Navier-Stokes Equations Solutions Completed

Abstract

"5% of the people think; 10% of the people think that they think; and the other 85% would rather die than think."----Thomas Edison

"The simplest solution is usually the best solution"---Albert Einstein

Over nearly a year and half ago, the Navier-Stokes equations in 3-D for incompressible fluid flow were analytically solved by the author. However, some of the solutions contained implicit terms. In this paper, the implicit terms have been expressed explicitly in terms of x, y, z and t. The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. It was mathematically shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known, and also, there would be no magnetohydrodynamics. In addition to the usual method of solving these equations, the N-S equations were also solved by a second method in which the three equations in the system were added to produce a single equation which was then integrated. The solutions by the two methods were identical, except for the constants involved. Ratios were used to split-up the equations; and the resulting sub-equations were readily integrable, and even, the nonlinear sub-equations were readily integrated. The examples in the preliminaries show everyday examples on using ratios to divide a quantity into parts, as well as possible applications of the solution method in mathematics, science, engineering, business, economics, finance, investment and personnel management decisions. The x–direction Navier-Stokes equation was linearized, solved, and the solution analyzed. This solution was followed by the solution of the Euler equation of fluid flow. The Euler equation represents the nonlinear part of the Navier-Stokes equation. Following the Euler solution, the Navier-Stokes equation was solved essentially by combining the solutions of the linearized equation and the Euler solution. For the Navier-Stokes equation, the linear part of the relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation; and the relation from the integration of the non-linear part satisfied the non-linear part of the equation. The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. The gravity term is the indispensable term in fluid flow, and it is involved in the parabolic and forward motion of fluids. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in the parabolic, periodic and decreasingly exponential motion. Periodicity increases with viscosity. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The fluid flow in the Navier-Stokes solution may be characterized as follows. The x–direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, some of the axes of symmetry of the parabolas would be at right angles to that of laminar flow. The characteristic curve for the integral of the x–nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the y–nonlinear term is similar parabolically to that of the x–nonlinear term. The integral of the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.

For a spin-off, the smooth solutions from above are specialized and extended to satisfy the requirements of the CMI Millennium Prize Problems, and thus prove the existence of smooth solutions of the Navier-Stokes equations.

Introduction

Solutions of the Navier-Stokes Equations

Case 1: Solutions of the Linearized Navier-Stokes Equations (x–direction)

Equation $$-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$$

Solutions $$\boxed{V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9;\;\; P(x)=d\rho g_x x}$$

Case 2: Solutions of the Euler Equations for Incompressible Fluid Flow ()

Equation $$\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})+\frac{\partial p_x}{\partial x}=\rho g_x \qquad x\text{–direction}$$

Solutions

$$\boxed{V_x(x,y,z,t)=fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};V_y\neq 0,V_z\neq 0;P(x)=d\rho g_xx;}$$

Case 3: Solutions of the Navier-Stokes Equations (Original) : x–direction

Equation: $$\boxed{-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x}$$

Solutions

$$\boxed{\begin{array}{l} V_x=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+C_9 \\ P(x)=d\rho g_xx;\quad (a+b+c+d+h+n+q=1)\quad V_y\neq 0,\ V_z\neq 0 \end{array}}$$

Summary for the fractional terms of the x–direction

$\frac{ng_xy}{V_y}$ and $\frac{qg_xz}{V_z}$ in terms of x, y, z and t (for Case 3)

$$\frac{ng_xy}{V_y}=\frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz})}{\beta_7g_z}$$

$$\frac{qg_xz}{V_z}=\frac{-(qg_xz)\{[(\beta_7g_zy)(-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-[CE]\}}{(\beta_7g_zy)(qg_xz-\beta_6g_zx)}$$

$$(CE=-(ng_xy)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz})$$

One observes above that the most important insight of the above solutions is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g_x, were zero, for Case 1, the first three terms, the seventh, and $P(x)$ would all be zero; for Case 2, the first four terms and $P(x)$ would all be zero; and for Case 3, the first three terms, the seventh, the eighth, the ninth, the tenth terms and $P(x)$ would all be zero. These results can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. It would not therefore be meaningful to write a Navier-Stokes equation for incompressible fluid flow without the gravity term, since there would be no fluid flow.

More Observations Comparison of the N-S solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): $V_f = V_0 + gt$; (C): $V_f^2 = V_0^2 + 2gx$; (D): $V = \sqrt{2gx}$; (E): $x = V_0 t + \frac{1}{2}gt^2$

The **liquid pressure**, P at the bottom of a liquid of depth h units is given by $P = \rho gh$

Observe the following about the Navier-Stokes Solutions (Case 3)

1. The first three terms are parabolic in x, y, and z; the minus sign shows the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; also note the "gt" in $V = gt$ of (B) of the motion equations and the $fg_x t$ in the Navier-Stokes solution.

2. The pressure, $P = \rho gh$ of the liquid pressure and the $P(x) = d\rho g_x x$ of the Navier-Stokes solution.

Note that, only the approach in this paper could yield $P(x) = d\rho g_x x$ by integrating $dp/dx = d\rho g_x$

3. Observe the "$\sqrt{2gx}$" in $V = \sqrt{2gx}$ of (D) and the $\sqrt{2hg_x x}$ in the Navier-Stokes solution.

In fact, the N-S solution term $\sqrt{2hg_x x}$ could have been obtained from $V_f^2 = V_0^2 + 2gx$,(C) , of the equation of motion by letting $V_0 = 0$ (for the convective term) ignoring the ratio term "h" of the N-S radicand. There are eight main terms (ignoring the arbitrary functions) in the N-S solution. Of these eight terms, six terms, namely, $-\frac{a\rho g_x}{2\mu}x^2$, $-\frac{b\rho g_x}{2\mu}y^2$, $-\frac{c\rho g_x}{2\mu}z^2$, $fg_x t$, $\sqrt{2hg_x x}$ and $d\rho g_x x$ are similar (except for the constants involved) to the terms in the equations of motion and fluid pressure. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

4. With regards to the variables x, y, and z, the parabolicity of the first three terms and the parabolicity of the eighth, ninth and tenth terms hint at inverse relations.. For examples, $V_x = x^2$ and $V_x = \pm\sqrt{x}$ are inverse relations of each other, $V_x = y^2$ and $V_x = \pm\sqrt{y}$ are inverse relations of each other, $V_x = z^2$ and $V_x = \pm\sqrt{z}$ are inverse relations of each other. The implications of knowing these relationships is that if one knows the steps, rules or formulas for designing for laminar flow, one can deduce the steps, rules or formulas for designing for turbulent flow by reversing the steps and using opposite operations in each step of the corresponding laminar flow design. Thus for every method, or formula for laminar flow, there is a corresponding method, formula for turbulent flow design (see also, "Power of Ratios" book by A. A. Frempong, p. 28).

