m C^{NN}_{ijkl}]^{UN} &= NN_{nakl}[-\alpha^{ab}_{ij}u_b]\nabla_m u^n \\
[\nabla_m C^{UU}_{ijkl}]^{UN} &= UU_{ac}[-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}\nabla_m u_d] \\
[\nabla_m C^{NU}_{ijkl}]^{NU} &= [\blacktriangledown_m NU_{ijc} - u_m \bullet NU_{ijc}][-\alpha^{cd}_{kl}u_d] \\
[\nabla_m C^{NN}_{ijkl}]^{NU} &= NN_{ijnc}[-\alpha^{cd}_{kl}u_d]\nabla_m u^n \\
[\nabla_m C^{UU}_{ijkl}]^{NU} &= UU_{ac}[-\alpha^{cd}_{kl}u_d][-\alpha^{ab}_{ij}\nabla_m u_b]
\end{aligned}$$

6. $\nabla \mathbf{R}$

See [12] for more details.

6.0.1. $\nabla[\tau \circ \mathbf{R}]$

$$\nabla[\tau \circ \mathbf{R}] \quad = \quad \nabla \mathbf{R}^{III} + \nabla \mathbf{R}^{II} + \nabla \mathbf{R}^{I}$$

The distribution of components reads $80 = 40 + 36 + 4$.

6.0.2. $\nabla[\rho \circ \mathbf{R}]$

$$\nabla[\rho \circ \mathbf{R}] \quad = \quad \nabla \mathbf{R}^{NN} + \nabla \mathbf{R}^{UN} + \nabla \mathbf{R}^{NU} + \nabla \mathbf{R}^{UU}$$

The distribution of components reads $80 = 24 + 32 + 24$.

6.1. π

$$\pi \circ [\nabla \mathbf{R}] \quad = \quad [\nabla \mathbf{R}]^{\yng(3,2)} + [\nabla \mathbf{R}]^{\yng(2,2,1)}$$

with

$$\begin{aligned}
[\nabla_m R_{ijkl}]^{\yng(3,2)} &= \nabla_m R_{ijkl} - \frac{1}{8}[\alpha^{eab}_{mij} \nabla_e R_{abkl} + \alpha^{ecd}_{mkl} \nabla_e R_{ijcd}] \\
[\nabla_m R_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{8}[\alpha^{eab}_{mij} \nabla_e R_{abkl} + \alpha^{ecd}_{mkl} \nabla_e R_{ijcd}] \\
&= \frac{1}{4}[\eta_{ijm}{}^{a} \nabla^b [*R]_{abkl} + \eta_{klm}{}^{a} \nabla^b [*R]_{abij}] \\
&= \frac{1}{8}[\eta_{ijm}{}^{b} \eta_{kl}{}^{cd} + \eta_{klm}{}^{b} \eta_{ij}{}^{cd}] \nabla^a [*R*]_{abcd} \\
&= - \frac{1}{8}[\alpha^{eab}_{mij} \alpha^{cd}_{kl} + \alpha^{eab}_{mkl} \alpha^{cd}_{ij}] g_{ac} \nabla^p [[*R*]_{pdbe} + g_{bd} [*R*]_{pe}] \\
&= 0
\end{aligned}$$

because of

$$\begin{aligned}
& 0 = \alpha^{eab}_{mij} \nabla_e R_{abkl} \\
\Leftrightarrow \quad & 0 = \nabla^p [*R]_{pjkl} \\
\Leftrightarrow \quad & 0 = \nabla^p [*R*]_{pjkl}
\end{aligned}$$

6.1.1. $\pi \circ \nabla[\tau \circ \mathbf{R}]$

$$\pi \circ \nabla[\tau \circ \mathbf{R}] \quad = \quad [\pi^{\yng(3,2)} + \pi^{\yng(2,2,1)}] \circ [\nabla \mathbf{R}^{III} + \nabla \mathbf{R}^{II} + \nabla \mathbf{R}^{I}]$$

with

$$\begin{aligned}
[\nabla_m R^{III}_{ijkl}]^{\yng(3,2)} &= \nabla_m R^{III}_{ijkl} - \frac{1}{8}[\alpha^{eab}_{mij} \nabla_e R^{III}_{abkl} + \alpha^{ecd}_{mkl} \nabla_e R^{III}_{ijcd}] \\
[\nabla_m R^{II}_{ijkl}]^{\yng(3,2)} &= \nabla_m R^{II}_{ijkl} - \frac{1}{8}[\alpha^{eab}_{mij} \alpha^{cd}_{kl} + \alpha^{ecd}_{mkl} \alpha^{ab}_{ij}] g_{ac} \nabla_e R^{II}_{bd}
\end{aligned}$$

$$
\begin{aligned}
[\nabla_m R^{I}_{ijkl}]^{\yng(3,2)} &= \nabla_m R^{I}_{ijkl} - \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{ecd}_{mkl}\alpha^{ab}_{ij}]\frac{1}{12}g_{ac}g_{bd}\nabla_e R \\
[\nabla_m R^{III}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{8}[\alpha^{eab}_{mij}\nabla_e R^{III}_{abkl} + \alpha^{ecd}_{mkl}\nabla_e R^{III}_{ijcd}] \\
&= \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]g_{ac}\nabla^p R^{III}_{pdbe} \\
[\nabla_m R^{II}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{ecd}_{mkl}\alpha^{ab}_{ij}]g_{ac}\nabla_e R^{II}_{bd} \\
&= - \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{eab}_{mkl}\alpha^{cd}_{ij}]g_{ac}[\frac{1}{2}\alpha^{rs}_{be}[\nabla_r R^{II}_{ds} + g_{ds}\nabla^p R^{II}_{pr}] + g_{bd}\nabla^p R^{II}_{pe}] \\
[\nabla_m R^{I}_{ijkl}]^{\yng(2,2,1)} &= \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{ecd}_{mkl}\alpha^{ab}_{ij}]\frac{1}{12}g_{ac}g_{bd}\nabla_e R \\
&= \frac{1}{8}[\alpha^{eab}_{mij}\alpha^{cd}_{kl} + \alpha^{ecd}_{mkl}\alpha^{ab}_{ij}]g_{ac}[\frac{1}{12}\alpha^{rs}_{be}g_{ds}\nabla_r R + \frac{1}{4}g_{bd}\nabla_e R]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= [\nabla_m R^{III}_{ijkl} + \nabla_m R^{II}_{ijkl} + \nabla_m R^{I}_{ijkl}]^{\yng(2,2,1)} \\
0 &= \nabla^p[[*R*]^{III}_{pjkl} + [*R*]^{II}_{pjkl} + [*R*]^{I}_{pjkl}]
\end{aligned}
$$

in particular

$$
\begin{aligned}
0 &= \alpha^{eab}_{mij}[\nabla_e R^{III}_{abkl} + \alpha^{cd}_{kl}g_{ac}\nabla_e R^{II}_{bd} + \alpha^{cd}_{kl}\frac{1}{12}g_{ac}g_{bd}\nabla_e R] \\
0 &= -\nabla^p R^{III}_{pjkl} + \alpha^{cd}_{kl}\frac{1}{2}[\nabla_c R^{II}_{jd} + g_{jd}\nabla^p R^{II}_{pc}] - \alpha^{cd}_{kl}\frac{1}{12}g_{jd}\nabla_c R
\end{aligned}
$$

