A REMARK BY ATIYAH ON DONALDSON'S THEORY, AP THEORY AND ADS/CFT DUALITY

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Abstract. Using Artin Presentation Theory, we mathematically augment a remark of Atiyah on physics and Donaldson's 4D theory which, conversely, explicitly introduces the theoretical physical relevance of AP Theory into Modern Physics. AP Theory is a purely discrete group-theoretic, in fact, a framed pure braid theory, which, in the sharpest possible *holographic* manner, *encodes* all closed, orientable 3-manifolds and their knot and linking theories, and a large class of compact, connected, simply-connected, *smooth* 4-manifolds with a connected boundary, whose physical relevance for Atiyah's remark we explain.

1. Introduction

The remark by Atiyah (see [A], p.5) we refer to is: *"One might say that the physicist's ambivalence to particles and fields is the essence of Donaldson's theory".*

We give a mathematically rigorous $(3+1)$-D smooth topological augmentation of this important prescient metamathematical remark:

The starting point of AP Theory is purely *discrete* group-theoretic, based on the Cayley-v.Dyck process, which is realized as follows: let $r_1, \ldots, r_m$ be m arbitrary elements, i.e., words, in the free group F_n, generated by $x_1, \ldots, x_n$.

Let N be the normal subgroup of F_n which is the intersection of *all* normal subgroups of F_n, which contain all the $r_1, \ldots, r_m$; then one says the factor group $F_n/N = G$ is presented by the *presentation* $< x_1, \ldots, x_n | r_1, \ldots, r_m >$.

It is important to notice here that the r_i of a presentation of the trivial group can be as complicated as that of any other group, and that, *a priori*, a concept of 'infinity' is used here, when saying: "intersection of *all* normal subgroups".

We will call this non-infinitesimal, discrete, *minimizing* '∞', the 'Cayley-v.Dyck' ∞ as opposed to the infinitesimal ∞ used in the construction of the classic analytical continuum, and its analytic ODE and PDE equations and moduli.

If furthermore $m = n$, a presentation $r {=} < x_1, \ldots, x_n | r_1, \ldots, r_n >$ is called an *Artin presentation* if, (in F_n!), the following group-theoretic equation holds:

$$x_1 \ldots x_n = r_1^{-1} x_1 r_1 \ldots r_n^{-1} x_n r_n$$

.

This is the *Artin Equation*, the fundamental equation of AP Theory, which characterizes pure framed braids of the 2-sphere. Also, an arbitrary group is isomorphic to the fundamental group of a closed, orientable 3-manifold , if and only if it has an Artin Presentation, see [G].

The $n \times n$ integer matrix, obtained by abelianization, from an Artin presentation r is denoted by $A(r)$; it is always symmetric, [W], p.248, and every symmetric, integer matrix appears in this manner.

Let Ω_n denote the compact 2-disk with n holes;, each one bounded by a circle; given a point on the bottom of the outer boundary of this 2-disk, and a left to right ordering of the holes, an Artin presentation determines an isotopy class, keeping the boundary of Ω_n fixed, of diffeomorphisms $h(r) : \Omega_n \to \Omega_n$, which restrict to the identity on the boundary of Ω_n, and every such isotopy class is determined by an Artin presentation, see [W], p.225, [G].

This is a theorem, not a postulate, see González-Acuña, [G], and is the first AP metamathematical bridge between the discreteness of r and the non-infinitesimal, but still smooth, *flat* $2D$ 'membranic' topology of $h(r) : \Omega_n \to \Omega_n$, the AP topological analogue of Planck's 'continuous to discrete' postulate.

Given r, with $h(r) : \Omega_n \to \Omega_n$ one constructs, via a relative open book construction, a (unique, up to diffeomorphism), connected, compact, simply-connected, *smooth* 4-manifold, $W^4(r)$, with a connected boundary, $M^3(r)$, such that the quadratic form of $W^4(r)$ is given by the symmetric matrix $A(r)$ and r presents the fundamental group, $\pi(r)$, of $M^3(r)$, [W], p.250.

We remark that the topological flatness of Ω_n, i.e., that its genus is 0, is necessary here, and that *all* closed, orientable 3-manifolds can be so constructed; for many examples with $\pi(r) = 1$, i.e., in fact *closed* $W^4(r)$, with non-trivial Seiberg-Witten invariants, which can be so constructed, see [CW1].

This construction relates the non-infinitesimal, but still smooth, $2D$ *flatness* of $h(r) : \Omega_n \to \Omega_n$ to the $4D$ infinitesimal differential geometry, of the $W^4(r)$ with its analytic *curvature*, connections, etc.