For the **velocity profile,** the x–direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize inverted parabolas. Flow distribution for laminar flow is parabolic with the axis of symmetry of the parabola in direction of the fluid flow. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axes of symmetry of some of the parabolas would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the integral of the x–nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the y–nonlinear term is similar parabolically to the integral of the x–nonlinear term. The characteristic curve for the integral of the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.

Options

The Navier-Stokes equations in three dimensions are three simultaneous equations in Cartesian coordinates for the flow of incompressible fluids. The equations are presented below:

$$\begin{cases} \mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}) & (\mathrm{N}_x) \\ \mu(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2})-\frac{\partial p}{\partial y}+\rho g_y=\rho(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}) & (\mathrm{N}_y) \\ \mu(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2})-\frac{\partial p}{\partial z}+\rho g_z=\rho(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}) & (\mathrm{N}_z) \end{cases}$$

Equation (N_x) will be the first equation to be solved; and based on its solution, one will be able to write down the solutions for the other two equations, (N_y), and (N_z).

Dimensional Consistency

The Navier-Stokes equations are dimensionally consistent as shown below:

$$\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})$$

Using MLT

$$\boxed{M(L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2}-L^{-2}T^{-2}+L^{-2}T^{-2})=M(L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2}+L^{-2}T^{-2})}$$

Using $kg\text{–}m\text{–}s$

$$\boxed{kg(m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}-m^{-2}s^{-2}+m^{-2}s^{-2}=kg(m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}+m^{-2}s^{-2}}$$

$$\mu(\overbrace{\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}}^{\text{viscosity}})-\overbrace{\frac{\partial p_x}{\partial x}}^{\text{pressure gradient}}+\overbrace{\rho g_x}=\underbrace{\rho(\overbrace{\frac{\partial V_x}{\partial t}}^{\substack{\text{variable acceleration}\\ \text{(local rate of change of } V_x)}}+\overbrace{V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}}^{\substack{\text{convective acceleration}\\ \text{(rate of change in } V_x \text{ due to motion)}}})}_{\text{inertia per volume}}$$

Option 1

Solution of 3-D Linearized Navier-Stokes Equation in the *x*-direction

The equation will be linearized by redefinition. The nine-term equation will be reduced to six terms.

Given: $$\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}) \qquad \text{(A)}$$

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x \qquad \text{(B)}$$

$$-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x \qquad \text{(C)}$$

Plan: One will split-up equation (C) into five sub-equations, solve them, and combine the solutions. On splitting-up the equations and proceeding to solve them, the non linear terms could be redefined and made linear. This linearization is possible if the gravitational force term is the subject of the equation as in equation (B). After converting the non-linear terms to linear terms by redefinition, one will have only six terms as in equation (C). One will show logically how equation (C) was obtained from equation (B), using a ratio method.

Three main steps are covered.
In main Step 1, one shows how equation (C) was obtained from equation (B)
In main Step 2, equation (C) will be split-up into five equations.
In main Step 3, each equation will be solved.
In main Step 4, the solutions from the five equations will be combined.
In main Step 5, the combined relation will be checked in equation (C). for identity.

Preliminaries

Requirements and procedure for solving a partial differential equation

1. Integrate the partial differential equation.
2. Find the partial derivatives from the integration relation from Step 1
3. Substitute the derivatives from Step 2 in the original partial differential equation and simplify. both sides of the equation.
4. If the left-hand side of the equation is equal to the right-hand side of the equation, then the integration relation from Step 1 is a solution to the partial differential equation.
   (Steps 2-4 can be summarized as checking for identity, or determining if the integration relation satisfies the original partial differential equation.)

Note: If one does not successfully check for identity, one cannot claim a solution.

A ratio method will be used to split-up the partial differential equations into sub-equation which are then integrated.

Example 1: A grandmother left \$45,000 in her will to be divided between eight grandchildren, Betsy, Comfort, Elaine, Ingrid, Elizabeth, Maureen, Ramona, Marilyn, in the ratio $\frac{1}{36}:\frac{1}{18}:\frac{1}{12}:\frac{1}{9}:\frac{5}{36}:\frac{1}{6}:\frac{7}{36}:\frac{2}{9}$. (**Note**: $\frac{1}{36}+\frac{1}{18}+\frac{1}{12}+\frac{1}{9}+\frac{5}{36}+\frac{1}{6}+\frac{7}{36}+\frac{2}{9}=1$)
How much does each receive?

Solution:

Betsy's share of \$45,000 $=\frac{1}{36}\times \$45{,}000=\$1{,}250$

Comfort's share of \$45,000 $=\frac{1}{18}\times \$45{,}000=\$2{,}500$

Elaine's share of \$45,000 $=\frac{1}{12}\times \$45{,}000=\$3{,}750$

Ingrid's share of \$45,000 $=\frac{1}{9}\times \$45{,}000=\$5{,}000$

Elizabeth's share of \$45,000 $=\frac{5}{36}\times \$45{,}000=\$6{,}250$

Maureen's share of \$45,000 $=\frac{1}{6}\times \$45{,}000=\$7{,}500$

Ramona's share of \$45,000 $=\frac{7}{36}\times \$45{,}000=\$8{,}750$

Marilyn's share of \$45,000 $=\frac{2}{9}\times \$45{,}000=\$10{,}000$

Check; Sum of shares $\boxed{= \$45{,}000}$
Sum of the fractions = 1

Example 2: Sir Isaac Newton left ρg_x units in his will to be divided between $-\mu\frac{\partial^2 V_x}{\partial x^2}$, $-\mu\frac{\partial^2 V_x}{\partial y^2}$, $-\mu\frac{\partial^2 V_x}{\partial z^2}$, $\frac{\partial p}{\partial x}$, $\rho\frac{\partial V_x}{\partial t}$, $\rho V_x\frac{\partial V_x}{\partial x}$, $\rho V_y\frac{\partial V_x}{\partial y}$, $\rho V_z\frac{\partial V_x}{\partial z}$ in the ratio $a:b:c:d:f:h:m:n$.
where $a+b+c+d+f+h+m+n=1$. How much does each receive?

