# Removing magic from the normal distribution and the Stirling and Wallis formulas.

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## §1. Introduction.

The Wallis formula is often written as

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{14}{13}\cdot\frac{14}{15}\cdots=\frac{\pi}{2} \tag{1.1a}$$

with the meaning that the sequence of partial products $p_1=\frac{2}{1}$, $p_2=\frac{2}{1}\cdot\frac{2}{3}$, $p_3=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}$, $p_4=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}$, $p_4=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}$, $\ldots$ of the left-hand side of (1.1a) converges to $\frac{\pi}{2}$. Since the partial product of the first $2n$ terms $p_{2n}=\left[\frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}}\right]^2$ we may rewrite (1.1a) as

$$\lim_{n\to+\infty}\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}=\sqrt{\pi}, \tag{1.1b}$$

which is often used to obtain the ubiquitous Stirling formula

$$\lim_{m\to+\infty}\frac{e^m\ m!}{m^m\sqrt{m}}=\sqrt{2\pi}, \tag{1.2}$$

described in McCartin, 2006 as "providing an intriguing connection between $\pi$ and $e$." To obtain (1.2) we rewrite (1.1b) as

$$\lim_{m\to+\infty}\frac{(f_m)^2}{f_{2m}}=\sqrt{2\pi},\quad f_m=\frac{e^m\ m!}{m^m\ \sqrt{m}}. \tag{1.3}$$

Since $f_m > 0$, $\dfrac{f_{m+1}}{f_m} = \dfrac{e}{\left(1+\frac{1}{m}\right)^{m+0.5}} = e\left(\dfrac{m}{m+1}\right)^{m+0.5} < 1$, sequence $f_m$ is positive and monotonically decreasing and as such must have a non-negative limit. To show that $\lim\limits_{m\to+\infty} f_m \neq 0$, consider sequence $g_m = \dfrac{(m-1)f_m}{m}$ which has the same limit as sequence $f_m$. Since $\dfrac{g_{m+1}}{g_m} = \dfrac{e\,m^2}{m^2-1}\left(\dfrac{m}{m+1}\right)^{m+0.5} > 1$, then $\lim\limits_{m\to+\infty} g_m > g_2 > 0$ and thus $\lim\limits_{m\to+\infty} f_m > 0$. Formula (2) is obtained by simple application of the laws of limits to (1.3) as follows $\sqrt{2\pi} = \lim\limits_{m\to+\infty} \dfrac{(f_m)^2}{f_{2m}} = \dfrac{\left(\lim\limits_{m\to+\infty} f_m\right)^2}{\lim\limits_{m\to+\infty} f_{2m}} = \lim\limits_{m\to+\infty} f_m$. Thus (1.2) and (1.1) are equivalent in the sense that each one of them implies the other.

Of many applications of the Stirling formula one is the derivation of the normal distribution as the limiting case of the binomial distribution. In a typical derivation found in many textbooks, one considers an infinite row of cells numbered by integers and a one-dimensional random walk of a point $\mathcal{P}$ that starts at the cell $K=0$ and at each step jumps from the point it occupies to either the right or left adjacent cell with probability $\dfrac{1}{2}$.

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ... | $-9$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |

The probability of finding point $\mathcal{P}$ at a cell $K$ after $N$ steps is given by the following table:

| $K=$ | ... | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N=0$ | ... | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2^0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| $N=1$ | ... | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2^1}$ | 0 | $\frac{1}{2^1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| $N=2$ | ... | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2^2}$ | 0 | $\frac{2}{2^2}$ | 0 | $\frac{1}{2^2}$ | 0 | 0 | 0 | 0 | 0 | 0 | ... |
| $N=3$ | ... | 0 | 0 | 0 | 0 | $\frac{1}{2^3}$ | 0 | $\frac{3}{2^3}$ | 0 | $\frac{3}{2^3}$ | 0 | $\frac{1}{2^3}$ | 0 | 0 | 0 | 0 | 0 | ... |
| $N=4$ | ... | 0 | 0 | 0 | $\frac{1}{2^4}$ | 0 | $\frac{4}{2^4}$ | 0 | $\frac{6}{2^4}$ | 0 | $\frac{4}{2^4}$ | 0 | $\frac{1}{2^4}$ | 0 | 0 | 0 | 0 | ... |
| $N=5$ | ... | 0 | 0 | $\frac{1}{2^5}$ | 0 | $\frac{5}{2^5}$ | 0 | $\frac{10}{2^5}$ | 0 | $\frac{10}{2^5}$ | 0 | $\frac{5}{2^5}$ | 0 | $\frac{1}{2^5}$ | 0 | 0 | 0 | ... |
| $N=6$ | ... | 0 | $\frac{1}{2^6}$ | 0 | $\frac{6}{2^6}$ | 0 | $\frac{15}{2^6}$ | 0 | $\frac{20}{2^6}$ | 0 | $\frac{15}{2^6}$ | 0 | $\frac{6}{2^6}$ | 0 | $\frac{1}{2^6}$ | 0 | 0 | ... |
| $N=7$ | ... | $\frac{1}{2^7}$ | 0 | $\frac{7}{2^7}$ | 0 | $\frac{21}{2^7}$ | 0 | $\frac{35}{2^7}$ | 0 | $\frac{35}{2^7}$ | 0 | $\frac{21}{2^7}$ | 0 | $\frac{7}{2^7}$ | 0 | $\frac{1}{2^7}$ | 0 | ... |

The nonzero entries are $\dfrac{1}{2^N}$ multiples of the binomial coefficients $\dfrac{N!}{\left(\frac{N+K}{2}!\right)\left(\frac{N-K}{2}!\right)}$, $|K| \leqslant N$ written as even functions of $K$. The probability $P(K,N)$ of finding point $\mathcal{P}$ at a cell labeled $K$ is given by

$$P(K,N)=\begin{cases}\dfrac{N!}{2^N\ \left(\frac{N+K}{2}!\right)\left(\frac{N-K}{2}!\right)}, & \text{if } |K|\leqslant N;\ N-K \text{ is even},\\ 0, & \text{otherwise.}\end{cases} \tag{1.4}$$

For $N$ sufficiently large and $|K|\ll N$, formula (2) allows us to approximate $N!\ \approx\ \sqrt{2\pi N}N^N e^{-N}$, $\dfrac{N\pm K}{2}!\ \approx\ \sqrt{\pi(N\pm K)}\ \left(\dfrac{N\pm K}{2}\right)^{\frac{N\pm K}{2}} e^{-\frac{N\pm K}{2}}$ which upon substitution into $\dfrac{N!}{2^N\ \left(\frac{N+K}{2}!\right)\left(\frac{N-K}{2}!\right)}$ lead to approximation $\dfrac{N!}{2^N\ \left(\frac{N+K}{2}!\right)\left(\frac{N-K}{2}!\right)}$ $\approx \sqrt{\dfrac{2}{\pi N}}\ e^{-\frac{K^2}{2N}}$ of the binomial distribution by the normal distribution .

The given derivation of the normal distribution is based on formula (1.2) with all difficulties buried in (1.2), or equivalently in (1.1). One would assume that formulas as fundamental as (1.1) and (1.2) had an intuitive proof, yet as pointed out in Gowers (2008), all proofs of (1.1) seem to contain a non-intuitive step with an identity or an estimate magically pulled out of a hat. Attempts to find a simple intuitive proof have led to a rather large number of publications some of which are listed in Bibliography, yet none seems to be fully intuitive. Most assume that formulas (1.1) are known and try to construct an appropriate proof. But is it possible to arrive at formulas (1.1) in a completely natural way without any magical steps? The author of this paper thinks it is and will show how in the next section based on the ideas outlined in Kovalyov (2009).