6.2. τ

$$
\tau \circ \nabla \mathbf{R} = [\nabla \mathbf{R}]^{III} + [\nabla \mathbf{R}]^{II} + [\nabla \mathbf{R}]^{I}
$$

with

$$
\begin{aligned}
[\nabla_m R_{ijkl}]^{III} &= \nabla_m R_{ijkl} - \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{\yng(2,2)}_{bdm} + g_{am}Y^{\yng(2,2)}_{bcd} + g_{cm}Y^{\yng(2,2)}_{dab}] \\
[\nabla_m R_{ijkl}]^{II} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{\yng(2,2)}_{bdm}]^{II} + g_{am}[Y^{\yng(2,2)}_{bcd}]^{II} + g_{cm}[Y^{\yng(2,2)}_{dab}]^{II}] \\
[\nabla_m R_{ijkl}]^{I} &= \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{\yng(2,2)}_{m} + g_{bm}g_{ac}Y^{\yng(2,2)}_{d} + g_{dm}g_{ac}Y^{\yng(2,2)}_{b}]
\end{aligned}
$$

and

$$
\begin{aligned}
[X^{\yng(2,2)}_{ikm}]^{II\,\yng(3)} &= \frac{1}{2}[\nabla_m R_{ik}]^{II\,\yng(3)} \\
[X^{\yng(2,2)}_{ikm}]^{II\,\yng(2,1)} &= \frac{4}{5}[\nabla_m R_{ik} + \frac{1}{4}\sigma^{ac}_{ik}\nabla^p R_{pacm}]^{II\,\yng(2,1)} \\
[Y^{\yng(2,2)}_{ikm}]^{II\,\yng(2,1)} &= \frac{1}{5}[\nabla^p R_{pikm} + \frac{1}{2}\alpha^{ce}_{km}\nabla_e R_{ic}]^{II\,\yng(2,1)} \\
X^{\yng(2,2)}_{k} &= \frac{1}{18}[\nabla_k R - \nabla^a R_{ak}] \\
Y^{\yng(2,2)}_{k} &= \frac{1}{36}[-\nabla_k R + 4\nabla^a R_{ak}]
\end{aligned}
$$

6.2.1. $\tau \circ \nabla[\tau \circ \mathbf{R}]$

$$\tau \circ \nabla[\tau \circ \mathbf{R}] \quad = \quad [\tau^{III} + \tau^{II}] \circ \nabla \mathbf{R}^{III} + [\tau^{II} + \tau^{I}] \circ \nabla \mathbf{R}^{II} + \tau^{I} \circ \nabla \mathbf{R}^{I}$$

with

$$\begin{array}{rcrcl}
[\nabla_m R^{III}_{ijkl}]^{III} & = & \nabla_m R^{III}_{ijkl} & - & \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{\boxplus\, III}_{bdm} - g_{am}Y^{\boxplus\, III}_{bcd} - g_{cm}Y^{\boxplus\, III}_{dab}] \\
[\nabla_m R^{III}_{ijkl}]^{II} & = & & & \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}X^{\boxplus\, III}_{bdm} + g_{am}Y^{\boxplus\, III}_{bcd} + g_{cm}Y^{\boxplus\, III}_{dab}] \\
[\nabla_m R^{II}_{ijkl}]^{II} & = & & & \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}[X^{\boxplus\, II}_{bdm}]^{II}] \\
[\nabla_m R^{II}_{ijkl}]^{I} & = & & & \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{\boxplus\, II}_{m} + g_{bm}g_{ac}Y^{\boxplus\, II}_{d} + g_{dm}g_{ac}Y^{\boxplus\, II}_{b}] \\
[\nabla_m R^{I}_{ijkl}]^{I} & = & & & \alpha^{ab}_{ij}\alpha^{cd}_{kl}[g_{ac}g_{bd}X^{\boxplus\, I}_{m}]
\end{array}$$

and

$$\begin{array}{rcl}
X^{\boxplus\, III}_{ikm} & = & \sigma^{ac}_{ik}Y^{\boxplus\, III}_{acm} \\
Y^{\boxplus\, III}_{ikm} & = & \frac{1}{5}\nabla^p R^{III}_{pikm} \\
[X^{\boxplus\, II}_{ikm}]^{II} & = & \frac{1}{2}[\nabla_m R^{II}_{ik}]^{II} \\
X^{\boxplus\, II}_{m} & = & -\frac{1}{2}Y^{\boxplus\, II}_{m} \\
Y^{\boxplus\, II}_{m} & = & \frac{1}{9}\nabla^a R^{II}_{am} \\
X^{\boxplus\, I}_{m} & = & \frac{1}{24}\nabla_m R
\end{array}$$

6.3. $\tau \circ \pi$

$$\tau \circ \pi \circ \nabla \mathbf{R} \quad = \quad [\tau^{III} + \tau^{II} + \tau^{I}] \circ [[\nabla \mathbf{R}]^{(3,2)} + [\nabla \mathbf{R}]^{(2,2,1)}]$$

with

$$\begin{array}{rcrcl}
[g^{jl}[\nabla_m R_{ijkl}]^{(3,2)}]^{II} & = & [\nabla_m R_{ik}]^{II} & - & \frac{1}{4}\sigma^{ac}_{ik}[\nabla^p R_{pacm} - \alpha^{xy}_{cm}\nabla_x R_{ay}]^{II} \\
[g^{mi}[\nabla_m R_{ijkl}]^{(3,2)}]^{II} & = & [\nabla^p R_{pjkl}]^{II} & - & \frac{1}{4}[\nabla^p R_{pjkl} - \alpha^{xy}_{kl}\nabla_x R_{jy}]^{II} \\
[g^{jl}[\nabla_m R_{ijkl}]^{(2,2,1)}]^{II} & = & & & \frac{1}{4}\sigma^{ac}_{ik}[\nabla^p R_{pacm} - \alpha^{xy}_{cm}\nabla_x R_{ay}]^{II} \\
[g^{mi}[\nabla_m R_{ijkl}]^{(2,2,1)}]^{II} & = & & & \frac{1}{4}[\nabla^p R_{pjkl} - \alpha^{xy}_{kl}\nabla_x R_{jy}]^{II} \\
g^{ik}g^{jl}[\nabla_m R_{ijkl}]^{(3,2)} & = & \nabla_m R & + & [\nabla^p R_{pm} - \frac{1}{2}\nabla_m R] \\
g^{mi}g^{jl}[\nabla_m R_{ijkl}]^{(3,2)} & = & \nabla^p R_{pk} & - & \frac{1}{2}[\nabla^p R_{pk} - \nabla_k R] \\
g^{ik}g^{jl}[\nabla_m R_{ijkl}]^{(2,2,1)} & = & & - & [\nabla^p R_{pm} - \frac{1}{2}\nabla_m R] \\
g^{mi}g^{jl}[\nabla_m R_{ijkl}]^{(2,2,1)} & = & & & \frac{1}{2}[\nabla^p R_{pk} - \frac{1}{2}\nabla_k R]
\end{array}$$