Solution $-\mu\frac{\partial^2 V_x}{\partial x^2}$'s share of ρg_x units $= a\rho g_x$ units

$-\mu\frac{\partial^2 V_x}{\partial y^2}$'s share of ρg_x units $= b\rho g_x$ units

$-\mu\frac{\partial^2 V_x}{\partial z^2}$'s share of ρg_x units $= c\rho g_x$ units

$\frac{\partial p}{\partial x}$'s share of ρg_x units $= d\rho g_x$ units

$\rho\frac{\partial V_x}{\partial t}$'s share of ρg_x units $= f\rho g_x$ units

$\rho V_x\frac{\partial V_x}{\partial x}$'s share of ρg_x units $= h\rho g_x$ units

$\rho V_y\frac{\partial V_x}{\partial y}$'s share of ρg_x units $= m\rho g_x$ units

$\rho V_z\frac{\partial V_x}{\partial z}$'s share of ρg_x units $= n\rho g_x$ units

Sum of shares = $\boxed{\rho g_x \text{ units}}$ **Note**: $a+b+c+d+f+h+m+n=1$

Example 3:
The returns on investments A, B, C, D are in the ratio $a:b:c:d$. If the total return on these four investments is P dollars, what is the return on each of these investments?

$$(a+b+c+d=1)$$

Solution Return on investment $A = aP$ dollars
Return on investment $B = bP$ dollars
Return on investment $C = cP$ dollars
Return on investment $D = dP$ dollars

Check

$aP + bP + cP + dP = P$
$P(a+b+c+d) = P$
$a+b+c+d = 1$ (dividing both sides by P)

Example 4: Solve the quadratic equation;

$$6x^2 + 11x - 10 = 0$$

Method 1 (a common method)
By factoring,

$6x^2 + 11x - 10 = 0$
$(3x-2)(2x+5) = 0$ and solving,
$(3x-2) = 0$ or $(2x+5) = 0$
$x = \frac{2}{3}, x = -\frac{5}{2}$.

Solution set: $\{-\frac{5}{2}, \frac{2}{3}\}$

Example 4, Method 2:
One will call this method the **multiplier method.**

Step 1: From $6x^2 + 11x - 10 = 0$ (1)

$6x^2 + 11x = 10$
$6x^2 = 10a$; (Here, a is a multiplier)
$3x^2 = 5a$ (2)
$11x = 10b$ (Here, b is a multiplier)
$11x = 10(1-a)$ $(a+b) = 1$
$11x = 10 - 10a$
$x = \frac{10-10a}{11}$
$3(\frac{10-10a}{11})^2 = 5a$ (Substituting for x in (2)
$3(\frac{100-200a+100a^2)}{121} = 5a$

Step 2: $300a^2 - 1205a + 300 = 0$

$$60a^2 - 241a + 60 = 0$$

$a = \frac{241 \pm \sqrt{241^2 - 4(60)(60)}}{120}$

$a = \frac{241 \pm \sqrt{43681}}{120}$

$a = \frac{241 \pm 209}{120}$

$a = \frac{241 \pm 209}{120} = \frac{241+209}{120}$ or $\frac{241-209}{120}$

$= \frac{450}{120}$ or $\frac{32}{120}$

$= \frac{15}{4}$ or $\frac{4}{15}$

Step 3: Since $a+b=1$, when $a = \frac{15}{4}$ or $3\frac{3}{4}$

$b = 1 - 3\frac{3}{4} = -2\frac{3}{4}$ or $-\frac{11}{4}$

when $a = \frac{4}{15}$, $b = 1 - \frac{4}{15} = \frac{11}{15}$

Step 4: When $b = -\frac{11}{4}$, $11x = 10(-\frac{11}{4})$

$x = -\frac{5}{2}$

When $b = \frac{11}{15}$, $11x = 10(\frac{11}{15})$

$x = \frac{10}{11}(\frac{11}{15})$; $x = \frac{2}{3}$

Again, one obtains the same solution set $\{-\frac{5}{2}, \frac{2}{3}\}$ as by the factoring method.
The objective of presenting examples 1, 2, 3,, and 4 was to convince the reader that the principles to be used in splitting the Navier-Stokes equations are valid.. In Examples 4, one could have used the quadratic formula directly to solve for x, without finding a and b first. The objective was to show that the introduction of a and b did not change the solution set of the original equation.
For the rest of the coverage in this paper, a multiplier is the same as a ratio term
The multiplier method is the same as the ratio method.

Main Step 1
Linearization of the Non-Linear Terms

Step 1: The main principle is to multiply the right side of the equation by the ratio terms
This step is critical to the removal of the non-linearity of the equation.
ρg_x is to be divided by the terms on the left-hand--side of the equation in the ratio
$a:b:c:d:f:h:m:n \quad (a+b+c+d+f+h+m+n=1$

$$-\mu\frac{\partial^2 v_x}{\partial x^2}-\mu\frac{\partial^2 v_x}{\partial y^2}-\mu\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial p}{\partial x}+\underbrace{\rho\frac{\partial V_x}{\partial t}+\overbrace{\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}}_{\text{all acceleration terms}}=\rho g_x \qquad (1)$$

Apply the principles involved in the ratio method covered in the preliminaries, to the nonlinear terms (the last three terms.)

Then $\rho V_z\frac{\partial V_x}{\partial z}=n\rho g_x$, where n is the ratio term corresponding to $\rho V_z\frac{\partial V_x}{\partial z}$.

$$V_z\frac{\partial v_x}{\partial z}=ng_x \qquad (2)$$

$V_z\frac{dV_x}{dz}=ng_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dz}{dt}\frac{dV_x}{dz}=ng_x \quad (V_z=\frac{dz}{dt}$, by definition)

$$\frac{dv_x}{dt}=ng_x \qquad (3)$$

Therefore, $\boxed{V_z\frac{\partial V_x}{\partial z}=\frac{dV_x}{dt}=ng_x}$ (4)

Step 2: Similarly, Let $\rho V_y\frac{\partial V_x}{\partial y}=m\rho g_x$ (m is the ratio term corresponding to $\rho V_y\frac{\partial V_x}{\partial y}$) (5)

$V_y\frac{dV_x}{dy}=mg_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dy}{dt}\frac{dV_x}{dy}=mg_x \quad (V_y=\frac{dy}{dt})$

$$\frac{dv_x}{dt}=mg_x \qquad (6)$$

Therefore, $\boxed{V_y\frac{dV_x}{dy}=\frac{dV_x}{dt}=mg_x}$ (7)

Step 3: Let $\rho V_x\frac{\partial v_x}{\partial x}=h\rho g_x$ where h is the ratio term corresponding to $\rho V_x\frac{\partial v_x}{\partial x}$.

$$V_x\frac{\partial V_x}{\partial x}=hg_x \qquad (8)$$

$V_x\frac{dV_x}{dx}=hg_x$ (One drops the partials symbol, since a single independent variable is involved)

$\frac{dx}{dt}\frac{dV_x}{dx}=hg_x \quad (V_x=\frac{dx}{dt})$

$\frac{dv_x}{dt}=hg_x$ (9) Therefore, $\boxed{V_x\frac{\partial V_x}{\partial x}=\frac{dV_x}{dt}=hg_x}$ (10)

From equations (4), (7), (10), $V_x \frac{\partial v_x}{\partial x} = V_y \frac{\partial v_x}{\partial y} = V_z \frac{\partial v_x}{\partial z} = \frac{dv_x}{dt}$ and

$$V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z} = \boxed{3\frac{dv_x}{dt}} \qquad (11)$$

Thus, the ratio of the linear term $\frac{\partial V_x}{\partial t}$ to the nonlinear sum $V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z}$ in equation (1) is 1 to 3. Unquestionably, there is a ratio between the sum of the nonlinear terms and the linear term $\frac{\partial v_x}{\partial t}$. This ratio must be verified experimentally.