## §2. Intuitive derivation of the Wallis formula and the normal distribution.

For simplicity's sake let us take

$$N=2n \text{ is even .} \tag{2.1}$$

Then

$$P(-2k,2n)=P(2k,2n)=\begin{cases}\dfrac{(2n)!}{2^{2n}\ (n!)^2}\cdot\dfrac{(n!)^2}{(n+k)!(n-k)!}=\dfrac{(2n)!}{2^{2n}\ (n!)^2}\displaystyle\prod_{j=1}^{k}\dfrac{n-(k-j)}{n+j}\\ =\dfrac{(2n)!}{2^{2n}\ (n!)^2}\displaystyle\prod_{j=1}^{k}\dfrac{1-\frac{k-j}{n}}{1+\frac{j}{n}}, \ \text{if } 0\leqslant k\leqslant n,\\ 0, \ \text{if } k>n.\end{cases} \tag{2.2}$$

Coefficients $P(2k,2n)$ also satisfy

$$\sum_{k=-n}^{n} P(2k,2n) = \sum_{k=-n}^{n} \frac{(2n)!}{2^{2n}(n+k)!(n-k)!} = \left(\frac{1}{2}+\frac{1}{2}\right)^{2n} = 1. \qquad (2.3)$$

The main idea of the derivation of (1.1b) is to estimate $P(2k,2n)$ in (2.2) by using rather intuitive approximations $1-\dfrac{k-j}{n} \approx e^{\frac{j-k}{n}}, 1+\dfrac{j}{n} \approx e^{\frac{j}{n}}, \dfrac{1-\frac{k-j}{n}}{1+\frac{j}{n}} \approx e^{-\frac{k}{n}}$ which lead to $\displaystyle\prod_{j=1}^{k} \frac{1-\frac{k-j}{n}}{1+\frac{j}{n}} \approx e^{-\frac{k^2}{n}}$ , $\dfrac{2^{2n}\ (n!)^2}{(2n)!} P(2k,2n) \approx e^{-\frac{k^2}{n}}$ and $\displaystyle\frac{2^{2n}\ (n!)^2}{(2n)!\sqrt{n}} \underbrace{\sum_{k=-n}^{n} P(2k,2n)}_{=1 \text{ due to } (2.3)} \approx \underbrace{\sum_{k=-n}^{n} \frac{1}{\sqrt{n}} e^{-\frac{k^2}{n}}}_{\substack{\text{approaches } \sqrt{\pi} \\ \text{as } n\to+\infty}}$ which, in turn, imply (1.1). The problem is that the approximations are valid only for $|j| \leqslant |k| \ll n$, $|k-j| \leqslant |k| \ll n$ and as $k$ gets close to $\pm n$ the approximations lose their validity for values of $j$ and $k-j$ close to $n$.

To turn these ideas into a rigorous proof we break up the set $-n \leqslant k \leqslant n$ into the core $|k| \leqslant n^{\varepsilon+0.5}$ and two tails $n^{\varepsilon+0.5} < |k| \leqslant n$, with $0 < \varepsilon < 0.5$ to be determined later, so that inside the core approximations $1-\dfrac{k-j}{n} \approx e^{\frac{j-k}{n}}$, $1+\dfrac{j}{n} \approx e^{\frac{j}{n}}$ are valid while the total probability of $\mathcal{P}$ being outside the core approaches zero as $n \to +\infty$ .

$$\underbrace{\boxed{-n}\ \cdots\cdots\ -n^{0.5+\varepsilon}}_{\substack{\text{left tail}\\ -n\leqslant k<-n^{0.5+\varepsilon}}}\ \underbrace{\cdots\cdots\ \boxed{-5}\boxed{-4}\boxed{-3}\boxed{-2}\boxed{-1}\boxed{0}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\ \cdots\cdots\ n^{0.5+\varepsilon}}_{\text{the core consists of integers } k \text{ such that } |k|\leqslant n^{0.5+\varepsilon}}\ \underbrace{\cdots\cdots\ \boxed{n}}_{\substack{\text{right tail}\\ n^{0.5+\varepsilon}<k\leqslant n}}$$

