N.S.B. Letters NSBL-QF-031

Thermal Contributions in Divergence-Free Quantum Field Theory

N.S. Baaklini
nsbqft@aol.com
http://nsbcosmic.neocities.org

Abstract

In the framework of divergence-free quantum field theory, we demonstrate how to compute the thermal free energy of bosonic and fermionic fields. While our computations pertain to one loop, they do indicate the method to be applied in higher-loops. In the course of our derivations, use is made of Poisson's summation formula, and the resulting expressions involve the zeta function. We note that the logarithmic terms involve temperature as an energy scale term.

1 Introduction

It should be well-known that the methods of statistical thermodynamics[1] can be applied to quantum field theory[2], where thermal contributions for free and interacting quantum fields can be described by partition functions that can computed using the path integrals of the corresponding Euclidean field theory. The one-loop contributions to the partition integrals of thermal field theories may be easily computed using the zeta-function technique[3],[4],[5]. The latter method is equivalent, at the one-loop level, to our divergence-free effective action framework[6] for quantum field theory. Our purpose in this article is to demonstrate the one-loop thermal computations for bosonic and fermionic field, and indicate how to to use resulting expressions for the propagators in other one- and higher-loop computations.

Recalling that our technique of regularizing the one-loop contribution commences with the representation of the natural logarithm such as:

$$\ln(A) = \varrho_\epsilon \left(-\frac{1}{\epsilon}\frac{1}{A^\epsilon} \right) \qquad \varrho_\epsilon = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\epsilon \right) \tag{1}$$

Here ϵ is a limiting parameter, and ϱ_ϵ is a corresponding pole-removing operator. The above expression is easily shown by writing $A^{-\epsilon} = e^{-\epsilon\ln(A)}$, and expanding in ϵ.

In our following computations, we shall apply the above representation to one-loop contributions in momentum space. We shall perform discrete summations using zeta functions, then apply the operator ϱ_ϵ, thus yielding divergence-free results.

2 Bosonic Contribution

The one-loop contribution to the free energy F of a bosonic quantum field is given by

$$-\beta F = -\frac{1}{2}\mathrm{tr}(\ln W_{ij}) \tag{2}$$

Here $\beta = 1/kT$, and W_{ij} is the bilinear kernel of the bosonic fields in compact notation. For a simple scalar component in momentum space, we have

$$\begin{cases} F = \frac{\delta^3(0)}{2\beta}\int_p \ln(p^2+M^2) \\ \Rightarrow \frac{V}{2\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(\frac{4\pi^2 n^2}{\beta^2}+\mathbf{p}^2+M^2\right) \end{cases} \tag{3}$$

Here, the discrete summation with n is over all integral values, $\mathbf{p}$ is the 3-momentum, and M is the mass associated with the quantum field. Notice how $\delta^3(0)$ is replaced by a finite volume V. Hence, up to a constant term, we can write for the free energy density:

$$\frac{1}{2\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(n^2+\frac{\beta^2\mathbf{p}^2}{4\pi^2}+\frac{\beta^2M^2}{4\pi^2}\right) \tag{4}$$

Scaling the 3-momentum integration variable like $\mathbf{p}\to(2\pi/\beta)\mathbf{p}$, we obtain

$$\frac{1}{2}\frac{1}{\beta^4}\sum_n \int \mathrm{d}^3p\,\ln\left(n^2+\mathbf{p}^2+\frac{\beta^2M^2}{4\pi^2}\right) \tag{5}$$

Now introducing the limiting representation for the logarithm, we write

$$-\frac{1}{2}\frac{1}{\beta^4}\frac{1}{\epsilon}\sum_n \int \mathrm{d}^3p\,\frac{1}{\left(n^2+\mathbf{p}^2+\frac{\beta^2M^2}{4\pi^2}\right)^{\epsilon}} \tag{6}$$

where we have suppressed the operator ϱ_ϵ. Notice that the effect of this operator on the above expression is simply to extract the coefficient of the term linear in ϵ, *after integration over 3-momentum, and executing the discrete summation.*

