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The Lorentz Force as Pure Geodesic Motion in Four-Dimensional Spacetime

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Abstract: We develop a linear element ds in ordinary four-dimensional spacetime which, when held stationary under worldline variations, leads to the gravitational equations of geodesic motion extended to include the Lorentz force law. We see that in the presence of an electromagnetic vector potential A^μ , all that is needed to obtain this result is to follow the well-known gauge theory prescription of replacing the kinetic momentum p^μ with a canonical momentum $\pi^\mu = p^\mu + eA^\mu$ in the mass / momentum relationship $m^2 = p_\sigma p^\sigma$, and then to apply variational calculus to obtain the motion of charged particles in this potential. We also show how by this same prescription, Maxwell's classical source-free field equations become embedded within the second Bianchi identity of Riemannian geometry.

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1. Introduction

In §9 of his landmark 1916 paper [1], Albert Einstein first derived the geodesic equation of motion $d^2x^\mu/ds^2 = -\Gamma^\mu_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)$ for a particle in a gravitational field based on the variation $0 = \delta \int_A^B ds$ of the linear metric element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ between any two spacetime events A and B at which the worldlines of different observers meet so that their clocks and measuring rods and scales can be coordinated at the outset A and then compared at the conclusion B . Notably absent from [1], however, was a similar geodesic development of the Lorentz force law $d^2x^\mu/ds^2 = (e/m)F^\mu_\alpha(dx^\alpha/ds)$. Subsequent papers by Kaluza [2] and Klein [3] did succeed in explaining the Lorentz force as a type of geodesic motion and even gave a geometric explanation for the electric charge itself, but only at the cost of adding a fifth dimension to spacetime and curling that dimension into a cylinder. To date, a century later, there still does not appear to have been any fully-successful attempt to obtain the Lorentz force from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime. In this letter, we show how this is done.

2. Basis and derivation

As the basis for obtaining the Lorentz force from a geodesic variation in four dimensions, we begin with the equation $m^2 = p_\sigma p^\sigma$ that describes the relativistic relationship between any mass m and its “kinetic” energy-momentum $p^\mu = mu^\mu = m(dx^\mu/ds)$. We then promote this kinetic momentum to a “canonical” momentum π^μ via the prescription $p^\mu \rightarrow \pi^\mu = p^\mu + eA^\mu$ taught by the local gauge (really, phase) theory of Hermann Weyl developed over 1918 to 1929 in [4], [5], [6], and so obtain $m^2 = p_\sigma p^\sigma \rightarrow m^2 = \pi_\sigma \pi^\sigma$. It will be appreciated that this prescription is the momentum space equivalent of $\partial_\mu = \partial/\partial x^\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ which is the gauge-covariant derivative specified in a configuration space for which the metric tensor of the tangent flat Minkowski space is $\text{diag}(\eta_{\mu\nu}) = (+1, -1, -1, -1)$. Consequently, deconstructing into a linear equation using the Dirac matrices $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$ in flat spacetime, one can employ $m^2 = \pi_\sigma \pi^\sigma$ to obtain Dirac’s equation $(i\gamma^\mu D_\mu - m)\psi = 0$ for an electron wavefunction ψ in an electromagnetic potential A_μ , which equation Dirac first derived in [7] for a free electron in a form equivalent to $(i\gamma^\mu \partial_\mu - m)\psi = 0$, i.e., without yet using $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$.