For large $n$ the total probability of $\mathcal{P}$ being in one of the two tails is

$$1-\sum_{|k|<n^{0.5+\varepsilon}} P\big(2k,2n\big) = \sum_{|k|\geqslant n^{0.5+\varepsilon}} P\big(2k,2n\big) = 2\sum_{k\geqslant n^{0.5+\varepsilon}} P\big(2k,2n\big) \leqslant 2^{-n^{2\varepsilon}} n, \qquad (2.4)$$

and hence goes to $0$ as $n \to +\infty$. Indeed, taking for simplicity's sake $k>0$ ,

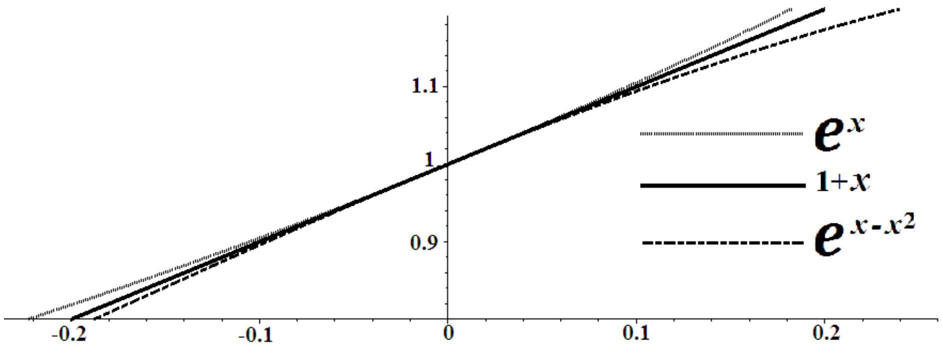
$$\sum_{k\geqslant n^{0.5+\varepsilon}} P\big(2k,2n\big) = \underbrace{\frac{(2n)!}{2^{2n}\ (n!)^2}}_{\substack{\text{this term is}\\ \text{less than } 0.5}} \sum_{n^{0.5+\varepsilon}<k\leqslant n} \prod_{j=1}^{k} \frac{n-(k-j)}{n+j} < 0.5 \sum_{n^{0.5+\varepsilon}<k\leqslant n} \prod_{j=1}^{k} \frac{n-k+j}{n+j} =$$

$$0.5 \sum_{n^{0.5+\varepsilon}<k\leqslant n} \prod_{j=1}^{k}\Bigg(1-\underbrace{\frac{k}{n+j}}_{\substack{\text{replacing}\\ j \text{ with } k\\ \text{makes it}\\ \text{larger}}}\Bigg) \leqslant 0.5 \sum_{n^{0.5+\varepsilon}<k\leqslant n} \prod_{j=1}^{k}\left(1-\frac{k}{n+k}\right) = 0.5 \sum_{n^{0.5+\varepsilon}<k\leqslant n} \underbrace{\left[\left(1+\frac{k}{n}\right)^{-\frac{k}{n}}\right]^{n}}_{\substack{\text{replacing } k \text{ with}\\ n^{0.5+\varepsilon} \text{ makes it}\\ \text{larger}}}$$

$$\leqslant 0.5\sum_{n^{0.5+\varepsilon}<k\leqslant n}\left[\left(1+\frac{1}{n^{0.5-\varepsilon}}\right)^{-n^{0.5-\varepsilon}}\right]^{n^{2\varepsilon}} \leqslant 0.5\Bigg[\underbrace{\left(1+\frac{1}{n^{0.5-\varepsilon}}\right)^{-n^{0.5-\varepsilon}}}_{\approx\frac{1}{e}<\frac{1}{2}\text{ for large } n}\Bigg]^{n^{2\varepsilon}} \underbrace{\sum_{n^{0.5+\varepsilon}<k\leqslant n} 1}_{<n} \leqslant \frac{0.5n}{2^{n^{2\varepsilon}}} .$$