Integrating over the 3-momentum $\mathbf{p}$, we obtain

$$-\frac{1}{2}\frac{\pi^{\frac{3}{2}}}{\beta^4}\frac{\Gamma(\epsilon-3/2)}{\Gamma(1+\epsilon)}\sum_n \frac{1}{\left(n^2+\frac{\beta^2M^2}{4\pi^2}\right)^{\epsilon-3/2}} \tag{7}$$

Now using the representation

$$\Gamma(\alpha)A^{-\alpha} = \int_0^\infty \mathrm{d}\lambda\,\lambda^{\alpha-1}e^{-\lambda A} \tag{8}$$

we obtain

$$-\frac{1}{2}\frac{\pi^{3/2}}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_n \int_0^\infty \mathrm{d}\lambda\,\lambda^{\epsilon-5/2}\,e^{-\lambda\left(n^2+\frac{\beta^2M^2}{4\pi^2}\right)} \tag{9}$$

Applying Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} e^{-tn^2} = \left(\frac{\pi}{t}\right)^{1/2} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2}{t}n^2} \tag{10}$$

we obtain

$$-\frac{1}{2}\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_n \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-3}\, e^{-\left(\frac{\pi^2}{\lambda}n^2+\lambda\frac{\beta^2M^2}{4\pi^2}\right)} \tag{11}$$

We need to compute the $n = 0$ term and the $n \neq 0$ terms separately. The $n = 0$ of the above expression gives

$$-\frac{1}{2}\frac{\pi^2}{\beta^4}\frac{\Gamma(\epsilon-2)}{\Gamma(1+\epsilon)}\left(\frac{\beta^2M^2}{4\pi^2}\right)^{2-\epsilon} = -\frac{M^4}{32\pi^2}\frac{1}{\epsilon(\epsilon-1)(\epsilon-2)}\left(\frac{\beta^2M^2}{4\pi^2}\right)^{-\epsilon} \tag{12}$$

Picking out the coefficient of the linear term in ϵ, we obtain for this term

$$-\frac{M^4}{32\pi^2}\left\{\frac{3}{2} - \ln\left(\frac{\beta^2M^2}{4\pi^2}\right)\right\} \tag{13}$$

This is the usual contribution in divergence-free quantum field theory. Notice, however, that whereas the logarithmic argument would have an arbitrary scale, it is scaled here with temperature.

We now manipulate the $n \neq 0$ terms. Because of the symmetry $n \leftrightarrow -n$, we write

$$-\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-3}\, e^{-\left(\frac{\pi^2}{\lambda}n^2+\lambda\frac{\beta^2M^2}{4\pi^2}\right)} \tag{14}$$

We need to expand with respect to the mass M. Now making the change of variable

$$\lambda = \frac{1}{\rho} \qquad \int_0^\infty \mathrm{d}\lambda = \int_0^\infty \frac{\mathrm{d}\rho}{\rho^2} \tag{15}$$

we obtain

$$-\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}\rho\, \rho^{1-\epsilon}\, e^{-\left(\pi^2n^2\rho+\frac{\beta^2M^2}{4\pi^2}\frac{1}{\rho}\right)} \tag{16}$$

Expanding with respect to M, we obtain

$$-\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta^2M^2}{4\pi^2}\right)^k \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}\rho\, \rho^{1-\epsilon-k}\, e^{-\pi^2n^2\rho} \tag{17}$$

Integrating over ρ,

$$-\frac{\pi^2}{\beta^4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta^2M^2}{4\pi^2}\right)^k \frac{\Gamma(2-\epsilon-k)}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty}\left(\frac{1}{\pi^2n^2}\right)^{2-\epsilon-k} \tag{18}$$

Now using the definition of the zeta function

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z} \tag{19}$$

we write the series

$$-\frac{\pi^2}{\beta^4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta^2M^2}{4\pi^2}\right)^k\left(\frac{1}{\pi^2}\right)^{2-\epsilon-k}\frac{\Gamma(2-\epsilon-k)}{\Gamma(1+\epsilon)}\zeta(4-2\epsilon-2k) \tag{20}$$