in particular

$$\begin{array}{rcl}
0 & = & [\nabla^p R_{pjkl} - \alpha^{xy}_{kl}\nabla_x R_{jy}]^{II} \\
0 & = & \nabla^p R_{pm} - \frac{1}{2}\nabla_m R
\end{array}$$

6.3.1. $\tau \circ \pi \circ \nabla[\tau \circ \mathbf{R}]$

$$
\begin{aligned}
\tau \circ \pi \circ \nabla[\tau \circ \mathbf{R}] \quad = \quad & [\tau^{III} + \tau^{II}] \circ [\nabla \mathbf{R}^{III}]^{\boxplus\!\square} \quad + \quad [\tau^{II} + \tau^{I}] \circ [\nabla \mathbf{R}^{II}]^{\boxplus\!\square} \quad + \quad \tau^{I} \circ [\nabla \mathbf{R}^{I}]^{\boxplus\!\square} \\
+ \quad & [\tau^{III} + \tau^{II}] \circ [\nabla \mathbf{R}^{III}]^{\substack{\boxplus\\\square}} \quad + \quad [\tau^{II} + \tau^{I}] \circ [\nabla \mathbf{R}^{II}]^{\substack{\boxplus\\\square}} \quad + \quad \tau^{I} \circ [\nabla \mathbf{R}^{I}]^{\substack{\boxplus\\\square}}
\end{aligned}
$$

with the corresponding traces

$$
\begin{aligned}
g^{jl}[\nabla_m R^{III}_{ijkl}]^{\substack{\boxplus\\\square}} &= \frac{1}{4}\sigma^{ac}_{ik}\nabla^p R^{III}_{pacm} \\
g^{mi}[\nabla_m R^{III}_{ijkl}]^{\substack{\boxplus\\\square}} &= \frac{1}{4}\nabla^p R^{III}_{pjkl} \\
[g^{jl}[\nabla_m R^{II}_{ijkl}]^{\substack{\boxplus\\\square}}]^{II} &= -\frac{1}{8}\sigma^{ac}_{ik}\alpha^{xy}_{cm}[\nabla_x R^{II}_{ay}]^{II} \\
[g^{mi}[\nabla_m R^{II}_{ijkl}]^{\substack{\boxplus\\\square}}]^{II} &= -\frac{1}{8}\alpha^{xy}_{kl}[\nabla_x R^{II}_{jy}]^{II} \\
g^{ik}g^{jl}[\nabla_m R^{II}_{ijkl}]^{\substack{\boxplus\\\square}} &= -\nabla^p R^{II}_{pm} \\
g^{mi}g^{jl}[\nabla_m R^{II}_{ijkl}]^{\substack{\boxplus\\\square}} &= \frac{1}{2}\nabla^p R^{II}_{pk} \\
g^{ik}g^{jl}[\nabla_m R^{I}_{ijkl}]^{\substack{\boxplus\\\square}} &= \frac{1}{4}\nabla_m R \\
g^{mi}g^{jl}[\nabla_m R^{I}_{ijkl}]^{\substack{\boxplus\\\square}} &= -\frac{1}{8}\nabla_k R
\end{aligned}
$$

in particular

$$
\begin{aligned}
0 &= -\nabla^p R^{III}_{pjkl} + \frac{1}{2}\alpha^{cd}_{kl}[\nabla_c R^{II}_{jd} + \frac{1}{3}g_{jd}\nabla^p R^{II}_{pc}] \\
0 &= -\nabla^p R^{II}_{pm} + \frac{1}{4}\nabla_m R
\end{aligned}
$$

6.4. ρ

$$
\begin{aligned}
\rho^{abcde}_{ijklm}\nabla_e R_{abcd} &= [\nabla_m R_{ijkl}]^{\mathcal{NN}} + [\nabla_m R_{ijkl}]^{\mathcal{WW}} + [\nabla_m R_{ijkl}]^{\mathcal{UN}} + [\nabla_m R_{ijkl}]^{\mathcal{NU}} \\
= \quad \nabla_m[\rho^{abcd}_{ijkl}R_{abcd}] &= \nabla_m R^{\mathcal{NN}}_{ijkl} + \nabla_m R^{\mathcal{WW}}_{ijkl} + \nabla_m R^{\mathcal{UN}}_{ijkl} + \nabla_m R^{\mathcal{NU}}_{ijkl}
\end{aligned}
$$

but

$$
\begin{aligned}
[\nabla \mathbf{R}]^{\mathcal{NN}} &= \nabla \mathbf{R}^{\mathcal{NN}} - \nabla \rho^{\mathcal{NN}} \circ \mathbf{R} \\
[\nabla \mathbf{R}]^{\mathcal{WW}} &= \nabla \mathbf{R}^{\mathcal{WW}} - \nabla \rho^{\mathcal{WW}} \circ \mathbf{R} \\
[\nabla \mathbf{R}]^{\mathcal{UN}} &= \nabla \mathbf{R}^{\mathcal{UN}} - \nabla \rho^{\mathcal{UN}} \circ \mathbf{R} \\
[\nabla \mathbf{R}]^{\mathcal{NU}} &= \nabla \mathbf{R}^{\mathcal{NU}} - \nabla \rho^{\mathcal{NU}} \circ \mathbf{R}
\end{aligned}
$$

in particular

$$
[\nabla_m R_{ijkl}]^{\mathcal{NN}} = \blacktriangledown_m \mathcal{NN}^{\boxplus}_{ijkl} - u_m \bullet \mathcal{NN}^{\boxplus}_{ijkl} + [-\alpha^{ab}_{ij}\nabla_m u_b]\mathcal{UN}^{\boxplus}_{akl} + [-\alpha^{cd}_{kl}\nabla_m u_d]\mathcal{NU}^{\boxplus}_{ijc} \tag{6.1}
$$