Inside the core $P(2k,2n)$ satisfies

$$e^{-\frac{k^2}{n}} e^{-n^{-0.5+3\varepsilon}} \leqslant \frac{2^{2n}\ (n!)^2}{(2n)!} P(2k,2n) \leqslant e^{-\frac{k^2}{n}} e^{n^{-0.5+3\varepsilon}}, \quad \text{if } |k| \leqslant n^{\varepsilon+0.5}. \tag{2.5}$$




Due to $P(-2k,2n) = P(2k,2n)$ it suffices to prove (2.5) for $k \geqslant 0$. To do so we employ inequality

$$e^{x-x^2} \leqslant 1+x \leqslant e^x, \quad \text{for } |x| \ll 1. \tag{2.6a}$$

To prove (2.6a) notice that functions $w_1(x) = e^x - 1 - x$, $w_2(x) = 1 + x - e^{x-x^2}$ are analytic and satisfy $w_1(0) = w_2(0) = w_1'(0) = w_2'(0) = 0$, $w_1''(0) = w_2''(0) = 1$ . Thus each of them must of the form $0.5x^2 + o(x^2) > 0$ for $x$ sufficiently small.

Inequality (2.6a) applied with $x = -\dfrac{k-j}{n}$ and $x = \dfrac{j}{n}$ gives us

$$e^{-\frac{k-j}{n}-\left(\frac{k-j}{n}\right)^2} \leqslant 1 - \frac{k-j}{n} \leqslant e^{-\frac{k-j}{n}}, \tag{2.6b}$$

$$e^{\frac{j}{n}-\left(\frac{j}{n}\right)^2} \leqslant 1 + \frac{j}{n} \leqslant e^{\frac{j}{n}}. \tag{2.6c}$$

Dividing (2.6b) by (2.6c) we obtain

$$e^{-\frac{k}{n}-\frac{(k-j)^2}{n^2}} \leqslant \frac{1-\frac{k-j}{n}}{1+\frac{j}{n}} \leqslant e^{-\frac{k}{n}+\frac{j^2}{n^2}}, \tag{2.6d}$$

which yields

$$e^{-\frac{k^2}{n}-\sum\limits_{j=1}^{k}\frac{(k-j)^2}{n^2}} = e^{-\sum\limits_{j=1}^{k}\left[\frac{k}{n}+\frac{(k-j)^2}{n^2}\right]} \leqslant \prod_{j=1}^{k}\frac{1-\frac{k-j}{n}}{1+\frac{j}{n}} \leqslant e^{\sum\limits_{j=1}^{k}\left[-\frac{k}{n}+\frac{j^2}{n^2}\right]} = e^{-\frac{k^2}{n}+\sum\limits_{j=1}^{k}\frac{j^2}{n^2}}. \tag{2.6e}$$

Using $\dfrac{1}{n^2}\sum_{j=1}^{k}(k-j)^2=\dfrac{k(k-1)(2k-1)}{6n^2}=\dfrac{2k^3-3k^2+k}{3n^2}\leqslant\dfrac{k^3}{6n^2}\leqslant\dfrac{(n^{0.5+\varepsilon})^3}{n^2}\leqslant n^{-0.5+3\varepsilon}$, $\dfrac{1}{n^2}\sum_{j=1}^{k}j^2=\dfrac{k(k+1)(2k+1)}{6n^2}=\dfrac{2k^3+3k^2+k}{6n^2}\leqslant\dfrac{k^3}{n^2}\leqslant\dfrac{(n^{0.5+\varepsilon})^3}{n^2}\leqslant n^{-0.5+3\varepsilon}$ valid for $0<k\leqslant n^{0.5+\varepsilon}$ we may further simplify (2.6e) to (2.5).