Moreover, the $4D$ smooth $W^4(r)$ are obtained from the discrete Artin presentation r, in the sharpest possible *holographic* manner: in AP Theory a connected, compact, smooth, simply-connected (3+1)-D manifold, with a connected boundary, is already determined by a certain type of presentation, an Artin presentation, of the fundamental group of that boundary.

One does not need a 'convenient place to put the hologram', compare to Maldacena, [M], p.63.

From the classical, *rooted in quantum field theory*, Donaldson Theorem, we obtain, *as a corollary*, the following non-trivial theorem, which can, nevertheless, be stated entirely in the discrete AP Theory:

THEOREM ([W]**, p.240)** *If $A(r)$ is a symmetric, integer, unimodular matrix, prevented by Donaldson's Theorem from representing the quadratic form of a closed, smooth, simply-connected 4-manifold, (such as, e.g., E_8), then the group $\pi(r)$ cannot be trivial ; in fact, it has a non-trivial representation into the Lie group SU(2).* See also [R], p.631.

It is important to point out here, that there exist (necessarily non-Artin) presentations, w, of the trivial group, such that $A(w) = E_8$, (see p.366 of [C1]). Since '$A(r)$ *not* unimodular' is also an abelian condition which implies $\pi(r)$ is non-trivial, one might at first think that '$A(r)$ *not* congruent over Z to the identity matrix' is also such a condition when $A(r)$ is unimodular and positive definite. But that is false by the above example.

It is the Artin Equation which brings in the deeper number theory of the symmetric integer matrix $A(r)$ and furthermore relates it to SU(2), via the above construction and Donaldson's Theorem, thus transcending, e.g., the Kirby Calculus. See also Remark 4 in section 5 ahead.

This theorem is, in particular, a non-trivial metamathematical bridge, a clasp, between the discrete Artin presentation r and the field-theoretic 4D curvature appearing in Donaldson's analytic theory.

Even if this theorem had another proof, using Donaldson's theorem gives the shortest and most immediate one.

We can again say: here non-infinitesimal, but still smooth, $2D$ *flatness* is related to infinitesimal using $4D$ *curvature* of the $W^4(r)$.

The $2D$ membranicity of the $h(r) : \Omega_n \to \Omega_n$ seems to give a smooth topological $4D$ "'metrical elasticity' of space, i.e., to generalized forces which oppose the curving of space"; as intuited by Sakharov, [S].

In full generality: this theorem relates two infinities, the discrete non-infinitesimal, minimizing Cayley-v.Dyck ∞ of discrete AP Theory and the classic infinitesimal ∞, used in continuous analytic field theory, curvature, connections, etc., in differential geometry.

This is our smooth $(3+1)$-D augmentation of Atiyah's remark and, conversely, is the explicit debut of AP Theory in theoretical Modern Physics.

Although, in the following, we do not explicitly, technically, use the above theorem, we think that its mere metamathematical existence, justifies taking seriously the many analogies between the mathematically rigorous AP Theory and still heuristic, but important, theories of Modern Physics.

We exploit these analogies to give short *very preliminary* views of the *'cosmic multiversal conceptual geography'*, so to speak, of the important, but still mostly heuristic, theories of modern physics: String/M-Theory in section 2, Quantum Gravity in section 3 and AdS/CFT Duality in section 4.

It will become clear that AP Theory, by not being a mere physical model, is a universal, *as holographic as possible*, $(3+1)$-D smooth topological *multiversal completion*, of these heuristic theories and should say something mathematically rigorous about them, at least in their multiversal versions and limits. Compare to Wilczek [Wk].

The set of Artin presentations on n generators form a group, which is isomorphic to $P_n \times Z^n$, where P_n is the pure braid group on n strands. If t and r are two elements of this group then $A(t \cdot r) = A(t) + A(r)$, i.e., $A(r)$ behaves like a logarithm of r, see [W], p.227.

AP Theory does not postulate, is canonically characteristic of dimension $(3+1)$ and needs no higher dimensions to be mathematically consistent; it is deterministic, background-independent, non-perturbative, parameter-free, and, a priori, uses no classical SUSY, nor Feynman graphs or integrals, nor metrics or analytic moduli, nor statistical, or category arguments, nor virtual particles or 'wormholes', nor 'bordism glueing', etc.

It has the powerful action of a graded group, the Torelli, and a sharp theory of topological change. [C], [W], [W1]. See section 3.

It shows that if one completely removes the infinitesimal analytic classical ∞, so basic for curvature/connection arguments, one still obtains a non-trivial smooth $4D$ theory, based on $2D$ smooth flatness instead of $4D$ curvature, with, furthermore, a Donaldson-like theorem, to legitimize it and provide a meta-mathematical anchor to classical theoretical physics.