So to obtain the Lorentz force from a geodesic variation in spacetime, we backtrack from $m^2 = \pi_\sigma \pi^\sigma$ to a linear metric element:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &\rightarrow ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu = g_{\mu\nu}(dx^\mu + ds(e/m)A^\mu)(dx^\nu + ds(e/m)A^\nu), \\ &= g_{\mu\nu}dx^\mu dx^\nu + 2(e/m)A_\sigma dx^\sigma ds + (e/m)^2 g_{\mu\nu}A^\mu A^\nu ds^2 \end{aligned} \quad (2.1)$$

which uses a canonical gauge prescription for the spacetime coordinates themselves, namely:

$$dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu. \quad (2.2)$$

This is just another variation of $p^\mu \rightarrow \pi^\mu = p^\mu + eA^\mu$ and $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$. Indeed, it is easily seen that if one multiplies $ds^2 = (dx_\sigma + ds(e/m)A_\sigma)(dx^\sigma + ds(e/m)A^\sigma)$ in (2.1) through by m^2/ds , the result is identical to the canonical $m^2 = \pi_\sigma \pi^\sigma$. Now, all we need do is apply a variation $0 = \delta \int_A^B ds$ to the linear element (2.1) and the Lorentz force naturally emerges as a geodesic equation of motion right alongside of the gravitational equation of motion.

Proceeding with this derivation which largely parallels that in the online [8], we first use (2.1) to construct the number

$$1 = \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma \frac{dx^\sigma}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu} A^\mu A^\nu}, \quad (2.3)$$

which we then use to write the variation as:

$$0 = \delta \int_A^B ds = \delta \int_A^B ds \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma \frac{dx^\sigma}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu} A^\mu A^\nu}. \quad (2.4)$$

Applying δ to the integrand and using (2.3) to clear the denominator, this yields:

$$0 = \delta \int_A^B ds = \frac{1}{2} \int_A^B ds \delta \left(g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma \frac{dx^\sigma}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu} A^\mu A^\nu \right). \quad (2.5)$$

Dropping the $1/2$ and using the product rule, while assuming that there is no variation in the charge-to-mass ratio – i.e., that $\delta(e/m) = 0$ – over the path from A to B , we now distribute δ using the product rule to obtain:

$$0 = \int_A^B ds \left(\delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\nu}{ds} + 2 \frac{e}{m} \delta A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} + (e/m)^2 (\delta g_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} \delta A^\mu A^\nu + g_{\mu\nu} A^\mu \delta A^\nu) \right). \quad (2.6)$$

One can use the chain rule in the small variation $\delta \rightarrow \partial$ limit to show that $\delta g_{\mu\nu} = \partial_\alpha g_{\mu\nu} \delta x^\alpha$ and $\delta A_\sigma = \partial_\alpha A_\sigma \delta x^\alpha$. So the bottom line equals $\delta x^\alpha (e/m)^2 (\partial_\alpha g_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} \partial_\alpha A^\mu A^\nu + g_{\mu\nu} A^\mu \partial_\alpha A^\nu)$. Likewise, we may recondense $\partial_\alpha (g_{\mu\nu} A^\mu A^\nu) = \partial_\alpha g_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} \partial_\alpha A^\mu A^\nu + g_{\mu\nu} A^\mu \partial_\alpha A^\nu$ via the product rule. Therefore, the entire integral on the bottom line contains a total derivative given by:

$$\int_A^B \delta x^\alpha \left(\frac{e}{m} \right)^2 \frac{\partial}{\partial x^\alpha} (g_{\mu\nu} A^\mu A^\nu) ds = \delta x^\alpha \left(\frac{e}{m} \right)^2 \frac{\partial s}{\partial x^\alpha} (g_{\mu\nu} A^\mu A^\nu) \Big|_A^B = 0. \quad (2.7)$$

This equals zero, because the two worldlines intersect at the boundary events A and B but have a slight variational difference between A and B otherwise, so that $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$ while $\delta x^\sigma \neq 0$ elsewhere. Consequently, the bottom line of (2.6) zeros out, leaving us with:

$$0 = \int_A^B ds \left(\delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\nu}{ds} + 2 \frac{e}{m} \delta A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} \right). \quad (2.8)$$

From here, again using $\delta g_{\mu\nu} = \partial_\alpha g_{\mu\nu} \delta x^\alpha$ and $\delta A_\sigma = \partial_\alpha A_\sigma \delta x^\alpha$, and also re-indexing and using the symmetry of $g_{\mu\nu}$ to combine the second and third terms above, we obtain:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + 2 \delta x^\alpha \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} \right). \quad (2.9)$$