Adding the above result for $n\neq 0$ to the preceding $n=0$ contribution, we obtain for the first few terms of the free energy density:

$$\frac{F}{V}=-\frac{M^4}{64\pi^2}\left\{\frac{3}{2}+2\gamma-\ln\left(\frac{M^2\beta^2}{\pi^2}\right)\right\}-\frac{\pi^2}{90\beta^4}+\frac{M^2}{24\beta^2}-\frac{\zeta'(-2)M^6\beta^2}{192\pi^2}+\cdots \tag{21}$$

Here $\gamma\approx 0.57722$ is the Euler constant, and $\zeta'(-2)\approx -0.03045$. Notice that for a massless quantum field ($M=0$), we have the familiar radiational term $\sim T^4$.

3 Fermionic Contribution

Turning to the case of a fermionic quantum field component, we obtain the one-loop result for the free energy density with an opposite sign:

$$\frac{F}{V}=-\frac{1}{2\beta}\sum_n\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left\{(n+\frac{1}{2})^2+\frac{\beta^2\mathbf{p}^2}{4\pi^2}+\frac{\beta^2M^2}{4\pi^2}\right\} \tag{22}$$

Notice, however, that the discrete energy eigenvalue is $(n+1/2)$ where n takes on all integral values. Now scaling the 3-momentum variable like $\mathbf{p}\to(2\pi/\beta)\mathbf{p}$, we obtain

$$-\frac{1}{2}\frac{1}{\beta^4}\sum_n\int\mathrm{d}^3p\,\ln\left\{(n+\frac{1}{2})^2+\mathbf{p}^2+\frac{\beta^2M^2}{4\pi^2}\right\} \tag{23}$$

Introducing our limiting representation for the logarithm,we write

$$\frac{1}{2}\frac{1}{\beta^4}\frac{1}{\epsilon}\sum_n\int\mathrm{d}^3p\,\frac{1}{\left\{(n+\frac{1}{2})^2+\mathbf{p}^2+\frac{\beta^2M^2}{4\pi^2}\right\}^{\epsilon}} \tag{24}$$

where, again, we have suppressed the operator ϱ_ϵ, however, we should remember to keep only the coefficient of the contribution which is linear in ϵ, after integrating over 3-momentum, and doing the discrete summation.

Integrating over the 3-momentum $\mathbf{p}$, we obtain

$$\frac{1}{2}\frac{\pi^{3/2}}{\beta^4}\frac{\Gamma(\epsilon-3/2)}{\Gamma(1+\epsilon)}\sum_n\frac{1}{\left\{\left(n+\frac{1}{2}\right)^2+\frac{\beta^2M^2}{4\pi^2}\right\}^{\epsilon-3/2}} \tag{25}$$

Now using the representation

$$\Gamma(\alpha)A^{-\alpha} = \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha-1}\, e^{-\lambda A} \tag{26}$$

we obtain

$$\frac{1}{2}\frac{\pi^{3/2}}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_n \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-5/2}\, e^{-\lambda\left\{\left(n+\frac{1}{2}\right)^2+\frac{\beta^2M^2}{4\pi^2}\right\}} \tag{27}$$

Applying the modified Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} e^{-t(n+\frac{1}{2})^2} = \left(\frac{\pi}{t}\right)^{1/2}\sum_{n=-\infty}^{\infty}(-1)^n e^{-\frac{\pi^2}{t}n^2} \tag{28}$$

we obtain

$$\frac{1}{2}\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_n(-1)^n \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-3}\, e^{-\left(\frac{\pi^2}{\lambda}n^2+\lambda\frac{\beta^2M^2}{4\pi^2}\right)} \tag{29}$$

The $n=0$ term of the above expression gives

$$\frac{1}{2}\frac{\pi^2}{\beta^4}\frac{\Gamma(\epsilon-2)}{\Gamma(1+\epsilon)}\left(\frac{\beta^2M^2}{4\pi^2}\right)^{2-\epsilon} = \frac{M^4}{32\pi^2}\frac{1}{\epsilon(\epsilon-1)(\epsilon-2)}\left(\frac{\beta^2M^2}{4\pi^2}\right)^{-\epsilon} \tag{30}$$