Note: One could have obtained equation (C) from equation (A) by redefining the nonlinear terms by **carelessly** disregarding the partial derivatives of the nonlinear terms in equation (1). However, the author did not do that, but logically, the terms became linearized.

Note also that the above linearization is possible only if ρg_x is the subject of the equation, and it will later be learned that a solution to the logically linearized Navier-Stokes equation is obtained only if ρg_x is the subject of the equation.

Step 4: Substitute the right side of equation (11) for the nonlinear terms on the left- side of

$$-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + \underbrace{\rho\frac{\partial V_x}{\partial t} + \overbrace{\rho V_x\frac{\partial V_x}{\partial x} + \rho V_y\frac{\partial V_x}{\partial y} + \rho V_z\frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}}_{\text{all acceleration terms}} = \rho g_x \qquad (12)$$

Then one obtains $$-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} \underbrace{+\rho\frac{\partial V_x}{\partial t} + 3\rho\frac{\partial V_x}{\partial x}}_{\text{all acceleration terms}} = \rho g_x$$

$$\boxed{-\mu\frac{\partial^2 v_x}{\partial x^2} - \mu\frac{\partial^2 v_x}{\partial y^2} - \mu\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho\frac{\partial v_x}{\partial t} = \rho g_x} \quad \text{(simplifying)} \qquad (13)$$

Now, instead of solving equation (1), previous page, one will solve the following equation

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} - K\frac{\partial^2 V_x}{\partial y^2} - K\frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho}\frac{\partial p}{\partial x} + 4\frac{\partial V_x}{\partial t} = g_x} \quad (k = \frac{\mu}{\rho}) \qquad (14)$$

Main Step 2

Step 5: In equation (14) divide g_x by the terms on the left side in the ratio $a : b : c : d : f$.

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} = ag_x; \quad -K\frac{\partial^2 V_x}{\partial y^2} = bg_x; \quad -K\frac{\partial^2 V_x}{\partial z^2} = cg_x; \quad \frac{1}{\rho}\frac{\partial p}{\partial x} = dg_x; \quad 4\frac{\partial V_x}{\partial t} = fg_x}$$

(a, b, c, d, f are the ratio terms and $a + b + c + d + f = 1$).

As proportions: $$\frac{-K\frac{\partial^2 V_x}{\partial x^2}}{a} = \frac{g_x}{1}; \quad \frac{-K\frac{\partial^2 V_x}{\partial y^2}}{b} = \frac{g_x}{1}; \quad \frac{-K\frac{\partial^2 V_x}{\partial z^2}}{c} = \frac{g_x}{1}; \quad \frac{\frac{1}{\rho}\frac{\partial p}{\partial x}}{d} = \frac{g_x}{1}; \quad \frac{4\frac{\partial V_x}{\partial t}}{f} = \frac{g_x}{1}$$

One can view each of the ratio terms a, b, c, d, f as a fraction (a real number) of $\boxed{g_x}$ contributed by each expression on the left-hand side of equation (14) above.

Main Step 3

Step 6: Solve the differential equations in Step 5.

Solutions of the five sub-equations

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2} = ag_x}$$
$$k\frac{\partial^2 V_x}{\partial x^2} = -ag$$
$$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{k}g$$
$$\frac{\partial V_x}{\partial x} = -\frac{ag}{k}x + C_1$$
$$V_{x1} = -\frac{ag}{2k}x^2 + C_1x + C_2$$

$$\boxed{-K\frac{\partial^2 V_x}{\partial y^2} = bg_x}$$
$$K\frac{\partial^2 V_x}{\partial y^2} = -bg$$
$$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{k}g$$
$$\frac{\partial V_x}{\partial y} = -\frac{bg}{k}y + C_3$$
$$V_{x2} = -\frac{bg}{2k}y^2 + C_3y + C_4$$

$$\boxed{-K\frac{\partial^2 V_x}{\partial z^2} = cg_x}$$
$$K\frac{\partial^2 V_x}{\partial z^2} = -cg$$
$$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{k}g$$
$$\frac{\partial V_x}{\partial z} = -\frac{cg}{k}z + C_5$$
$$V_{x3} = -\frac{cg}{2k}z^2 + C_5z + C_6$$

$$\boxed{\frac{1}{\rho}\frac{\partial p}{\partial x} = dg_x}$$
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = dg$$
$$\frac{\partial p}{\partial x} = d\rho g$$
$$p = d\rho gx + C_7$$
$$\boxed{4\frac{\partial V_x}{\partial t} = fg_x}$$
$$\frac{\partial V_x}{\partial t} = \frac{f}{4}g_x$$
$$V_{x4} = \frac{fg_x}{4}t$$

Main Step 4

Step 7: One combines the above solutions

$$\begin{aligned}
V_x &= V_{x1} + V_{x2} + V_{x3} + V_{x4} \\
&= -\frac{ag_x}{2k}x^2 + C_1x + C_2 - \frac{bg_x}{2k}y^2 + C_3y + C_4 - \frac{cg_x}{2k}z^2 + C_5z + C_6 + \frac{fg_x}{4}t + C_7 \\
&= -\frac{ag_x}{2k}x^2 + C_1x - \frac{bg_x}{2k}y^2 + C_3y - \frac{cg_x}{2k}z^2 + C_5z + \frac{fg_x}{4}t + C_9 \\
&= -\frac{ag_x}{2k}x^2 - \frac{bg_x}{2k}y^2 - \frac{cg_x}{2k}z^2 + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9 \\
&= -\frac{ag_x}{2k}x^2 - \frac{bg_x}{2k}y^2 - \frac{cg_x}{2k}z^2 + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9 \\
&= -\frac{g_x}{2k}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9
\end{aligned}$$

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9$$

$$P(x) = d\rho g_x x$$

$$\boxed{\begin{aligned}
&V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4} \\
&V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9 \\
&P(x) = d\rho g_x x
\end{aligned}}$$