Next, we integrate by parts. First, we use the product rule to replace $g_{\mu\nu} (d\delta x^\mu / ds) (dx^\nu / ds) = (d / ds) (\delta x^\mu g_{\mu\nu} (dx^\nu / ds)) - \delta x^\mu (d / ds) (g_{\mu\nu} (dx^\nu / ds))$ and likewise $(dA_\sigma / ds) \delta x^\sigma = (d / ds) (A_\sigma \delta x^\sigma) - A_\sigma d\delta x^\sigma / ds$. But the terms containing the total derivatives will vanish for the same reasons that the terms in (2.7) vanished as a result of the boundary conditions $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$. As a result, (2.9) now becomes:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2 \delta x^\mu \frac{d}{ds} \left(g_{\mu\nu} \frac{dx^\nu}{ds} \right) + 2 \delta x^\alpha \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2 \delta x^\sigma \frac{e}{m} \frac{dA_\sigma}{ds} \right). \quad (2.10)$$

Applying the d / ds derivative contained in the second term above then yields:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2 \delta x^\mu g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - 2 \delta x^\mu \frac{dg_{\mu\nu}}{ds} \frac{dx^\nu}{ds} + 2 \delta x^\alpha \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2 \delta x^\sigma \frac{e}{m} \frac{dA_\sigma}{ds} \right), \quad (2.11)$$

for the first time revealing the acceleration $d^2 x^\nu / ds^2$ in the second term above.

Next, we use the chain rules $dg_{\mu\nu} / ds = \partial_\alpha g_{\mu\nu} (dx^\alpha / ds)$ and $dA_\sigma / ds = \partial_\alpha A_\sigma (dx^\alpha / ds)$ to rewrite the third and fifth terms above, thus obtaining:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2 \delta x^\mu g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - 2 \delta x^\mu \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} \right. \\ \left. + 2 \delta x^\alpha \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2 \delta x^\sigma \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\alpha}{ds} \right). \quad (2.12)$$

In the bottom line above, we may rename indexes $\alpha \leftrightarrow \sigma$ in the last term, to find that we may rewrite $\delta x^\alpha \partial_\alpha A_\sigma (dx^\sigma / ds) - \delta x^\sigma \partial_\sigma A_\alpha (dx^\alpha / ds) = \delta x^\alpha F_{\alpha\sigma} (dx^\sigma / ds)$ using the electromagnetic field strength tensor $F_{\alpha\sigma} = \partial_\alpha A_\sigma - \partial_\sigma A_\alpha$, which has now appeared as a result of the variation. So the above now simplifies to:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \quad (2.13)$$

Now we rename indexes so that the δx terms all contain the index α , that is, so all of these terms are δx^α . We then factor this out and interchange the first and second terms, obtaining:

$$0 = \int_A^B ds \delta x^\alpha \left(-2g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2\frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \quad (2.14)$$

For material worldlines, $ds \neq 0$. Likewise, while $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$ at the boundaries, between these boundaries where the variation occurs, $\delta x^\sigma \neq 0$. Thus, multiplying through by $1/2$, for (2.14) to be true the integrand must be zero, and so we have:

$$0 = -g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2} \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds}. \quad (2.15)$$

Now we move the acceleration term to the left, split the term with $\partial_\mu g_{\alpha\nu} = \frac{1}{2} \partial_\mu g_{\alpha\nu} + \frac{1}{2} \partial_\mu g_{\alpha\nu}$ into two halves, rename some indexes while using the symmetry of $g_{\alpha\nu}$, and finally multiply through by $g^{\beta\alpha}$ and then raise indexes. This all yields:

$$\frac{d^2 x^\beta}{ds^2} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \quad (2.16)$$

