A proof of a well-known representation of Catalan's constant*

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Abstract

The formula, $G = 3\displaystyle\int_0^1 \arctan\left(\frac{x(1-x)}{2-x}\right)\frac{1}{x}\mathrm{d}x$, G being the Catalan's constant, have been popularized by James McLaughlin in September 2007[1].
We present here an elementary proof of it.

1 Introduction

From the following formulae [2],

$$G = \frac{3}{2}\int_{2+\sqrt{3}}^{+\infty} \frac{\log x}{1+x^2}\mathrm{d}x$$

$$G = \int_1^{+\infty} \frac{\log x}{1+x^2}\mathrm{d}x$$

we deduce:

$$G = 3\int_1^{2+\sqrt{3}} \frac{\log x}{1+x^2}\mathrm{d}x$$

In what follows, our aim is to prove:

$$\boxed{\int_0^1 \frac{1}{x}\arctan\left(\frac{x(1-x)}{2-x}\right).\mathrm{d}x = \int_1^{2+\sqrt{3}} \frac{\log x}{1+x^2}\mathrm{d}x} \tag{1.1}$$

*Thanks go to `http://www.les-mathematiques.net` for help and inspiration and to the author of Bigints LaTeX package, Merciadri Luca

2 Preamble: Some results.

The proof relies on the following identity.
Let u, v be two real numbers such that $u < 0$, $v > 0$ and $u + v > 0$.
Let $w := \dfrac{u^2 + v^2}{u + v}$.

$$\boxed{\forall x \in [0,1], \arctan\left(\frac{x(1-x)}{\frac{w}{u+v} - x}\right) = \arctan\left(\frac{ux}{w - vx}\right) + \arctan\left(\frac{vx}{w - ux}\right)} \quad (2.1)$$

Proof. It's straightforward to prove that $\dfrac{uvx^2}{(w-ux)(w-vx)}$ is well defined and negative when conditions on u, v, x are satisfied.

To prove it's greater than or equal to -1 notice that:

$uvx^2 + (w - ux)(w - vx) = 2uvx^2 - (u^2 + v^2)x + w^2$

The use of arctangent addition formula terminates the proof. □

Two lemmas are required to achieve the proof.

Lemma 2.1. *Let f be a continuous and differentiable function that is defined for all real numbers, and $f(0) = 0$.*
If $d > 0$ and, $d > c > 0$ or $c < 0$ then:

$$\int_0^1 \frac{1}{x} f\left(\frac{ax}{d - cx}\right) \mathrm{d}x = a \int_0^{\frac{a}{d-c}} \frac{f(x)}{x(a + cx)} \mathrm{d}x$$

Lemma 2.2. *Let u, v two real numbers such that $c > 0$ and $0 \leq u < \dfrac{1}{c}$.*

$$\int_0^u \frac{1}{1+x^2} \log\left(\frac{1 + \frac{1}{c}x}{1 - cx}\right) \mathrm{d}x = -\log c . \arctan u - \int_{\frac{1-cu}{c+u}}^{\frac{1}{c}} \frac{\log x}{1 + x^2} \mathrm{d}x$$

Proof. Use change of variable in the integral in the right-hand side:

$$y = \frac{1-cx}{c+x}$$

and recall that:

$$\arctan u = \int_0^u \frac{1}{1+x^2}\mathrm{d}x$$

□

3 Proof of the main result

Hereafter, N is an integer greater than or equal to 2.

Assume $u = 1-\sqrt{N}$, $v = 1+\sqrt{N}$ therefore, $u < 0$, $v > 0$, $u+v = 2 > 0$ and $w = \dfrac{u^2+v^2}{u+v} = 1+N$.

For all $x \in [0,1]$, according to 2.1 :

$$\arctan\left(\frac{x(1-x)}{\frac{N+1}{2} - x}\right) = \arctan\left(\frac{(1-\sqrt{N})x}{N+1-\left(1+\sqrt{N}\right)x}\right) + \arctan\left(\frac{(1+\sqrt{N})x}{N+1-\left(1-\sqrt{N}\right)x}\right)$$

Assume $a = 1-\sqrt{N}$, $c = 1+\sqrt{N}$, $d = N+1$, therefore, $d > c > 0$ and $\dfrac{a}{d-c} = -\dfrac{1}{\sqrt{N}}$.