Main Step 5
Checking in equation (C)

Step 8: Find the derivatives, using

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{f g_x}{4} t + C_9$$

$$\boxed{P(x) = d\rho g_x x}$$

$$\frac{\partial V_x}{\partial x} = -\frac{\rho g_x}{2\mu}(2ax) + C_1$$

1. $\boxed{\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}}$

4. $\boxed{\frac{\partial p}{\partial x} = d\rho g_x}$; 5. $\boxed{\frac{\partial V_x}{\partial t} = \frac{f g_x}{4}}$

$$\frac{\partial V_x}{\partial y} = -\frac{\rho g_x}{\mu}(by) + C_3$$

2. $\boxed{\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}}$

$$\frac{\partial V_x}{\partial z} = -\frac{\rho g_x}{\mu}(cz)$$

3. $\boxed{\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu};}$

Step 9: Substitute the derivatives from Step 8 in $-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p_x}{\partial x} + 4\rho\frac{\partial V_x}{\partial t} = \rho g_x$ to check for identity (to determine if the relation obtained satisfies the original equation).

$$-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho\frac{\partial V_x}{\partial t} = \rho g_x$$

$$-\mu(-\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu}) + d\rho g_x + 4\rho\frac{f}{4}g_x \stackrel{?}{=} \rho g_x$$

$$a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + \rho f g_x \stackrel{?}{=} \rho g_x$$

$$a g_x + b g_x + c g_x + d g_x + f g_x \stackrel{?}{=} g_x$$

$$g_x(a + b + c + d + f) \stackrel{?}{=} g_x$$

$$g_x(1) \stackrel{?}{=} g_x \quad (a + b + c + d + f = 1)$$

$$g_x \stackrel{?}{=} g_x \quad \text{Yes}$$

Scrapwork

$$\boxed{\begin{aligned}\frac{\partial^2 V_x}{\partial x^2} &= -\frac{a\rho g_x}{\mu};\\ \frac{\partial^2 V_x}{\partial y^2} &= -\frac{b\rho g_x}{\mu};\\ \frac{\partial^2 V_x}{\partial z^2} &= -\frac{c\rho g_x}{\mu};\end{aligned}}$$

$\boxed{\frac{\partial p}{\partial x} = d\rho g_x;}$ $\boxed{\frac{\partial V_x}{\partial t} = \frac{f g_x}{4}}$

An identity is obtained and therefore, the solution of equation (C), p.5, is given by

$$\boxed{V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{f g_x}{4} t + C_9; \;\; P(x) = d\rho g_x x}$$

The above solution is unique, because all possible equations were integrated but only a single equation, the equation with the gravity term as the subject of the equation produced the solution.

Solution Summary for V_x, V_y and V_z

For V_x $\quad a+b+c+d+f=1$

$$\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho g_x=\rho\left(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}\right)$$

$$-K\frac{\partial^2 V_x}{\partial x^2}-K\frac{\partial^2 V_x}{\partial y^2}-K\frac{\partial^2 V_x}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}+4\frac{\partial V_x}{\partial t}=g_x$$

$$\begin{aligned}V_x &= V_{x1}+V_{x2}+V_{x3}+V_{x4}\\ &=-\frac{ag_x}{2k}x^2+C_1x+C_2-\frac{bg_x}{2k}y^2+C_3y+C_4-\frac{cg_x}{2k}z^2+C_5z+C_6+\frac{fg_x}{4}t+C_7+C_8\\ &=-\frac{ag_x}{2k}x^2+C_1x-\frac{bg_x}{2k}y^2+C_3y-\frac{cg_x}{2k}z^2+C_5z+\frac{fg_x}{4}t+C_9\\ &=-\frac{ag_x}{2k}x^2-\frac{bg_x}{2k}y^2-\frac{cg_x}{2k}z^2+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9\end{aligned}$$

$$V_y(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9$$

$$P(x)=d\rho g_x x$$

For V_y $\quad h+j+m+n+q=1$

$$\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right)-\frac{\partial p}{\partial y}+\rho g_y=\rho\left(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}\right)$$

$$-K\frac{\partial^2 V_y}{\partial x^2}-K\frac{\partial^2 V_y}{\partial y^2}-K\frac{\partial^2 V_y}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial y}+4\frac{\partial V_y}{\partial t}=g_y$$

$$\boxed{\begin{aligned}&V_y=-\frac{hg_y}{2k}x^2+C_1x-\frac{jg_y}{2k}y^2+C_3y-\frac{mg_y}{2k}z^2+C_5z+\frac{ng_y}{4}t\\ &V_y(x,y,z,t)=-\frac{\rho g_y}{2\mu}(hx^2+jy^2+mz^2)+C_1x+C_3y+C_5z+\frac{qg_y}{4}t+C\\ &P(y)=n\rho g_y y\end{aligned}}$$

For V_z $\quad r+s+u+v+w=1$

$$\mu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2}\right)-\frac{\partial p}{\partial z}+\rho g_z=\rho\left(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}\right)$$

$$-k\frac{\partial^2 V_z}{\partial x^2}-k\frac{\partial^2 V_z}{\partial y^2}-k\frac{\partial^2 V_z}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial z}+4\frac{\partial V_z}{\partial t}=g_z$$

$$\boxed{\begin{aligned}&V_z=-\frac{rg_z}{2k}x^2+C_1x-\frac{sg_z}{2k}y^2+C_3y-\frac{ug_z}{2k}z^2+C_5z+\frac{wg_z}{4}t\\ &V_z(x,y,z,t)=-\frac{\rho g_z}{2\mu}(rx^2+sy^2+uz^2)+C_1x+C_3y+C_5z+\frac{wg_z}{4}t+C\\ &P(z)=v\rho g_z z\end{aligned}}$$

Discussion About Linearized N-S Solutions

A solution to equation $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$ (C) is

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9 \\ P(x)=d\rho g_x x; \quad (a+b+c+d+f=1) \end{array}}.$$

This relation gives an identity when checked in Equation (C) above.

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g, were zero, the first three terms, the seventh term, and $P(x)$ would all be zero.. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The above result will be the same when one covers the general case, Option 4.

The above parabolic solution is also encouraging. It reminds one of the parabolic curve obtained when a stone is projected upwards at an acute angle to the horizontal..