But of course, we recognize that the Christoffel symbols $-\Gamma^\beta{}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$. As a consequence, the above reduces to:

$$\frac{d^2 x^\beta}{ds^2} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \quad (2.17)$$

In the presence of gravitational and electromagnetic fields, this contains both the equations of gravitational motion and the Lorentz force law, obtained via the geodesic variation of the canonical invariant metric length element (2.1). In the absence of gravitation, i.e., for $g_{\mu\nu} = \eta_{\mu\nu}$ over the spacetime region being considered thus $\Gamma^\beta{}_{\mu\nu} = 0$, this reduces to the Lorentz force law.

As a result, we have proved that by using Weyl's canonical prescription in form of $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ from (2.2) to define the linear metric element by $ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu$ as shown in (2.1), the Lorentz force law of electrodynamics may indeed be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry.

3. Einstein's Equation and Maxwell's Equations

Because the metric length $ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu$ of (2.1) under a variation $0 = \delta \int_A^B ds$ simultaneously provides a geodesic description of motion in a gravitational field and in an electromagnetic field, and because the prescription $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ is no more than a variant of Weyl's gauge prescriptions $p^\mu \rightarrow \pi^\mu = p^\mu + eA^\mu$ in momentum space and $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ in configuration space and leads directly as well to Dirac's equation $(i\gamma^\mu D_\mu - m)\psi = 0$ for an interacting fermion, this may fairly be regarded as a classical metric-level unification of electrodynamics with gravitation, using four spacetime dimensions only. But the equations of motion in a field are only half the matter. We also need to know the equations for the fields themselves in relation to their sources. Thus we now ask, can the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ which specifies the gravitational field, be shown to contain in some fashion, Maxwell's field equations for electric and (the absence of) magnetic sources?

Because the Lorentz force (2.17) is obtained by dilating or contracting the differential coordinate elements via $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ without in any way altering the metric tensor $g_{\mu\nu}$ as is done, for example, in Kaluza-Klein theory, one might incorrectly conclude that the electromagnetic interaction does not affect spacetime curvature as represented by the Riemann tensor $R^\alpha_{\beta\mu\nu}$ with the field dynamics specified by $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. However, one must keep in mind that the Riemann tensor may be defined via $R^\alpha_{\beta\mu\nu}V_\alpha \equiv [\partial_{;\nu}, \partial_{;\mu}]V_\beta$ as a measure of the extent to which the gravitationally-covariant derivatives $\partial_{;\mu}V_\beta = \partial_\mu V_\beta - \Gamma^\alpha_{\mu\beta}V_\alpha$ operating on a vector V_β do not commute. Likewise, the field strength tensor $F_{\nu\mu}$ may be defined via $ieF_{\nu\mu}V_\beta \equiv [D_\nu, D_\mu]V_\beta$ as a measure of the extent to which the gauge-covariant derivatives $D_\mu V_\beta = (\partial_\mu + ieA)V_\beta$ do not commute when operating on this same vector V_β . Indeed, this latter definition results in $F_{\nu\mu} = D_{[\nu}A_{\mu]} = \partial_{[\nu}A_{\mu]} + ie[A_\nu, A_\mu]$ for a *non-abelian* gauge theory where $[A_\nu, A_\mu] \neq 0$, which simplifies to $F_{\nu\mu} = \partial_{[\nu}A_{\mu]}$ for an abelian theory such as electrodynamics in which $[A_\nu, A_\mu] = 0$.