$$
[\nabla_m R_{ijkl}]^{\mathcal{WW}} = [\blacktriangledown_m \mathcal{WW}^{\boxplus}_{ac} - u_m \bullet \mathcal{WW}^{\boxplus}_{ac} + \nabla_m u^n[\mathcal{UN}^{\boxplus}_{anc} + \mathcal{NU}^{\boxplus}_{nac}]][-\alpha^{ab}_{ij}u_b][-\alpha^{cd}_{kl}u_d] \tag{6.2}
$$

$$
[\nabla_m R_{ijkl}]^{\mathcal{UN}} = [\blacktriangledown_m \mathcal{UN}^{\boxplus}_{akl} - u_m \bullet \mathcal{UN}^{\boxplus}_{akl} + \nabla_m u^n \mathcal{NN}^{\boxplus}_{nakl} + [-\alpha^{cd}_{kl}\nabla_m u_d]\mathcal{WW}^{\boxplus}_{ac}][-\alpha^{ab}_{ij}u_b] \tag{6.3}
$$

$$
[\nabla_m R_{ijkl}]^{\mathcal{NU}} = [\blacktriangledown_m \mathcal{NU}^{\boxplus}_{ijc} - u_m \bullet \mathcal{NU}^{\boxplus}_{ijc} + \nabla_m u^n \mathcal{NN}^{\boxplus}_{ijnc} + [-\alpha^{ab}_{ij}\nabla_m u_b]\mathcal{WW}^{\boxplus}_{ac}][-\alpha^{cd}_{kl}u_d] \tag{6.4}
$$

and

$$\begin{aligned} 0 &= \nabla^l [*R*]_{ijkl} \\ &= \nabla_m [*R*]_{abc}{}^l \rho^{abcm}_{ijkl} \\ &= \nabla_m [*R*]_{abcl} [n^{ml} - u^m u^l] n^a_i n^b_j n^c_k \\ &\quad -\nabla_m [*R*]_{abcl} n^{ml} n_{ia} n_{jb} u_k u_c \\ &\quad -\nabla_m [*R*]_{pqcl} [n^{ml} - u^m u^l] \alpha^{ab}_{ij} n^p_a u_b u^q n^c_k \\ &\quad +\nabla_m [*R*]_{pqcl} n^{ml} \alpha^{ab}_{ij} n^p_a u_b u^q u_k u^c \end{aligned}$$

or

$$0 = n^a_i n^b_j n^c_k n^{ml} \nabla_m [*R*]_{abcl} - u^l [n^a_i n^b_j n^c_k u^m \nabla_m [*R*]_{abcl}] \tag{6.5}$$

$$0 = u_k [n^a_i n^b_j u^c n^{ml} \nabla_m [*R*]_{abcl}] \tag{6.6}$$

$$0 = u_j [n^a_i u^b n^c_k n^{ml} \nabla_m [*R*]_{abcl}] - u_j u^l [n^a_i u^b n^c_k u^m \nabla_m [*R*]_{abcl}] \tag{6.7}$$

$$0 = u_j u_k [n^a_i u^b u^c n^{ml} \nabla_m [*R*]_{abcl}] \tag{6.8}$$

Equating the coefficients of (6.5) - (6.8) with those of (6.1) - (6.4) leads to

$$\begin{aligned} 0 &\neq \blacktriangledown_m NN^{\boxplus}_{ijkl} - u_m \bullet NN^{\boxplus}_{ijkl} + [-\alpha^{ab}_{ij} \nabla_m u_b] UN^{\boxplus}_{akl} + [-\alpha^{cd}_{kl} \nabla_m u_d] NU^{\boxplus}_{ijc} \\ &\quad -u_l [\blacktriangledown_m NU^{\boxplus}_{ijk} - u_m \bullet NU^{\boxplus}_{ijk} + \nabla_m u^n NN^{\boxplus}_{ijnk} + [-\alpha^{ab}_{ij} \nabla_m u_b] UU^{\boxplus}_{ak}] \\ 0 &\neq u_k [\blacktriangledown_m NU^{\boxplus}_{ijl} - u_m \bullet NU^{\boxplus}_{ijl} + \nabla_m u^n NN^{\boxplus}_{ijnl} + [-\alpha^{ab}_{ij} \nabla_m u_b] UU^{\boxplus}_{al}] \\ 0 &\neq -u_j [\blacktriangledown_m UN^{\boxplus}_{ikl} - u_m \bullet UN^{\boxplus}_{ikl} + \nabla_m u^n NN^{\boxplus}_{nikl} + [-\alpha^{cd}_{kl} \nabla_m u_d] UU^{\boxplus}_{ic}] \\ &\quad +u_j u_l [\blacktriangledown_m UU^{\boxplus}_{ik} - u_m \bullet UU^{\boxplus}_{ik} + \nabla_m u^n [UN^{\boxplus}_{ink} + NU^{\boxplus}_{nik}]] \\ 0 &\neq -u_j u_k [\blacktriangledown_m UU^{\boxplus}_{il} - u_m \bullet UU^{\boxplus}_{il} + \nabla_m u^n [UN^{\boxplus}_{inl} + NU^{\boxplus}_{nil}]] \end{aligned}$$

Substitution $R \to [*R*]$ and taking the trace over m, l gives

$$\begin{aligned} 0 &= \blacktriangledown^l [*UU*]^{\boxplus}_{ijkl} - [-\alpha^{ab}_{ij} \blacktriangledown^l u_b] [*NU*]^{\boxplus}_{akl} - [-\alpha^{cd}_{kl} \blacktriangledown^l u_d] [*UN*]^{\boxplus}_{ijc} \\ &\quad + \bullet [*UN*]^{\boxplus}_{ijk} - \bullet u^n [*UU*]^{\boxplus}_{ijnk} - [-\alpha^{ab}_{ij} \bullet u_b] [*NN*]^{\boxplus}_{ak} \\ 0 &= \blacktriangledown^l [*UN*]^{\boxplus}_{ijl} - \blacktriangledown^l u^n [*UU*]^{\boxplus}_{ijnl} - [-\alpha^{ab}_{ij} \blacktriangledown^l u_b] [*NN*]^{\boxplus}_{al} \\ 0 &= \blacktriangledown^l [*NU*]^{\boxplus}_{ikl} - \blacktriangledown^l u^n [*UU*]^{\boxplus}_{nikl} - [-\alpha^{cd}_{kl} \blacktriangledown^l u_d] [*NN*]^{\boxplus}_{ic} \\ &\quad + \bullet [*NN*]^{\boxplus}_{ik} - \bullet u^n [*NU*]^{\boxplus}_{ink} - \bullet u^n [*UN*]^{\boxplus}_{nik}] \\ 0 &= \blacktriangledown^l [*NN*]^{\boxplus}_{il} - \blacktriangledown^m u^n [*NU*]^{\boxplus}_{inl} - \blacktriangledown^m u^n [*UN*]^{\boxplus}_{nil} \end{aligned}$$