More Observations Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): $V_f=V_0+gt$; (C): $V_f^2=V_0^2+2gx$; (D): $V=\sqrt{2gx}$; (E): $x=V_0t+\frac{1}{2}gt^2$

Liquid Pressure,

The liquid pressure, P at the bottom of a liquid of depth h units is given by $P=\rho gh$

$x-$ direction linearized Navier–Stokes equation:

$$-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p_x}{\partial x}+4\rho\frac{\partial V_x}{\partial t}=\rho g_x$$

$x-$ direction Navier–Stokes solution:

$$\boxed{V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\tfrac{1}{4}fg_xt+C_9; \;\; P(x)=d\rho g_x x}$$

Observe the following above:

1. Observe that the first three terms of the solution are parabolic in x, y, and z; the minus sign showing the inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; Also note the "gt" in $V=gt$ of (B) of the motion equations and the fg_xt in the Navier-Stokes solution.

2. Observe the $P=\rho gh$ of the liquid pressure and the $P(x)=d\rho g_x x$ of the Navier-Stokes solution. Note that d is a ratio term.

There are five main terms in the solution of the linearized Navier-Stokes equation. All of these five terms, namely, $-\frac{a\rho g_x}{2\mu}x^2$, $-\frac{b\rho g_x}{2\mu}y^2$, $-\frac{c\rho g_x}{2\mu}z^2$, fg_xt, and $d\rho g_x x$ are similar (except for the constants involved) to the terms in the equations of motion and fluid pressure of elementary physics. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these five terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the linearized Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.

The author also tried the following possible approaches: (D), (E) and (F), but none of the possible solutions completely satisfied the corresponding original equations (D), (E) or (F) .

$$\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}+\mu\frac{\partial^2 V_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial V_x}{\partial t}=\frac{\partial p}{\partial x} \quad \text{(D)} \quad \text{(One uses the subject } \boxed{\frac{\partial p}{\partial x}}$$

$$\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t} \quad \text{(E), (One uses the subject } \boxed{\frac{\partial V_x}{\partial t}}$$

$$-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial x^2} \quad \text{(F)} \quad \text{(One uses subject } \boxed{\frac{\partial^2 V_x}{\partial x^2}}$$

Integration Results Summary

Case 1: $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$ (C)

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_x}{4}t+C_9 \\ P(x)=d\rho g_x x; \qquad (a+b+c+d+f=1) \end{array}}$$

<----Solution

Case 2: $\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}+\mu\frac{\partial^2 V_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial V_x}{\partial t}=\frac{\partial p}{\partial x}$ (D). (One uses the subject $\boxed{\frac{\partial p}{\partial x}}$

$$\boxed{\begin{array}{l} V_x(x,y,z.t)=\frac{\lambda_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+\lambda_p x+C_3y+C_5z-\frac{f\lambda}{4\rho}t+C \\ P(x)=\frac{1}{d}\rho g_x x \end{array}}$$

Case 3: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$ (E). (One uses the subject $\boxed{\frac{\partial V_x}{\partial t}}$

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=(C_1\cos\lambda_x x+C_2\sin\lambda_x x)e^{-(\lambda^2/\beta)t}+(C_3\cos\lambda_y y+C_4\sin\lambda_y y)e^{-(\lambda_y{}^2/\omega)t} \\ +(C_5\cos\lambda_z z+C_6\sin\lambda_z z)e^{-(\lambda_z{}^2/\varepsilon)t}+\frac{g}{4f}t+\lambda x+C_8 \\ P(x)=\lambda x=d\rho g_x x \end{array}}$$

Case 4: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial x^2}$ (F). (One uses the subject $\boxed{\frac{\partial^2 V_x}{\partial x^2}}$

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=(A\cos\lambda y+B\sin\lambda y)\left(Ce^{(\frac{\lambda\sqrt{a}}{a})x}+De^{-(\frac{\lambda\sqrt{a}}{a})x}\right) \\ +(E\cos\lambda z+F\sin\lambda z)\left(He^{(\frac{\lambda\sqrt{b}}{b})x}+Le^{(-\frac{\lambda\sqrt{b}}{b})x}\right)-\frac{\rho g_x x^2}{2c\mu}+Ax+B+(A_1\cos\lambda x+B_1\sin\lambda x)e^{-(\lambda^2/\alpha)t} \\ +\frac{\lambda}{2\mu f}x^2+C_2x+C_3); \qquad P(x)=d\rho g_x x \end{array}}$$

Note: Relations for equations with subjects g_x and $\frac{\partial p}{\partial x}$ are almost identical.

By comparing possible solutions for equations (C) and (D), $\lambda_x=-\rho g_x$ in relation for (D).

$$\boxed{V_x(x,y,z,t)=\frac{\lambda_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+\lambda_p x+C_3y+C_5z-\frac{f\lambda}{4\rho}t+C; \quad P(x)=\frac{1}{d}\rho g_x x}$$

Comparative analysis of the possible solutions when checked in each corresponding equation

Equation	Equation Subject	Number of terms of possible solutions **not** satisfying original equation
Case 1: $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$	ρg_x	None Case 1 yields the solution
Case 2: $\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}+\mu\frac{\partial^2 V_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial V_x}{\partial t}=\frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	One term
Case 3: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	At least 2 terms
Case 4: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	At least 2 terms
Case 5: $-\frac{\partial^2 V_x}{\partial x^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	At least 2 terms
Case 6: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial x^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	At least 2 terms

Note above that only Case 1 is the solution, and this may imply that the solution to the Navier-Stokes equation is unique. Out of six possible subjects, only one subject produced a solution. The above results show that a relation obtained by the integration of a partial differential equation must be checked in the corresponding equation for identity before the relation becomes a solution, Cases 2, 3, 4, 5 and 6, are not solutions but integration relations. For example, it would be incorrect to say that the equation in Case 3 has a periodic solution; but it would be correct to say that the equation in Case 3 has a periodic relation, since the relation obtained by integration does not satisfy its corresponding equation. It would be correct to say that the equation in Case 1 has a parabolic solution or a parabolic relation.
Below are detailed explanation of results of the identity checking process.

Outcome 1: With g_x included and with g_x as the subject of the equation. The solution is straightforward and the possible solution checks well in the original equation (C). Also, if g_x or ρg_x is not the subject of the equation, the linearization of the nonlinear terms could not be justified.

Outcome 2: With g_x included but with $\frac{\partial V_x}{\partial t}$ as the subject of the equation.

There are two problems when checking . **1.** For $\frac{\partial V_x}{\partial t}=-\frac{1}{4\rho}\frac{\partial p}{\partial x}\rightarrow-\frac{\lambda t}{4\rho d}$**;** **2.** $\frac{g_x}{4}=\frac{\partial V_x}{\partial t}\rightarrow\frac{g_x t}{4f}$

With d and f in the denominators, the multipliers sum $a+b+c+d+f=1$ is false.