Therefore, if we apply Weyl's canonical prescription to the gravitationally-covariant derivatives by employing:

$$\partial_{;\mu} V_\beta = \partial_\mu V_\beta - \Gamma^\alpha_{\mu\beta} V_\alpha \rightarrow D_{;\mu} V_\beta = (\partial_\mu + ieA_\mu) V_\beta - \Gamma^\alpha_{\mu\beta} V_\alpha, \quad (3.1)$$

which is the same prescription that in the form $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ of (2.2) yielded the Lorentz force law in (2.17), and if we then use *these derivatives* (3.1) to canonically *define* the Riemann tensor as:

$$R^\alpha_{\beta\mu\nu} V_\alpha \equiv [D_{;\nu}, D_{;\mu}] V_\beta, \quad (3.2)$$

it can be expected as a consequence of $ieF_{\nu\mu} V_\beta \equiv [D_\nu, D_\mu] V_\beta$ that the electrodynamic fields $F_{\nu\mu}$ and potentials A_μ will appear in the Riemann tensor and therefore in the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. Further, because $F_{\nu\mu} = D_{[\nu} A_{\mu]}$ encompasses both *abelian and non-abelian field strengths*, the gravitational field equations using $R_{\beta\mu} = R^\alpha_{\beta\mu\alpha}$ and $R = R^\sigma_\sigma$ can be made not only to govern abelian electrodynamics, *but also non-abelian such as weak and strong interactions*. So let us expressly calculate $R^\alpha_{\beta\mu\nu}$ using (3.2) and see what results.

We first calculate:

$$\begin{aligned} D_{;\nu} (D_{;\mu} V_\beta) &= (\partial_\nu + ieA_\nu) \left((\partial_\mu + ieA_\mu) V_\beta - \Gamma^\alpha_{\mu\beta} V_\alpha \right) \\ &\quad - \Gamma^\tau_{\mu\nu} \left((\partial_\tau + ieA_\tau) V_\beta - \Gamma^\alpha_{\tau\beta} V_\alpha \right) - \Gamma^\tau_{\nu\beta} \left((\partial_\mu + ieA_\mu) V_\tau - \Gamma^\alpha_{\mu\tau} V_\alpha \right) \end{aligned} \quad (3.3)$$

as well as the like expression interchanging $\mu \leftrightarrow \nu$, then subtract the latter from the former and reduce using index renaming and the symmetries of the objects in the resulting equations, as needed. Many terms cancel, and with the vector V_α still attached on the right, what remains is:

$$\begin{aligned} R^\alpha_{\beta\mu\nu} V_\alpha &\equiv [D_{;\nu}, D_{;\mu}] V_\beta = D_{;\nu} (D_{;\mu} V_\beta) - D_{;\mu} (D_{;\nu} V_\beta) \\ &= \left(-\partial_\nu \Gamma^\alpha_{\mu\beta} + \partial_\mu \Gamma^\alpha_{\nu\beta} + \Gamma^\tau_{\nu\beta} \Gamma^\alpha_{\mu\tau} - \Gamma^\tau_{\mu\beta} \Gamma^\alpha_{\nu\tau} + ie\delta^\alpha_\beta F_{\nu\mu} \right) V_\alpha \end{aligned} \quad (3.4)$$

This indeed contains a non-abelian field strength $F_{\nu\mu} = \partial_{[\nu} A_{\mu]} + ie[A_\nu, A_\mu]$ that becomes abelian in the event $[A_\nu, A_\mu] = 0$. With the group structure constants f^{abc} of $if^{abc} T^c = [T^a, T^b]$ for the non-abelian generators T^a explicitly displayed, $F^a_{\nu\mu} = \partial_{[\nu} A^a_{\mu]} - ef^{abc} [A^b_\nu, A^c_\mu]$ is a fully real object. Thus, with some index renaming and lowered to covariant form, the Riemann tensor is:

$$R_{\alpha\beta\mu\nu} = g_{\alpha\tau} R^\tau_{\beta\mu\nu} = -g_{\alpha\tau} \partial_\nu \Gamma^\tau_{\beta\mu} + g_{\alpha\tau} \partial_\mu \Gamma^\tau_{\beta\nu} + g_{\alpha\tau} \Gamma^\sigma_{\beta\nu} \Gamma^\tau_{\sigma\mu} - g_{\alpha\tau} \Gamma^\sigma_{\beta\mu} \Gamma^\tau_{\sigma\nu} - ieg_{\alpha\beta} F_{\mu\nu} \quad (3.5)$$

