Some Trigonometric Series for Pi

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Abstract

In this note we show a collection of series for constant Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

The series are special cases of a general formula:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}r^n\cos(n\,x)\sin(n\,y)$$

Resumen

En esta nota mostramos una colección de fórmulas para la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Las fórmulas son casos particulares de una fórmula general:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}r^n\cos(n\,x)\sin(n\,y)$$

Introducción

Recordamos una clásica serie de Fourier para π :

$$\pi = 4\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)x\right)}{2n+1} \ \ ,0 < x < \pi$$

Si ponemos $x = \frac{\pi}{2}$, obtenemos:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

que es la conocida serie de Leibniz-Gregory-Madhava.

En esta nota mostramos una colección de casos particulares de la fórmula:

$$\pi = A\sum_{n=1}^{\infty}\frac{1}{n}r^n\cos(n\,x)\sin(n\,y)$$

A	x	y	r
8	$\pi/6$	$\pi/6$	$\frac{2}{1+\sqrt{3}}$
8	$\pi/8$	$\pi/8$	$1/\sqrt{2}$
8	$\pi/12$	$\pi/12$	$\frac{2}{1+\sqrt{3}}$
8	$\pi/5$	$\pi/5$	$\frac{4}{-1+\sqrt{5}+\sqrt{2\left(5+\sqrt{5}\right)}}$
8	$\pi/4$	$\pi/3$	$\sqrt{6+3\sqrt{3}-\sqrt{59-34\sqrt{3}}}$
8	$\pi/6$	$\pi/3$	$\frac{3}{2}+\sqrt{3}-\frac{1}{2}\sqrt{17+8\sqrt{3}}$
8	$\pi/6$	$\pi/4$	$\sqrt{2-\sqrt{3}}$
8	$\pi/8$	$\pi/4$	$\frac{1}{2}\left(\frac{2}{\sqrt{2-\sqrt{2}}}-2^{3/4}\right)$
8	$\pi/8$	$\pi/6$	$\frac{1}{2}\sqrt{2+\sqrt{8}}-\frac{1}{2}\sqrt{6+\sqrt{2}-4\sqrt{3}}$
8	$\pi/3$	$\pi/2$	$\frac{\sqrt{5}-1}{2}$
8	$\pi/4$	$\pi/2$	$\sqrt{2-\sqrt{3}}$

8	$\pi/5$	$\pi/2$	$\frac{-1-\sqrt{5}+\sqrt{2(11+\sqrt{5})}}{4}$
8	$\pi/6$	$\pi/2$	$\frac{\sqrt{7}-\sqrt{3}}{2}$
8	$\pi/8$	$\pi/2$	$\frac{\sqrt{6+\sqrt{2}}-\sqrt{2+\sqrt{2}}}{2}$

A	x	y	r
12	$\pi/4$	$\pi/4$	$1/\sqrt{3}$
12	$\pi/4$	$\pi/3$	$\sqrt{2}-1$
12	$\pi/4$	$\pi/2$	$\sqrt{4-\sqrt{15}}$
12	$\pi/6$	$\pi/2$	$\frac{\sqrt{13}-3}{2}$
12	$\pi/6$	$\pi/3$	$\sqrt{3}-\sqrt{2}$

A	x	y	r
16	$\pi/4$	$\pi/4$	$\sqrt{2}-1$
16	$\pi/3$	$\pi/3$	$(\sqrt{2}-1)\left(1+\sqrt{2-\sqrt{3}}\right)$
16	$\pi/6$	$\pi/6$	$(\sqrt{2}-1)\left(-1+\sqrt{2+\sqrt{3}}\right)$
16	$\pi/12$	$\pi/12$	$1-\sqrt{\frac{3}{2}+\frac{1}{\sqrt{2}}}$
16	$\pi/12$	$\pi/4$	$\frac{\sqrt{2}+\sqrt{6}-2\sqrt{6-4\sqrt{2}+\sqrt{3}}}{4}$
16	$\pi/6$	$\pi/4$	$\frac{\sqrt{3}-\sqrt{7-4\sqrt{2}}}{2}$

A	x	y	r
24	$\pi/6$	$\pi/4$	$\frac{(\sqrt{3}-1)(3-\sqrt{5})}{2\sqrt{2}}$
24	$\pi/6$	$\pi/3$	$\frac{\sqrt{3}-\sqrt{23-12\sqrt{3}}}{2(\sqrt{3}-1)}$

Otro caso particular es:

$$\pi = 8\sum_{n=1}^{\infty}\frac{\cos(2n)\sin(2n)}{n(\cos 4+\sin 4)^{n}}$$

Referencias

1. Valdebenito , E. : Pi Handbook, manuscript,unpublished,1989,(20000 fórmulas)