2. Artin presentations as cosmic strings.

First we point out that AP Theory, a priori, has no relation whatsoever with Sullivan's very abstract algebraic 'topological string theory', some immediate basic differences being, that AP Theory is canonically characteristic of dimension $(3+1)$, has a very sharp concept of holography and has a Donaldson-like theorem.

Classic String Theory demands that mathematical points be substituted by extended objects, however, as such, it still uses the analytic classic ∞, which is based on such points, when defining strings as background-dependent arcs and loops.

AP Theory immediately adresses Schwarz's 'third basic principle', about why particles, which *'in practice, are smeared over a region of space due to quantum effects, but their description in the basic equations is as mathematical points'*; see [Sch]. See also [M], p.58, for the inadequacy of analytic equations, in certain crucial situations of physics.

Hence it is natural, due to the difficulties appearing in classic string theory, to refer to an Artin presentation r as a *background independent* 'string' and the associated isotopy classes $h(r) : \Omega_n \to \Omega_n$ as its set of (membranic) 'excitations', thus avoiding the many nomographical and analytical problems of classical background-dependent String/M-theory.

In AP Theory, classical strings and their background-dependent 'worldsheets', become as obsolete as 'planetary orbits' for electrons in early classical QM.

Thus AP strings are the non-infinitesimal products of the extremal minimality of the Cayley-v.Dyck process and the fact that a different ∞, with no analytic equations, only non-infinitesimal smooth topology is being used, makes it natural to call them 'cosmic'.

This is how we see *'the signature of string theory'* in AP Theory, see p. 8 of [Wi].

This also supports the speculations of Susskind in [Su]: black holes in AP string theory are the 'excited string states' $h(r) : \Omega_n \to \Omega_n$ and their 'single string states' are the r, the discrete, but not point-like 'particles' of AP Theory. See also [H], p.9, footnote.

This also seems to support 't Hooft's *'long standing belief that black holes are the extrapolations of elementary particles to high mass'*, see [Su], last page.

In AP Theory, the only remnants of the classical 'entropy of black holes' are the purely mathematical entropies of the diffeomorphisms $h(r) : \Omega_n \to \Omega_n$.

The $W^4(r)$ should have no classical black holes on them, caused by curvature singularities, since they are constructed with the $2D$ *flat*, non-infinitesimal, but still smooth $h(r) : \Omega_n \to \Omega_n$.

Since the discrete Artin presentation determines the *membrane* $h(r) : \Omega_n \to \Omega_n$, we can call AP Theory also a topological M-theory, thus AP theory is a mathematically rigorous smooth topological multiversal background-independent String/M-Theory, with a non-trivial Donaldson Theorem.

In AP Theory, the hologram r, the string particle r and the black hole r are identified to a single thing, the Artin presentation r, in a consistent mathematically rigorous manner, still leaving a non-trivial theory.

Atiyah's remark makes sense with the r considered as *string particles.*

In AP Theory, AP String/M-Theory lives in harmony with AP QM and AP GR.

3. AP Theory as a smooth topological $4D$ Quantum Gravity Theory.

The extreme AP holography is at the basis of exhibiting AP Theory as a union, a mixture of AP topological GR and QM, i.e., a topological Quantum Gravity Theory, without the numerics of the classical Planck constant, nor the analytic curvature of classical GR. Thus AP Theory avoids the inadequacy of classical analytic equations, and their nomography, when trying to reconcile that matter obeys the laws of classical QM and gravity, the laws of classical GR, see [M], p.58.

Besides the first analogy of the AP r and $h(r) : \Omega_n \to \Omega_n$ (the AP analogues of the wave-packets of QM), with Planck's 'continuous to discrete' postulate, and, more topologically, de Broglie's wave-particle duality, AP Theory has much stronger analogies to QFT, where the classic Donaldson Theory is rooted in:

The important QM concepts of ∞ and non-commutativity are retained, *dynamically so*: In AP Theory there exists, a graded by the positive integers, (i.e., cone-like), ∞-*generated* at each stage, group of interactions/transitions, which 'acts' in unison on the r, the $h(r) : \Omega_n \to \Omega_n$ and the $W^4(r)$, and their connected boundaries, the $M^3(r)$. It is called the Torelli group, [C], [W], which also consists of Artin presentations; its elements are characterized by $A(r) = 0$. In the latter case these transitions are smooth topology changing, but homology-preserving, (they preserve the matrix $A(r)$), in a subtle and interesting manner, see [C] and [W1]. At each stage n, this group is isomorphic to the *commutator subgroup* (of the pure braid group P_n on n strands), which is always ∞-generated, when $n > 2$.