The terms containing Christoffels are no different from the usual. But the new term $-ieg_{\alpha\beta} F_{\mu\nu}$ resulting from the same gauge prescription $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ that likewise brought

the Lorentz force law into (2.17) changes several aspects of $R_{\alpha\beta\mu\nu}$. First, while for the last two indexes $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$ as usual, for the first two indexes $R_{\alpha\beta\mu\nu} \neq -R_{\beta\alpha\mu\nu}$ due to the presence of the symmetric $g_{\alpha\beta}$ next to the antisymmetric $F_{\mu\nu}$ in $g_{\alpha\beta}F_{\mu\nu}$. Thus, $R_{\alpha\beta\mu\nu}$ is *non-symmetric* in α, β . Second, noting that $F^a_{\nu\mu}$ is in fact real when the f^{abc} are accounted for, the term $ieg_{\alpha\beta}F_{\mu\nu}$ provides an *imaginary* aspect to $R_{\alpha\beta\mu\nu}$, so that overall, $R_{\alpha\beta\mu\nu}$ is now a *complex* object. Third, as a consequence of both these matters, the *real* part of $R_{\alpha\beta\mu\nu}$ has the usual symmetries, while the *imaginary* part has the mixed symmetry of $g_{\alpha\beta}F_{\mu\nu}$.

It is readily seen from (3.5) after some re-indexing that the Ricci tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha} = -\partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} + \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\alpha} + ieF_{\mu\nu} \quad (3.6)$$

is likewise non-symmetric, with the usual, real gravitational terms being symmetric and the new, imaginary electrodynamic term being antisymmetric in μ, ν . Finally, because $F^{\sigma}_{\sigma} = 0$, the Ricci scalar is the usual:

$$R = g^{\mu\nu}R_{\mu\nu} = -g^{\sigma\tau}\partial_{\alpha}\Gamma^{\alpha}_{\sigma\tau} + g^{\sigma\tau}\partial_{\tau}\Gamma^{\alpha}_{\sigma\alpha} + g^{\sigma\tau}\Gamma^{\beta}_{\sigma\alpha}\Gamma^{\alpha}_{\beta\sigma} - g^{\sigma\tau}\Gamma^{\beta}_{\sigma\tau}\Gamma^{\alpha}_{\beta\alpha} \quad (3.7)$$

with no residual terms from electrodynamics.

Using (3.5) in the second Bianchi identity with gravitationally-covariant derivatives only (the usual $\partial_{;\sigma}$, not $D_{;\sigma}$), the Christoffel terms will zero out as usual as a result of the Riemannian geometry, and all that will remain is the field strength in the form:

$$\partial_{;\sigma}R^{\alpha}_{\beta\mu\nu} + \partial_{;\mu}R^{\alpha}_{\beta\nu\sigma} + \partial_{;\nu}R^{\alpha}_{\beta\sigma\mu} = -ie\delta^{\alpha}_{\beta}(\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}) = 0 \quad (3.8)$$

It will be seen that this imaginary part of this identity is synonymous with Maxwell's equation for vanishing magnetic charges.

Using (3.6) and (3.7), we may also form the contravariant field equation:

$$\begin{aligned} -\kappa T^{\sigma\tau} &= R^{\sigma\tau} - \frac{1}{2}g^{\sigma\tau}R \\ &= -g^{\sigma\mu}g^{\tau\nu}\partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} + g^{\sigma\mu}g^{\tau\nu}\partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} + g^{\sigma\mu}g^{\tau\nu}\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} - g^{\sigma\mu}g^{\tau\nu}\Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} + ieF^{\sigma\tau} \\ &\quad + \frac{1}{2}g^{\sigma\tau}g^{\mu\nu}\partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} - \frac{1}{2}g^{\sigma\tau}g^{\mu\nu}\partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} - \frac{1}{2}g^{\sigma\tau}g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} + \frac{1}{2}g^{\sigma\tau}g^{\mu\nu}\Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} \end{aligned} \quad (3.9)$$