or

$$\begin{aligned} 0 &= \eta_{kl}{}^c [\blacktriangledown^l UU^{\boxplus}_{ic} + \blacktriangledown^l u^n [\eta_{ni}{}^b [*NU]^{\boxplus}_{bc} + \eta_{nc}{}^d [UN*]^{\boxplus}_{id}]] \\ &\quad + \bullet [UN*]^{\boxplus}_{ik} + \bullet u^n [\eta_{kn}{}^c UU^{\boxplus}_{ic} + \eta_{ni}{}^b [*NN*]^{\boxplus}_{bk}] \\ 0 &= \blacktriangledown^l [UN*]^{\boxplus}_{il} + \blacktriangledown^l u^n [\eta_{ln}{}^c UU^{\boxplus}_{ic} + \eta_{ni}{}^b [*NN*]^{\boxplus}_{bl}] \\ 0 &= \eta_{kl}{}^c [\blacktriangledown^l [*NU]^{\boxplus}_{ic} + \blacktriangledown^l u^n [\eta_{in}{}^a UU^{\boxplus}_{ac} + \eta_{nc}{}^d [*NN*]^{\boxplus}_{id}]] \\ &\quad + \bullet [*NN*]^{\boxplus}_{ik} - \bullet u^n [\eta_{nk}{}^c [*NU]^{\boxplus}_{ic} + \eta_{ni}{}^a [UN*]^{\boxplus}_{ak}] \\ 0 &= \blacktriangledown^l [*NN*]^{\boxplus}_{il} - \blacktriangledown^l u^n [\eta_{nl}{}^c [*NU]^{\boxplus}_{ic} + \eta_{ni}{}^a [UN*]^{\boxplus}_{al}] \end{aligned}$$

6.5. $\nabla[\omega \circ \rho \circ \mathbf{R}]$

$$
\begin{aligned}
\nabla_m[\omega \circ \rho \circ R]_{ijkl} &= \nabla_m \sum_\alpha Z_{\alpha ij} Z_{\alpha kl} \\
&= \sum_\alpha [\nabla_m Z_{\alpha ij} Z_{\alpha kl} + Z_{\alpha ij} \nabla_m Z_{\alpha kl}] \\
&=: \sum_\alpha [h_{\alpha mij} Z_{\alpha kl} + Z_{\alpha ij} h_{\alpha mkl}]
\end{aligned}
$$

The identity $\nabla^p[*R*]_{pjkl} = 0$ becomes

$$
\begin{aligned}
0 &= \nabla^p \sum_\alpha *Z_{\alpha pj} *Z_{\alpha kl} \\
&=: \sum_\alpha [*h_{\alpha j} *Z_{\alpha kl} + *Z_{\alpha pj} [*h_\alpha]^p{}_{kl}] \\
&= \sum_\alpha [*h_{\alpha j} *Z_{\alpha kl} - h_{\alpha j} Z_{\alpha kl} + \alpha_{kl}^{cd} [\frac{1}{2} g_{cj} [h_\alpha]_d{}^{pq} Z_{\alpha pq} + [h_\alpha]_{dj}{}^p Z_{\alpha pc} + g_{cj} h_\alpha^b Z_{\alpha bd}]]
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= g^{mi}[\nabla_m R_{ijkl}^{III}]^{\boxplus} + [g^{mi}[\nabla_m R_{ijkl}^{II}]^{\boxplus}]^{II} \\
0 &= g^{mi} g^{jl} [\nabla_m R_{ijkl}^{II}]^{\boxplus} + g^{mi} g^{jl} [\nabla_m R_{ijkl}^{I}]^{\boxplus}
\end{aligned}
$$

can be written as

$$
\begin{aligned}
\nabla^p R_{pjkl}^{III} &= \frac{1}{2}[-\alpha_{kl}^{cd} \nabla_c R_{jd}^{II}]^{II} &&= \frac{1}{2} \sum_\alpha [*h_{\alpha j} *Z_{\alpha kl} - h_{\alpha j} Z_{\alpha kl} + \alpha_{kl}^{cd} Z_{\alpha j}{}^b h_{\alpha dbc}]^{II} \\
\nabla^p R_{pk}^{II} &= \frac{1}{4} \nabla_k R &&= \sum_\alpha [-*h_\alpha^b *Z_{\alpha bk} - h_\alpha^b Z_{\alpha bk}]
\end{aligned}
$$

If

$$
Z_{\alpha ij} := \alpha_{ij}^{ab} \nabla_a P_{\alpha b}
$$

then

$$
\begin{aligned}
h_{\alpha abc} &= \nabla_a \nabla_b P_{\alpha c} - \nabla_a \nabla_c P_{\alpha b} \\
h_{\alpha k} &= \nabla^p \nabla_p P_{\alpha k} + R_k{}^p P_{\alpha p} \\
*h_{\alpha k} &= \eta_{ak}{}^{pq} \nabla^a \nabla_p P_{\alpha q} \\
&= -\frac{1}{2} \eta_k{}^{apq} R_{apq}{}^x P_{\alpha x} \\
&= -[*R]_k{}^q{}_q{}^x P_{\alpha x} \\
&= 0
\end{aligned}
$$

so

$$
\begin{aligned}
0 &= \sum_\alpha *Z_{\alpha pj} [*h_\alpha]^p{}_{kl} \\
\nabla^p R_{pjkl}^{III} &= \frac{1}{2} \sum_\alpha [-h_{\alpha j} Z_{\alpha kl} + \alpha_{kl}^{cd} Z_{\alpha j}{}^b h_{\alpha dbc}]^{II} \\
\nabla^p R_{pk}^{II} &= \sum_\alpha [-h_\alpha^b Z_{\alpha bk}]
\end{aligned}
$$

Part I.

See [5], [6], [7] and [8] for comparison.

