Two Proofs for the existence of integral solutions ($a_1, a_2,\ldots\ldots,a_n$) of the equation $a_1p_1^m + a_2p_2^m+\ldots\ldots+ a_np_n^m =0$ for any positive integer "m", for sequence of primes $p_1,p_2,\ldots,p_n$

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Abstract: We prove using Bezout's identity that $a_1p_1^m + a_2p_2^m+\ldots\ldots+ a_np_n^m =0$ has integral solutions for $a_1, a_2,\ldots\ldots,a_n$, where $p_1,p_2,\ldots,p_n$ is a sequence of primes and m is any positive integer.

Proof for n>2

If $p_1, p_2, p_3,\ldots\ldots\ldots,p_n$ be "n" distinct primes in a sequence and n>2 and m is any positive integer, there exists integers $a_1, a_2, a_3,\ldots\ldots\ldots,a_n$ such that ,

$a_1p_1^m + a_2p_2^m+\ldots\ldots+ a_np_n^m =0$

Since $p_1, p_2, p_3,\ldots\ldots\ldots,p_n$ are n distinct primes, therefore the terms $p_1^m, p_2^m, p_3^m,\ldots\ldots\ldots,p_n^m$ are pair wise co-prime and gcd $(p_1^m, p_2^m, p_3^m,\ldots\ldots\ldots,p_n^m)=1$
This also implies gcd $(p_1^m, p_2^m, p_3^m,\ldots\ldots\ldots,p_{n-1}^m)=1$

Therefore using Bezout's identity there must exist (n-1) integers
$b_1, b_2, b_3,\ldots\ldots\ldots,b_{n-1}$ such that

$b_1p_1^m + b_2p_2^m+\ldots\ldots+ (b_{n-1})(p_{n-1})^m=1$

Multiplying both sides with $(- a_np_n^m)$ where we choose a_n is a non-zero integer,

$(- a_np_n^m)\, b_1p_1^m+(- a_np_n^m)\, b_2p_2^m+\ldots\ldots\ldots\ldots+ (- a_np_n^m)\, (b_{n-1})(p_{n-1})^m = (- a_np_n^m)$

Replacing $(- a_np_n^m)\, b_1$ by a_1,
$(- a_np_n^m)\, b_2$ by a_2,
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$(- a_np_n^m)\, (b_{n-1})$ by a_{n-1}

We have
$a_1p_1^m + a_2p_2^m+\ldots\ldots+ a_{n-1}p_{n-1}^m=(- a_np_n^m)$

or

$a_1p_1^m + a_2p_2^m+\ldots\ldots+ a_{n-1}p_{n-1}^m+ a_np_n^m =0$
where $a_1, a_2, a_3,\ldots\ldots\ldots,a_n$ are integers.

Alternate proof for n>3

Consider again the same equation

$\mathbf{a_1p_1^m + a_2p_2^m + \ldots\ldots + a_{n-1}p_{n-1}^m + a_np_n^m = 0}$

We derive an alternate simple proof for the existence of integral solutions $a_1, a_2, \ldots.. a_n$ where n is a positive integer and n>3, and m is any positive integer for the equation.

Consider a sequence of primes $p_1, p_2, \ldots. p_k, p_{k+1}, \ldots\ldots\ldots p_n$
Let k be a positive integer greater than 1 but less than (n-1), where n>3.

Then consider the sequence of primes $p_1, p_2, \ldots. p_k$

Since $\gcd(p_1, p_2, \ldots p_k) = 1$

Therefore $\gcd(p_1^m, p_2^m, \ldots p_k^m) = 1$

It follows from Bezout's identity that integers $a_1, a_2, \ldots\ldots.., a_k$ exist such that

$\mathbf{a_1p_1^m + a_2p_2^m + \ldots\ldots. + a_kp_k^m = 1}$ **…………………………..(A)**

Similarly $\gcd(p_{k+1}, p_{k+2}, \ldots p_n) = 1$

Therefore $\gcd(p_{k+1}^m, p_{k+2}^m, \ldots p_n^m) = 1$

It follows from Bezout's identity that integers $b_{k+1}, b_{k+2}, \ldots\ldots.., b_n$ exist such that

$\mathbf{b_{k+1}p_{k+1}^m + b_{k+2}p_{k+2}^m + \ldots\ldots. + b_np_n^m = 1}$ **…………………………..(B)**

Subtracting (B) from (A) we obtain:

$(a_1p_1^m + a_2p_2^m + \ldots\ldots. + a_kp_k^m) - (b_{k+1}p_{k+1}^m + b_{k+2}p_{k+2}^m + \ldots\ldots. + b_np_n^m) = 0$

Replacing $-b_{k+1}, -b_{k+2}, \ldots\ldots.., -b_n$ by $a_{k+1}, a_{k+2}, \ldots\ldots.., a_n$

we obtain

$\mathbf{a_1p_1^m + a_2p_2^m + \ldots\ldots. + a_kp_k^m + a_{k+1}p_{k+1}^m + a_{k+2}p_{k+2}^m + \ldots\ldots. + a_np_n^m}$

where $a_1, a_2, \ldots\ldots.., a_k, a_{k+1}, a_{k+2}, \ldots\ldots.., a_n$ are integers.