Outcome 3 : With g_x excluded, and $\frac{\partial V_x}{\partial t}$ as the subject of the equation, there is one problem:

$-\frac{1}{4\rho}\frac{\partial p}{\partial x}=\frac{\partial V_x}{\partial t}\rightarrow-\frac{\lambda t}{4\rho d}$ .With d in the denominator $a+b+c+d+f=1$ is false

Outcome 4 : With g_x included, and $\frac{\partial^2 V_x}{\partial x^2}$ as the subject of the equation, there are at least, two problems in the checking with the multipliers c and f in the denominators. Checking for $a+b+c+d+f=1$ is impossible.

Outcomes 5 and 6 are similar to Outcome 4.

Characteristic curves of the integration results

Equations	Equation Subject	Curve characteristics
Case 1: $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$	ρg_x	Parabolic and Inverted
Case 2: $\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}+\mu\frac{\partial^2 V_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial V_x}{\partial t}=\frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	Parabolic
Case 3: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	Quasiperiodic ,and decreasingly exponential
Case 4: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	Quasiperiodic,, parabolic, and decreasingly exponential
Case 5: $-\frac{\partial^2 V_x}{\partial x^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	Quasiperiodic, parabolic, and decreasingly exponential
Case 6: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial x^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	Quasiperiodic, parabolic, and decreasingly exponential

The following are possible interpretations of the roles of the terms based on the types of curves produced when using the terms as subjects of the equations.

1. g_x and $\frac{\partial p}{\partial x}$ are involved in the parabolic motion; g_x is responsible for the forward motion.

2. $\frac{\partial V_x}{\partial t}$ is involved in the quasiperiodic, and decreasingly exponential behavior.

3. $\frac{\partial^2 V_x}{\partial x^2}$, $\frac{\partial^2 V_x}{\partial y^2}$ and $\frac{\partial^2 V_x}{\partial z^2}$ are involved in the parabolic, quasiperiodic, and decreasingly exponential motion. As μ increases, the quasiperiodicity increases

Definitions and Classification of Equations

$$\boxed{-K\frac{\partial^2 V_x}{\partial x^2}-K\frac{\partial^2 V_x}{\partial y^2}-K\frac{\partial^2 V_x}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}+4\frac{\partial V_x}{\partial t}=g_x} \qquad (k=\frac{\mu}{\rho})$$

One may classify the equations involved in Option 1 according to the following:

Driver Equation: A differential equation whose integration relation satisfies its corresponding equation.

Supporter equation: A differential equation which contains the same terms as the driver equation but whose integration relation does not satisfy its corresponding equation but provides useful information about the driver equation.

Note that the driver equation and a supporter equation differ only in the subject of the equation.

Equation	**Equation Subject**	**Type of equation**	**# of terms of relation not satisfying original equation**
Case 1: $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})=\rho g_x$	ρg_x	Driver Equation	None
Case 2: $\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}+\mu\frac{\partial^2 V_x}{\partial z^2}+\rho g_x-4\rho\frac{\partial V_x}{\partial t}=\frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	Supporter equation	One term
Case 3: $\frac{K}{4}\frac{\partial^2 V_x}{\partial x^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial y^2}+\frac{K}{4}\frac{\partial^2 V_x}{\partial z^2}-\frac{1}{4\rho}\frac{\partial p}{\partial x}+\frac{g_x}{4}=\frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	Supporter equation	At least 2 terms
Case 4: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	Supporter equation	At least 2 terms
Case 5: $-\frac{\partial^2 V_x}{\partial x^2}-\frac{\partial^2 V_x}{\partial z^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	Supporter equation	At least 2 terms
Case 6: $-\frac{\partial^2 V_x}{\partial y^2}-\frac{\partial^2 V_x}{\partial x^2}-\frac{\rho g_x}{\mu}+\frac{4\rho}{\mu}\frac{\partial V_x}{\partial t}+\frac{1}{\mu}\frac{\partial p}{\partial x}=\frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	Supporter equation	At least 2 terms

The uniqueness of the above solution will guide one to save time and not try to solve some forms of Euler or Navier-Stokes equation which do not produce solutions. That is, one will solve only the equations with the gravity term as the subject. This uniqueness will also guide one to solve the magnetohydrodynamic equations.

Applications of the splitting technique in science, engineering, business fields

The approach used in solving the equations allows for how the terms interact with each other. The author has not seen this technique anywhere, but the results are revealing and promising.

Fluid flow design considerations:

1. Maximize the role of g_x forces, followed by**; 2.** $\frac{\partial p}{\partial x}$ forces; then 3. $\frac{\partial V_x}{\partial t}$

Make g_x happy by always providing a workable slope.

For long distance flow design such as for water pipelines, water channels, oil pipelines. whenever possible, the design should facilitate and maximize the role of gravity forces, and if design is impossible to facilitate the role of gravity forces, design for $\frac{\partial p}{\partial x}$ to take over flow.

The performance of $\frac{\partial^2 V_x}{\partial x^2}$ should be studied further, since its role is the most complicated: periodic, parabolic, and decreasingly exponential.

Tornado Effect Relief

Perhaps, machines can be designed and built to chase and neutralize or minimize tornadoes during touch-downs. The energy in the tornado at touch-down can be harnessed for useful purposes.

Business and economics applications.

1. Figuratively, if g_x is the president of a company, it will have good working relationships with all the members of the board of directors, according to the solution of the Navier-Stokes equation. If g_x is present at a meeting g_x must preside over the meeting for the best outcome.

2. If g_x is absent from a meeting, let $\frac{\partial p}{\partial x}$ preside over the meeting, and everything will workout well. However, if g_x is present, g_x must preside over the meeting.

To apply the results of the solutions of the Navier-Stokes equations in other areas or fields, the properties, characteristics and functions of g_x, $\frac{\partial p}{\partial x}$, $\frac{\partial v_x}{\partial t}$ must be studied to determine analogous terms in those areas of possible applications. Other areas of applications include investments choice decisions, financial decisions, personnel management and family relationships.

Option 2
Solutions of 4-D Linearized Navier-Stokes Equations

One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

If one adds $\mu\frac{\partial^2 V_x}{\partial s^2}$ and $\rho V_s\frac{\partial V_x}{\partial s}$ to the 3-D x–direction equation, one obtains the 4-D Navier--Stokes equation $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2})+\frac{\partial p}{\partial x}+4\rho(\frac{\partial V_x}{\partial t})+\rho V_s\frac{\partial V_x}{\partial s}=\rho g_x$

After linearization, $-\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2})+\frac{\partial p}{\partial x}+5\rho(\frac{\partial V_x}{\partial t})=\rho g_x$ and its solution is

$$\boxed{\begin{array}{l} V_x(x,y,z,s,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2+es^2)+C_1x+C_3y+C_5z+C_7s+\frac{fg_x}{5}t+C_9 \\ P(x)=d\rho g_x x \qquad (a+b+c+d+e+f=1) \end{array}}.$$

For n–dimensions one can repeat the above as many times as one wishes.