7. T

Set

$$\frac{[*R*]}{\kappa} \quad =: \quad T$$

That gives

$$\begin{aligned} & & \pi \circ [*\mathbf{R}*] &= \pi \circ \kappa\mathbf{T} \\ & & \tau \circ [*\mathbf{R}*] &= \tau \circ \kappa\mathbf{T} \\ *[\rho \circ \mathbf{R}]* &= & \rho \circ [*\mathbf{R}*] &= \rho \circ \kappa\mathbf{T} \\ *[\rho \circ \tau \circ \mathbf{R}]* &= & \rho \circ \tau \circ [*\mathbf{R}*] &= \rho \circ \tau \circ \kappa\mathbf{T} \end{aligned}$$

in particular

$$\begin{aligned} & & [*\mathbf{R}*]^{\boxplus} &= \kappa\mathbf{T}^{\boxplus} \\ -\mathbf{R}^{III} &= & [*\mathbf{R}*]^{III} &= \kappa\mathbf{T}^{III} \\ -\mathbf{R}^{II} &= & [*\mathbf{R}*]^{II} &= \kappa\mathbf{T}^{II} \\ -\mathbf{R}^{I} &= & [*\mathbf{R}*]^{I} &= \kappa\mathbf{T}^{I} \\ [*[\mathbf{R}^{N\!N}]*] &= & [*\mathbf{R}*]^{U\!U} &= \kappa\mathbf{T}^{E\!E} \\ [*[\mathbf{R}^{U\!U}]*] &= & [*\mathbf{R}*]^{N\!N} &= \kappa\mathbf{T}^{H\!H} \\ [*[\mathbf{R}^{U\!N}]*] &= & [*\mathbf{R}*]^{N\!U} &= \kappa\mathbf{T}^{H\!E} \\ [*[\mathbf{R}^{N\!U}]*] &= & [*\mathbf{R}*]^{U\!N} &= \kappa\mathbf{T}^{E\!H} \end{aligned}$$

with the definitions

$$\begin{aligned} u^j u^l T_{ijkl} &=: E\!E^{\boxplus}_{ik} \\ n^a_i n^b_j u^l T_{abkl} &=: H\!E^{\boxplus}_{ijc} \\ u^j n^c_k n^d_l T_{ijcd} &=: E\!H^{\boxplus}_{akl} \\ n^a_i n^b_j n^c_k n^d_l T_{abcd} &=: H\!H^{\boxplus}_{ijkl} \end{aligned}$$

The same holds for the covariant derivatives

$$\nabla\frac{[*\mathbf{R}*]}{\kappa} \quad = \quad \nabla\mathbf{T}$$

so

$$\begin{aligned} \pi \circ \nabla[*\mathbf{R}*] &= \pi \circ \kappa\nabla\mathbf{T} \\ \tau \circ \nabla[*\mathbf{R}*] &= \tau \circ \kappa\nabla\mathbf{T} \\ \rho \circ \nabla[*\mathbf{R}*] &= \rho \circ \kappa\nabla\mathbf{T} \end{aligned}$$

in particular

$$
\begin{aligned}
[\nabla[*\mathbf{R}*]]^{\yng(3,2)} &= \kappa[\nabla\mathbf{T}]^{\yng(3,2)} \\
[\nabla[*\mathbf{R}*]]^{\yng(2,2,1)} &= \kappa[\nabla\mathbf{T}]^{\yng(2,2,1)} \\
[\nabla[*\mathbf{R}*]]^{III} &= \kappa[\nabla\mathbf{T}]^{III} \\
[\nabla[*\mathbf{R}*]]^{II} &= \kappa[\nabla\mathbf{T}]^{II} \\
[\nabla[*\mathbf{R}*]]^{I} &= \kappa[\nabla\mathbf{T}]^{I} \\
[\nabla[*\mathbf{R}*]]^{NN} &= \kappa[\nabla\mathbf{T}]^{HH} \\
[\nabla[*\mathbf{R}*]]^{UN} &= \kappa[\nabla\mathbf{T}]^{EH} \\
[\nabla[*\mathbf{R}*]]^{NU} &= \kappa[\nabla\mathbf{T}]^{HE} \\
[\nabla[*\mathbf{R}*]]^{UU} &= \kappa[\nabla\mathbf{T}]^{EE}
\end{aligned}
$$

and for the corresponding identity

$$
0 = \nabla^p \frac{[*R*]_{pjkl}}{\kappa} = \nabla^p T_{pjkl}
$$

in particular

$$
\begin{aligned}
0 &= \nabla^p T^{III}_{pjkl} + \alpha^{cd}_{kl}\frac{1}{2}[\nabla_c T^{II}_{jd} + g_{jd}\nabla^p T^{II}_{pc}] + \alpha^{cd}_{kl}\frac{1}{12}g_{jd}\nabla_c T \\
0 &= \nabla^p T^{III}_{pjkl} + \frac{1}{2}\alpha^{cd}_{kl}[\nabla_c T^{II}_{jd}]^{II} \\
0 &= \nabla^l T^{II}_{jl} + \frac{1}{4}\nabla_j T \\
0 &= \eta_{kl}{}^{c}[\blacktriangledown^l[*HH*]^{\boxplus}_{ic} - \blacktriangledown^l u^n[\eta_{ni}{}^{b}[EH*]^{\boxplus}_{bc} + \eta_{nc}{}^{d}[*HE]^{\boxplus}_{id}]] \\
&\quad - \bullet\,[*HE]^{\boxplus}_{ik} + \bullet u^n[\eta_{kn}{}^{c}[*HH*]^{\boxplus}_{ic} + \eta_{ni}{}^{b}EE^{\boxplus}_{bk}] \\
0 &= -\blacktriangledown^l[*HE]^{\boxplus}_{il} + \blacktriangledown^l u^n[\eta_{ln}{}^{c}[*HH*]^{\boxplus}_{ic} + \eta_{ni}{}^{b}EE^{\boxplus}_{bl}] \\
0 &= \eta_{kl}{}^{c}[-\blacktriangledown^l[EH*]^{\boxplus}_{ic} + \blacktriangledown^l u^n[\eta_{in}{}^{a}[*HH*]^{\boxplus}_{ac} + \eta_{nc}{}^{d}EE^{\boxplus}_{id}]] \\
&\quad + \bullet\,EE^{\boxplus}_{ik} + \bullet u^n[\eta_{nk}{}^{c}[EH*]^{\boxplus}_{ic} + \eta_{ni}{}^{a}[*HE]^{\boxplus}_{ak}] \\
0 &= \blacktriangledown^l EE^{\boxplus}_{il} + \blacktriangledown^l u^n[\eta_{nl}{}^{c}[EH*]^{\boxplus}_{ic} + \eta_{ni}{}^{a}[*HE]^{\boxplus}_{al}]
\end{aligned}
$$

The difference between

$$
[*R*]_{ijkl} = \kappa T_{ijkl}
$$

and the Einstein equations

$$
[*R*]_{ik} = \kappa T_{ik}
$$

can be emphasized by considering $\tau \circ [*\mathbf{R}*]$ in both cases:

$$
\tau \circ [*\mathbf{R}*] \quad \left\{ \begin{array}{ccc} [*\mathbf{R}*]^{III} & = & \kappa\mathbf{T}^{III} \\ [*\mathbf{R}*]^{II} & = & \kappa\mathbf{T}^{II} \\ [*\mathbf{R}*]^{I} & = & \kappa\mathbf{T}^{I} \end{array} \right\} \quad \kappa\tau \circ \mathbf{T}
$$

and

$$
\tau \circ [*\mathbf{R}*] \quad \left\{ \begin{array}{ccc} [*\mathbf{R}*]^{III} & = & [*\mathbf{R}*]^{III} \\ [*\mathbf{R}*]^{II} & = & \kappa\mathbf{T}^{II} \\ [*\mathbf{R}*]^{I} & = & \kappa\mathbf{T}^{I} \end{array} \right\} \quad \kappa\mathbf{T}^{I} + \kappa\mathbf{T}^{II} + [*\mathbf{R}*]^{III}
$$

Part II.