This means that the energy tensor also becomes a complex object, with a real aspect that remains symmetric and is gravitational and an imaginary aspect that is antisymmetric and is electrodynamic. Of course, the Bianchi identity (3.8) contracts to $\partial_{;\sigma}(R^{\sigma\tau} - \frac{1}{2}g^{\sigma\tau}R) = 0$ which implies that $\partial_{;\sigma}T^{\sigma\tau} = 0$ is locally conserved and which is why the field equation connects $-\kappa T^{\sigma\tau}$

to $R^{\sigma\tau} - \frac{1}{2}g^{\sigma\tau}R$. So when we take $\partial_{;\sigma}T^{\sigma\tau} = 0$, we know that all the Christoffel terms will zero out, and what will remain is:

$$-\kappa\partial_{;\sigma}T^{\sigma\tau} = ie\partial_{;\sigma}F^{\sigma\tau} = ieJ^\tau = 0 \quad (3.10)$$

Consequently, we see that the imaginary part of the local conservation law for the energy tensor also contains Maxwell's equation for vanishing electric sources, $J^\tau = 0$.

Overall, we find that if we use the canonical gauge extension $\partial_{;\mu} \rightarrow D_{;\mu} = \partial_{;\mu} + ieA_\mu$ to obtain the Riemann tensor via the definition $R^\alpha{}_{\beta\mu\nu}V_\alpha \equiv [D_{;\nu}, D_{;\mu}]V_\beta$ of (3.2), Maxwell's source-free field equations indeed become embedded in an imaginary, antisymmetric aspect of the gravitational field equation while the real aspect of the gravitational field equation remains symmetric. We shall not at this moment, examine the next logical question as to a geometric understanding of non-vanishing electric sources $J^\tau \neq 0$. Certainly, we expect that using the canonical extension $\partial_{;\mu} \rightarrow D_{;\mu} = \partial_{;\mu} + ieA_\mu$ for the derivatives in (3.8) and (3.10) may play a role in exploring this question.

4. Conclusion

It has been shown how the Lorentz force law may be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry as a consequence, at bottom, of simply applying Weyl's gauge prescription $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ to the dilate or contract the spacetime coordinate elements by $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$. It has also been shown how this same prescription embeds Maxwell's source-free equations into an imaginary, antisymmetric aspect that is added to the gravitational field equations. Studying sources in Maxwell theory will likely require continuing to apply $\partial_{;\mu} \rightarrow D_{;\mu} = \partial_{;\mu} + ieA_{;\mu}$ to the second Bianchi identity and its contracted variant that relates to the local conservation of the energy tensor.

As a consequence of what has been shown here, it may well be possible to unify gravitation not only with electrodynamics – but because $F^{\sigma\tau}$ in (3.9) encompasses a non-abelian field strength $F^a{}_{\nu\mu} = \partial_{[\nu}A^a{}_{\mu]} - ef^{abc}[A^b{}_\nu, A^c{}_\mu]$ – with the remaining weak and strong interactions as well, all consistently with quantum mechanics because the canonical gauge prescriptions $p^\mu \rightarrow \pi^\mu = p^\mu + eA^\mu$ and $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$ and now $dx^\mu \rightarrow d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ remain at the root of the entire development. The main questions that would remain following such a unification, would be as to the specific non-abelian gauge groups that operate physically at any given energy ranging up to the Planck mass, and how the symmetry of those groups becomes broken at lower energies down to the phenomenological group $SU(3)_C \times SU(2)_W \times U(1)_Y \rightarrow SU(3)_C \times U(1)_{em}$ and the fermions on which these groups act. The author has previously published on these questions, and even shown how the three generations of quarks and leptons originate, and why their left-chiral projections engage in CKM mixing, at [9].

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