Multiplying (2.5) by $\dfrac{1}{\sqrt{n}}$ and summing up in $k$ we obtain

$$\left[n^{-0.5}\sum_{|k|\leqslant n^{0.5+\varepsilon}}e^{-\frac{k^2}{n}}\right]e^{-n^{-0.5+3\varepsilon}}\leqslant\frac{2^{2n}\ (n!)^2}{(2n)!\sqrt{n}}\sum_{|k|\leqslant n^{0.5+\varepsilon}}P(2k,2n)\leqslant\left[n^{-0.5}\sum_{|k|\leqslant n^{0.5+\varepsilon}}e^{-\frac{k^2}{n}}\right]e^{n^{-0.5+3\varepsilon}}\quad(2.7a)$$

which upon division by $\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}P(2k,2n)$ becomes

$$\frac{\left[n^{-0.5}\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}e^{-\frac{k^2}{n}}\right]e^{-n^{-0.5+3\varepsilon}}}{\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}P(2k,2n)}\leqslant\frac{2^{2n}\ (n!)^2}{(2n)!\sqrt{n}}\leqslant\frac{\left[n^{-0.5}\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}e^{-\frac{k^2}{n}}\right]e^{n^{-0.5+3\varepsilon}}}{\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}P(2k,2n)}.\qquad(2.7b)$$

If we take

$$0<\varepsilon<\frac{1}{6}\qquad(2.8)$$

then $\lim\limits_{n\to+\infty}e^{n^{-0.5+3\varepsilon}}=\lim\limits_{n\to+\infty}e^{-n^{-0.5+3\varepsilon}}=1$, $\lim\limits_{n\to+\infty}\sum\limits_{|k|\leqslant n^{0.5+\varepsilon}}P(2k,2n)=1$ due to (2.3) and (2.4), $\lim\limits_{n\to+\infty}\dfrac{1}{\sqrt{n}}\sum\limits_{\substack{|k|\leqslant n^{0.5+\varepsilon},\\ k\text{ is even}}}e^{-\frac{k^2}{n}}=\int\limits_{-\infty}^{+\infty}e^{-t^2}dt=\sqrt{\pi}$ and as $n\to+\infty$ the limits of the first and third terms in (2.7b) exist and are equal to $\sqrt{\pi}$. By the well-known theorem of Calculus the limit of the middle term $\dfrac{2^{2n}\ (n!)^2}{(2n)!\sqrt{n}}$ must also exist and be equal to $\sqrt{\pi}$ thus proving (1.1b). Notice that the identity $\int\limits_{-\infty}^{+\infty}e^{-t^2}dt=\sqrt{\pi}$ comes from multivariable calculus, its proof follows from the string of identities $\left[\int\limits_{-\infty}^{+\infty}e^{-t^2}dt\right]^2=\int\limits_{-\infty}^{+\infty}e^{-x^2}dx\int\limits_{-\infty}^{+\infty}e^{-y^2}dy=\iint\limits_{\mathrm{R}^2}e^{-x^2-y^2}dxdy=$ $\int\limits_{0}^{+\infty}\int\limits_{0}^{2\pi}e^{r^2}rdrd\phi=\underbrace{\int\limits_{0}^{+\infty}e^{r^2}rdr}_{=\frac{1}{2}}\underbrace{\int\limits_{0}^{2\pi}d\phi}_{=2\pi}=\pi.$

We may rewrite (2.5) as

$$e^{-n^{-0.5+3\varepsilon}} \frac{(2n)!\sqrt{n\pi}}{2^{2n}\ (n!)^2} \leqslant P(2k,2n)\sqrt{n\pi}\ e^{\frac{k^2}{n}} \leqslant e^{n^{-0.5+3\varepsilon}} \frac{(2n)!\sqrt{n\pi}}{2^{2n}\ (n!)^2}. \qquad (2.9)$$

Since the first and third terms of (2.9) approach $1$ as $n \to +\infty$ , we conclude that

$$\lim_{n\to+\infty} P(2k,2n)\sqrt{\pi n}\ e^{\frac{k^2}{n}} = 1, \qquad (2.10a)$$

and consequently

$$P(2k,2n) \approx \frac{1}{\sqrt{\pi n}}\ e^{-\frac{k^2}{n}}, \qquad (2.10b)$$

providing us with the simplest case of the Central Limit Theorem.

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