Repeating the reasoning of 1.2 in the developed notation.

$$g^{jl}[*R*]^{II}_{ijkl} \quad = \quad g^{jl}\kappa T^{II}_{ijkl} \quad = \quad g^{jl}[\sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}]^{II}$$

Removing the projector II gives

$$g^{jl}[*R*]_{ijkl} \quad = \quad g^{jl}\kappa T_{ijkl} \quad = \quad g^{jl}[\sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}]$$

Removing the trace gives

$$[*R*]_{ijkl} \quad = \quad \kappa T_{ijkl} \quad = \quad \sqrt{\kappa}F_{ij}\sqrt{\kappa}F_{kl}$$

8. $\omega \circ \rho \circ \mathbf{T}$

Setting

$$\frac{[*R*]}{\kappa} \quad =: \quad T$$

gives

$$\omega \circ \rho \circ [*\mathbf{R}*] \quad = \quad \kappa\, \omega \circ \rho \circ \mathbf{T}$$

in particular

$$\begin{aligned}
*N^{\boxplus\, NN}_{\alpha ij} * N^{\boxplus\, NN}_{\alpha kl} &= [-\alpha^{ab}_{ij}u_b]*N^{\boxplus\, NN}_{\alpha a}[-\alpha^{cd}_{kl}u_d]*N^{\boxplus\, NN}_{\alpha c} \\
&=: [-\alpha^{ab}_{ij}u_b]\sqrt{\kappa}E^{\boxplus\, EE}_{\alpha a}[-\alpha^{cd}_{kl}u_d]\sqrt{\kappa}E^{\boxplus\, EE}_{\alpha c} &&=: \sqrt{\kappa}E^{\boxplus\, EE}_{\alpha ij}\sqrt{\kappa}E^{\boxplus\, EE}_{\alpha kl} \\
*U^{\boxplus\, UU}_{\alpha ij} * U^{\boxplus\, UU}_{\alpha kl} &= [-\eta_{ij}{}^{a}U^{\boxplus\, UU}_{\alpha a}][-\eta_{kl}{}^{c}U^{\boxplus\, UU}_{\alpha c}] \\
&=: \eta_{ij}{}^{a}\sqrt{\kappa}*H^{\boxplus\, HH}_{\alpha a}\eta_{kl}{}^{c}\sqrt{\kappa}*H^{\boxplus\, HH}_{\alpha c} &&=: \sqrt{\kappa}H^{\boxplus\, HH}_{\alpha ij}\sqrt{\kappa}H^{\boxplus\, HH}_{\alpha kl} \\
*U^{\boxplus\, UN}_{\alpha ij} * N^{\boxplus\, UN}_{\alpha kl} &= [-\eta_{ij}{}^{a}U^{\boxplus\, UN}_{\alpha a}][-\alpha^{cd}_{kl}u_d]*N^{\boxplus\, UN}_{\alpha c} \\
&=: \eta_{ij}{}^{a}\sqrt{\kappa}*H^{\boxplus\, HE}_{\alpha a}[-\alpha^{cd}_{kl}u_d]\sqrt{\kappa}E^{\boxplus\, HE}_{\alpha c} &&=: \sqrt{\kappa}H^{\boxplus\, HE}_{\alpha ij}\sqrt{\kappa}E^{\boxplus\, HE}_{\alpha kl} \\
*N^{\boxplus\, UN}_{\alpha ij} * U^{\boxplus\, UN}_{\alpha kl} &= [-\alpha^{ab}_{ij}u_b]*N^{\boxplus\, UN}_{\alpha a}[-\eta_{kl}{}^{c}U^{\boxplus\, UN}_{\alpha c}] \\
&=: [-\alpha^{ab}_{ij}u_b]\sqrt{\kappa}E^{\boxplus\, HE}_{\alpha a}\eta_{kl}{}^{c}\sqrt{\kappa}*H^{\boxplus\, HE}_{\alpha c} &&=: \sqrt{\kappa}E^{\boxplus\, HE}_{\alpha ij}\sqrt{\kappa}H^{\boxplus\, HE}_{\alpha kl}
\end{aligned}$$

and

$$*Z_{\alpha ij}*Z_{\alpha kl} \quad =: \quad \sqrt{\kappa}F_{\alpha ij}\sqrt{\kappa}F_{\alpha kl}$$

The identity $\nabla^p T_{pjkl} = 0$ becomes

$$\begin{aligned}
0 &= \nabla^p \sum_\alpha F_{\alpha pj}F_{\alpha kl} \\
&=: \sum_\alpha [j_{\alpha j}F_{\alpha kl} + F_{\alpha pj}[j_\alpha]^p{}_{kl}]
\end{aligned}$$

or

$$
\begin{aligned}
-\nabla^p T^{III}_{pjkl} &= \frac{1}{2}[-\alpha^{cd}_{kl}\nabla_c T^{II}_{jd}]^{II} &&= \frac{1}{2}\sum_\alpha [j_{\alpha j}F_{\alpha kl} - *j_{\alpha j}*F_{\alpha kl} + \alpha^{cd}_{kl}*F_{\alpha j}{}^b *j_{\alpha dbc}]^{II} \\
\nabla^p T^{II}_{pk} &= -\frac{1}{4}\nabla_k T &&= \sum_\alpha [-j^b_\alpha F_{\alpha bk} - *j^b_\alpha *F_{\alpha bk}]
\end{aligned}
$$

If

$$
F_{\alpha ij} := \alpha^{ab}_{ij}\nabla_a A_{\alpha b}
$$

then

$$
\begin{aligned}
j_{\alpha abc} &= \nabla_a\nabla_b A_{\alpha c} - \nabla_a\nabla_c A_{\alpha b} \\
j_{\alpha k} &= \nabla^p\nabla_p A_{\alpha k} + R_k{}^p A_{\alpha p} \\
*j_{\alpha k} &= \eta_{ak}{}^{pq}\nabla^a\nabla_p A_{\alpha q} \\
&= -\frac{1}{2}\eta_k{}^{apq}R_{apq}{}^x A_{\alpha x} \\
&= -[*R]_k{}^q{}_q{}^x A_{\alpha x} \\
&= 0
\end{aligned}
$$

so

$$
\begin{aligned}
-\nabla^p T^{III}_{pjkl} &= \frac{1}{2}[-\alpha^{cd}_{kl}\nabla_c T^{II}_{jd}]^{II} &&= \frac{1}{2}\sum_\alpha [j_{\alpha j}F_{\alpha kl} + \alpha^{cd}_{kl}*F_{\alpha j}{}^b *j_{\alpha dbc}]^{II} \\
\nabla^p T^{II}_{pk} &= -\frac{1}{4}\nabla_k T &&= \sum_\alpha [-j^b_\alpha F_{\alpha bk}]
\end{aligned}
$$

Part III.