These interactions/transitions are *'immediately there'* and need not be separately postulated, compare to [Wi].

These, at each stage ∞-generated, powerful dynamical symmetries are the mathematically rigorous AP equivalent of classical *postulated* SUSY.

The *whole* of AP Theory is the multiversal AP version of so-called '$\mathcal{N} = 4$ Super $4D$ YM Theory, in the planar limit', the classical maximally supersymmetric gauge theory in four dimensions, the simplest classical non-trivial QFT.

On the other hand, with respect to GR in AP Theory, with the holographic construction of the *smooth* $(3 + 1)$-D $W^4(r)$, AP Theory, without using analytic classical curvature, retains smooth topologically and seamlessly, some remnant of Einstein's differential geometric gravity, due to the construction of these $4D$ smooth $W^4(r)$ and the Theorem above.

Referring to our previous section, we can say: in AP Theory, GR is the *structure formation* theory of M-Theory.

In AP Theory, the r form the QM part, the $W^4(r)$, the GR gravity part, as in the more restricted case of the original AdS/CFT duality on anti-de Sitter space, where gravity lies in the bulk and QFT on the boundary.

Thus AP Theory, with its ultimate holography, extends this duality to the most universal *multiversal* $(3 + 1)$-D smooth topological case, which we discuss in the next section.

4. AP Theory as the ultimate multiversal AdS/CFT Duality.

The classical AdS/CFT duality, also known as gauge/gravity duality, see [H], is a *conjectured* mathematical relationship between two kinds of physical theories, quantum gravity in anti-de Sitter spaces on the left side and QFT on the other.

It's holography is just from the dimension on the left, (the bulk), to one less dimension on the right, (the boundary).

This non-multiversal duality, also called Maldacena Duality, was arrived at using the framework of string theory on the left side and motivated by so-called black-holes, [H], p.4, and suggests that GR arises from some 'underlying microscopic theory', [H], p.4.

With its as sharp as possible holography, AP Theory, with the multiversal version of the classic left-side being the $W^4(r)$ and the other side being the discrete Artin presentations, r, provides a very clear, mathematically rigorous, *multiversal* analogue of AdS/CFT duality, and raises the question whether 'something has gone wrong along the way', [H], p.12, when attempting to prove the latter rigorously.

Are anti-de Sitter spaces 'procrustean beds' in AdS/CFT, due to not going to the multiversal AP metamathematical limit? Compare to [Wi1].

We conjecture that a rigorous mathematical proof of the AdS/CFT is furthermore obstructed by insisting that string theory is just on the left side, whereas in AP Theory, due to its as sharp as possible holography, considered as the multiverse version of AdS/CFT, string theory is at the beginning, i.e., really on *both* sides, see section 2 above. Furthermore, in AP Theory, we have a 'complete independent' definition of the quantum gravity side of the correspondence, unlike in the classical AdS/CFT duality, [H], p.4.

5. Remarks, Questions.

1. The ultimate holography of AP Theory, that the extended 'particles', the r, determine the smooth 'fields', the $h(r) : \Omega_n \to \Omega_n$, is parallel to the 'disturbing' dualism that Newton's material points (studied with ODE) appeared side by side with Maxwell's continuous fields (studied with PDE), see Einstein, [E], seems to suggest that classical PDE 'quantizes' metamathematically into the discrete group-theoretic Fox PDE Calculus, which is *not* merely formal, see [CW]. This seems to enhance the Maxwellian nature of AP Theory, see [D], and is relevant to Klainerman's worries about classical PDEs being a 'unified subject', [K].

2. Does the sheer mathematical existence of AP Theory, at least topologically, answer A. Zee's 'IQ question', [Z]: *'What is to Gravity as YM is to EM?'*

3. Since the 'flatness' of the $h(r) : \Omega_n \to \Omega_n$ is responsible for the 'linearity' in the *group ring* of $\pi(r)$, as in [CW], p.2, can AP Theory help explain the appearence of the polytope geometry of the so-called amplituhedron? See [AT].

4. Since the Theorem in the introduction above relates the Number Theory of the matrix $A(r)$ to Representation Theory into $SU(2)$, it can be considered to be a very, very primitive version of Langlands Theory. Nevertheless, we can ask: just like the also very primitive Rohlin Theorem, can it have any serious *disruptive* effects specially on Geometric Langlands Theory, just like Rohlin's Theorem has on low-dimensional Handlebody theory? See [Si].

5. How far is AP Theory from being, at least smooth $(3 + 1)$-D topologically, 'a mathematically complete example of a quantum gauge field theory in four-dimensional space-time'? See [JW], pp.3, 5.

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