Let $\alpha = \dfrac{\sqrt{N}-1}{\sqrt{N}+1}$.

Using lemma 2.1 and change of variable $y = -x$ one obtains:

$$\int_0^1 \frac{1}{x}\arctan\left(\frac{(1-\sqrt{N})x}{N+1-\left(1+\sqrt{N}\right)x}\right)\mathrm{d}x = -\int_0^{\frac{1}{\sqrt{N}}} \frac{\arctan x}{x\left(1+\frac{1}{\alpha}x\right)}\mathrm{d}x$$

Assume $a = 1+\sqrt{N}$, $c = 1-\sqrt{N}$, $d = N+1$, therefore, $d > 0$, $c < 0$ and $\dfrac{a}{d-c} = \dfrac{1}{\sqrt{N}}$.

Using lemma 2.1 :

$$\int_0^1 \frac{1}{x}\arctan\left(\frac{(1+\sqrt{N})x}{N+1-\left(1-\sqrt{N}\right)x}\right)\mathrm{d}x = \int_0^{\frac{1}{\sqrt{N}}} \frac{\arctan x}{x(1-\alpha x)}\mathrm{d}x$$

Since $\frac{1}{\alpha} > 1$, the following identity holds for all real numbers in $]0,1]$:

$$\frac{1}{x(1-\alpha x)} - \frac{1}{x\left(1+\frac{1}{\alpha}x\right)} = \frac{\alpha}{1-\alpha x} + \frac{\frac{1}{\alpha}}{1+\frac{1}{\alpha}x}$$

Using integration by parts, one gets:

$$\int_0^{\frac{1}{\sqrt{N}}} \frac{\alpha\arctan x}{1-\alpha x}\mathrm{d}x = \quad -\log\left(1-\frac{\alpha}{\sqrt{N}}\right)\arctan\left(\frac{1}{\sqrt{N}}\right) + \int_0^{\frac{1}{\sqrt{N}}} \frac{\log(1-\alpha x)}{1+x^2}\mathrm{d}x$$

$$\int_0^{\frac{1}{\sqrt{N}}} \frac{\frac{1}{\alpha}\arctan x}{1+\frac{1}{\alpha}x}\mathrm{d}x = \log\left(1+\frac{1}{\alpha\sqrt{N}}\right)\arctan\left(\frac{1}{\sqrt{N}}\right) - \int_0^{\frac{1}{\sqrt{N}}} \frac{1}{1+x^2}\log\left(1+\frac{1}{\alpha}x\right)\mathrm{d}x$$

The following equality holds:

$$\frac{1+\frac{1}{\alpha\sqrt{N}}}{1-\frac{1}{\sqrt{N}}\alpha} = \frac{1}{\alpha}$$

thus, one obtains:

$$\int_0^1 \frac{1}{x}\arctan\left(\frac{x(1-x)}{\frac{N+1}{2}-x}\right)\mathrm{d}x = -\arctan\left(\frac{1}{\sqrt{N}}\right)\log\alpha - \int_0^{\frac{1}{\sqrt{N}}} \frac{1}{1+x^2}\log\left(\frac{1+\frac{1}{\alpha}x}{1-\alpha x}\right)\mathrm{d}x$$

The following equality holds:

$$\frac{1-\dfrac{\alpha}{\sqrt{N}}}{\alpha+\dfrac{1}{\sqrt{N}}}=1$$

thus, applying lemma 2.2 to the integral in the right-hand side one obtains:

$$\boxed{\int_0^1 \frac{1}{x}\arctan\left(\frac{x(1-x)}{\dfrac{N+1}{2}-x}\right)dx=\int_1^{\frac{\sqrt{N}+1}{\sqrt{N}-1}}\frac{\log x}{1+x^2}dx} \tag{3.1}$$

The formula 1.1 follows by taking $N=3$.

Remark. *An alternative way to prove 3.1 is to consider* N *as a real number parameter strictly greater than* 1.

4 References

References

[1] `https://listserv.nodak.edu/cgi-bin/wa.exe?A0=NMBRTHRY`

[2] David M. Bradley, *Representations of Catalan's constant* (2001), formulae (32) and (17).