9. κ

Let the units *meter* 'm', *second* 's', *kilogram* 'kg' and the *electrostatic unit of charge* 'esu' be defined. Let π denote in this (and only in this) section the number π. Set

$$
\begin{aligned}
c &:= 299792458 \cdot \frac{\mathrm{m}}{\mathrm{s}} \qquad (9.1)\\
G &:= 6,67259 \cdot 10^{-20} \cdot \left[\frac{\mathrm{m}}{\mathrm{esu}}\right]^2 \left[\frac{\mathrm{m}}{\mathrm{s}}\right]^4 \\
\text{alternatively} \qquad G &:= 6,67259 \cdot 10^{-11} \cdot \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2} \\
K_{[1]} &:= \sqrt{\frac{c^4}{8\pi G}} \cdot \frac{\mathrm{m}}{\mathrm{esu}} \\
&\approx 6,940 \cdot 10^{25} \\
K_{[2]} &:= \frac{c^2}{8\pi G} \cdot \frac{\mathrm{m}}{\mathrm{kg}} \\
&\approx 5,359 \cdot 10^{25} \\
K_{[3]} &:= \frac{c^4}{8\pi G} \cdot \frac{\mathrm{m}}{\mathrm{kg}} \cdot \frac{\mathrm{s}^2}{\mathrm{m}^2} \\
&\approx 4,817 \cdot 10^{42}
\end{aligned}
$$

The equations

$$
\begin{aligned}
R_{ijkl} &= \frac{1}{2}\alpha_{ij}^{ab}\alpha_{kl}^{cd}[\partial_a\partial_c g_{bd} + g_{pq}\Gamma^p_{ac}\Gamma^q_{bd}] \\
R_{ijkl} &= \kappa T_{ijkl} \\
T_{ijkl} &= \sum_{\alpha=1}^{3} F_{\alpha ij}F_{\alpha kl} + \Delta[F_{ij}F_{kl}] \\
T_{ijkl} &= T^{HH}_{ijkl} + T^{EH}_{ijkl} + T^{HE}_{ijkl} + T^{EE}_{ijkl}
\end{aligned}
$$

lead to

$$
\begin{aligned}
[R_{ijkl}] &= \frac{1}{\mathrm{m}^2} \\
[R_{ijkl}] &= [\kappa][T_{ijkl}] \\
[F_{\alpha ij}] &= \sqrt{[T_{ijkl}]} \\
[T_{ijkl}] &= [EE_{ik}] + [EH_{ijk}] + [HE_{ijk}] + [HH_{ijkl}]
\end{aligned}
$$

9.1. X

$$
\begin{aligned}
[T_{ijkl}] &=: \frac{\mathrm{X}}{\mathrm{m}^4} \\
[\kappa] &= \frac{\mathrm{m}^2}{\mathrm{X}}
\end{aligned}
$$

$$
\begin{aligned}
[F_{\alpha ij}] &= \frac{\sqrt{\mathrm{X}}}{\mathrm{m}^2} \\
[E\!\!\!\!E_{ik}] &= \frac{\mathrm{X}}{\mathrm{m}^4} \\
\left[\int E\!\!\!\!E^a{}_a \frac{1}{6}\eta_{ijk} dx^i dx^j dx^k\right] &= \frac{\mathrm{X}}{\mathrm{m}}
\end{aligned}
$$

9.2. esu

$$
\begin{aligned}
[F_{\alpha ij}] &=: K_{[1]} \cdot \frac{\mathrm{esu}}{\mathrm{m}^2} \\
[T_{ijkl}] &= K_{[1]}^2 \cdot \frac{\mathrm{esu}^2}{\mathrm{m}^4} \\
[\kappa] &= \frac{\mathrm{m}^2}{K_{[1]}^2 \cdot \mathrm{esu}^2} \\
[E\!\!\!\!E_{ik}] &= K_{[1]}^2 \cdot \frac{\mathrm{esu}^2}{\mathrm{m}^4} \\
\left[\int E\!\!\!\!E^a{}_a \frac{1}{6}\eta_{ijk} dx^i dx^j dx^k\right] &= K_{[1]}^2 \cdot \frac{\mathrm{esu}^2}{\mathrm{m}}
\end{aligned}
$$

9.3. kg

$$
\begin{aligned}
\left[\int E\!\!\!\!E^a{}_a \frac{1}{6}\eta_{ijk} dx^i dx^j dx^k\right] &=: K_{[2]} \cdot \mathrm{kg} \\
[T_{ijkl}] &= K_{[2]} \cdot \frac{\mathrm{kg}}{\mathrm{m}^3} \\
[\kappa] &= \frac{1}{K_{[2]}} \cdot \frac{\mathrm{m}}{\mathrm{kg}} \\
[F_{\alpha ij}] &= \sqrt{K_{[2]} \cdot \frac{\mathrm{kg}}{\mathrm{m}^3}} \\
[E\!\!\!\!E_{ik}] &= K_{[2]} \cdot \frac{\mathrm{kg}}{\mathrm{m}^3}
\end{aligned}
$$

Alternatively

$$
\begin{aligned}
\left[\int E\!\!\!\!E^a_a \frac{1}{6}\eta_{ijk} dx^i dx^j dx^k\right] &=: K_{[2]} \cdot c^2 \cdot \mathrm{kg} \\
&= K_{[3]} \cdot \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \\
[T_{ijkl}] &= K_{[3]} \cdot \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \cdot \frac{1}{\mathrm{m}^3} \\
[\kappa] &= \frac{1}{K_{[3]}} \cdot \frac{\mathrm{m}}{\mathrm{kg}} \cdot \frac{\mathrm{s}^2}{\mathrm{m}^2} \\
[F_{\alpha ij}] &= \sqrt{K_{[3]} \cdot \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \cdot \frac{1}{\mathrm{m}^3}} \\
[E\!\!\!\!E_{ik}] &= K_{[3]} \cdot \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \cdot \frac{1}{\mathrm{m}^3}
\end{aligned}
$$

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