Picking out the coefficient of the term that is linear in ϵ, we obtain for the above

$$\frac{M^4}{64\pi^2}\left\{\frac{3}{2}-\ln\left(\frac{\beta^2M^2}{4\pi^2}\right)\right\} \tag{31}$$

Turning to the $n\neq 0$ terms, and from the symmetry under $n\leftrightarrow -n$, we have

$$\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty}(-1)^n \int_0^\infty \mathrm{d}\lambda\, \lambda^{\epsilon-3}\, e^{-\left(\frac{\pi^2}{\lambda}n^2+\lambda\frac{\beta^2M^2}{4\pi^2}\right)} \tag{32}$$

Now making the change of variable

$$\lambda = \frac{1}{\rho} \qquad \int_0^\infty \mathrm{d}\lambda = \int_0^\infty \frac{\mathrm{d}\rho}{\rho^2} \tag{33}$$

we obtain

$$\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty}(-1)^n \int_0^\infty \mathrm{d}\rho\, \rho^{1-\epsilon}\, e^{-\left(\pi^2n^2\rho+\frac{\beta^2M^2}{4\pi^2}\frac{1}{\rho}\right)} \tag{34}$$

Expanding with respect to M, we obtain

$$\frac{\pi^2}{\beta^4}\frac{1}{\Gamma(1+\epsilon)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta^2M^2}{4\pi^2}\right)^k\sum_{n=1}^{\infty}(-1)^n \int_0^\infty \mathrm{d}\rho\, \rho^{1-\epsilon-k}\, e^{-\pi^2n^2\rho} \tag{35}$$

Integrating over ρ,

$$\frac{\pi^2}{\beta^4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\beta^2M^2}{4\pi^2}\right)^k\frac{\Gamma(2-\epsilon-k)}{\Gamma(1+\epsilon)}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\pi^2n^2}\right)^{2-\epsilon-k} \tag{36}$$

Including the $n=0$ result and looking at the first few terms of the above series, we obtain for the free energy density of a free fermionic field component:

$$\frac{M^4}{64\pi^2}\left(\frac{3}{2}+2\gamma-\ln\left(\frac{\beta^2M^2}{4\pi^2}\right)\right)-\frac{7\pi^2}{720\beta^4}+\frac{M^2}{48\beta^2}+\frac{7\zeta'(-2)M^6\beta^2}{192\pi^2}+\cdots \tag{37}$$

where $\zeta'(-2)\approx -0.03045$.

4 Discussion

The one-loop divergence-free thermal computations that we have presented in the preceding sections can easily be extended to higher loops, in any desired field theoretical model. In the above, we have dealt with a regular operator like $1/\Delta^\alpha$. Higher-loop computations would involve propagators that can always be combined to take such a form, but they might also involve momenta-dependent vertices, such as in gauge and gravitational theories, but it would not be too difficult to make the required extensions.

It would be interesting to examine the corrections to the thermal free energy due to field interactions. Most interesting would be the effects of quantum gravity. The corresponding high-temperature contributions may have important applications in stellar and cosmological physics.

References

[1] N.S. Baaklini, "Introduction to Statistical Thermodynamics", *N.S.B. Letters*, **NSBL-ST-010**

[2] N.S. Baaklini, "Introduction to Thermal Quantum Field Theory", *N.S.B. Letters*, **NSBL-QF-030**

[3] A. Salam & J. Strathdee, *Nucl. Phys.*, **B90** (1975) 203

[4] J. Dowker & R. Critchley, *Phys. Rev.*, **D13** (1976) 3224

[5] S.W. Hawking, *Commun. Math. Phys.*, **55** (1977) 133

[6] N.S. Baaklini, "Effective Action Framework for Divergence-Free Quantum Field Theory", *N.S.B. Letters*, **NSBL-QF-010**; http://www.vixra.org/abs/1312.0056