Option 3
Solutions of the Euler Equations of Fluid flow

In the Navier-Stokes equation, if $\mu = 0$, one obtains the Euler equation. From

$$\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})$$, one obtains

Euler equation : $(\mu = 0)$ $-\frac{\partial p}{\partial x}+\rho g_x=\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})$ or

$$\boxed{\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})+\frac{\partial p_x}{\partial x}=\rho g_x}$$ <---driver equation.

Euler equation $(\mu = 0)$: $\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=g_x$ <---driver equation

Split the equation using the ratio terms f, h, n, q, d,, and solve. $(f+h+n+q+d=1)$

1. $\frac{\partial V_x}{\partial t}=fg_x$ $V_{x4}=fg_x t$ $V_{x4}=fg_x t$	**2.** $V_x\frac{\partial V_x}{\partial x}=hg_x$ $V_x\frac{dV_x}{dx}=hg_x$ $V_x dV_x=hg_x dx$ $\frac{V_x^2}{2}=hg_x x$ or $V_x^2=2hg_x x$ $V_x=\pm\sqrt{2hg_x x}$	**3.** $V_y\frac{\partial V_x}{\partial y}=ng_x$ $V_y\frac{dV_x}{dy}=ng_x$ $V_y dV_x=ng_x \mathrm{dy}$ $V_y V_x=ng_x \mathrm{y}+\psi_y(V_y)$ $V_{x6}=\frac{ng_x \mathrm{y}}{V_y}+\frac{\psi_y(V_y)}{V_y}$ $V_y\neq 0$	**4.** $V_z\frac{\partial V_x}{\partial z}=qg_x$ $V_z\frac{dV_x}{dz}=qg_x$ $V_z dV_x=qg_x \mathrm{dz}$; $V_z V_x=qg_x \mathrm{z}+\psi_z(V_z)$ $V_{x7}=\frac{qg_x \mathrm{z}}{V_z}+\frac{\psi_z(V_z)}{V_z}$ $V_z\neq 0$	**5.** $\frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x$ $\frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x$ $\frac{\partial p}{\partial x}=d\rho g_x$ $p=d\rho g_x x+C_7$

$$\boxed{\begin{array}{l} V_x(x,y,z,t)=fg_x t\pm\sqrt{2hg_x x}+\frac{ng_x \mathrm{y}}{V_y}+\frac{qg_x \mathrm{z}}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+C \\ P(x)=d\rho g_x x \qquad (f+h+n+q+d=1)\ V_y\neq 0,\ V_z\neq 0 \end{array}}$$

Find the test derivatives to check in the original equation.

1. $\frac{\partial V_x}{\partial t}=fg_x$	**2.** $V_x^2=2hg_x x$; $2V_x\frac{\partial V_x}{\partial x}=2hg_x$; $\frac{\partial V_x}{\partial x}=\frac{hg_x}{V_x}$, $V_x\neq 0$	3. $\frac{\partial V_x}{\partial y}=\frac{ng_x}{V_y}$ $V_y\neq 0$	4. $\frac{\partial V_x}{\partial z}=\frac{qg_x}{V_z}$ $V_z\neq 0$	5. $\frac{\partial p}{\partial x}=d\rho g_x$

$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=g_x$ (Above, $\psi_y(V_y)$ and $\psi_z(V_z)$ are arbitrary functions)

$$fg_x+V_x\frac{hg_x}{V_x}+V_y\frac{ng_x}{V_y}+V_z\frac{qg_x}{V_z}+\frac{1}{\rho}d\rho g_x\overset{?}{=}g_x$$

$$fg_x+hg_x+ng_x+qg_x+dg_x\overset{?}{=}g_x$$

$$g_x(f+h+n+q+d)\overset{?}{=}g_x$$

$$g_x(1)\overset{?}{=}g_x \qquad (f+h+n+q+d=1)$$

$$g_x\overset{?}{=}g_x$$ Yes

The relation obtained satisfies the Euler equation. Therefore the solution to the Euler equation

$$:\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z})=\rho g_x \text{ is}$$

$$\boxed{\begin{aligned}&V_x(x,y,z,t)=fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};\ P(x)=d\rho g_xx\\ &V_y\neq 0,\ V_z\neq 0;\ \ (d+f+h+n+q=1)\end{aligned}}\quad x\text{-direction}$$

Similarly, the equations and solutions for the other two directions are respectively

$$\text{For } V_y,\ \frac{\partial p}{\partial y}+\rho\frac{\partial v_y}{\partial t}+\rho V_x\frac{\partial v_y}{\partial x}+\rho V_y\frac{\partial v_y}{\partial y}+\rho V_z\frac{\partial V_y}{\partial z}=\rho g_y$$

$$\boxed{\begin{aligned}&V_y(x,y,z,t)=\lambda_5g_yt\pm\sqrt{2\lambda_7g_y\mathrm{y}}+\frac{\lambda_6g_yx}{V_x}+\frac{\lambda_8g_y\mathrm{z}}{V_z}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_z(V_z)}{V_z};\ P(y)=\lambda_4\rho g_yy\\ &V_x\neq 0,\ V_z\neq 0;\ \ (\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1)\end{aligned}}\quad y\text{-direction}$$

$$\text{For } V_z:\ \frac{\partial p}{\partial z}+\rho\frac{\partial v_z}{\partial t}+\rho V_x\frac{\partial v_z}{\partial x}+\rho V_y\frac{\partial v_z}{\partial y}+\rho V_z\frac{\partial v_z}{\partial z}=\rho g_z$$

$$\boxed{\begin{aligned}&V_z(x,y,z,t)=\beta_5g_zt\pm\sqrt{2\beta_8g_zz}+\frac{\beta_6g_zx}{V_x}+\frac{\beta_7g_zy}{V_y}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y}\ ;\ P(z)=\beta_4\rho g_zz\\ &V_x\neq 0,\ V_y\neq 0;\ \ (\beta_4+\beta_5+\beta_6+\beta_7+\beta_8=1)\end{aligned}}\quad z\text{-direction}$$

One will next solve the above system of solutions for V_x , V_y, V_z in order to express $\frac{ng_xy}{V_y}$ and $\frac{q_eg_xz}{V_z}$ in terms of x, y, z, and t.

Solving for V_x , V_y, V_z $\frac{ng_xy}{V_y}$, and $\frac{q_eg_xz}{V_z}$

$$\begin{cases}V_x=fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}. & (A)\\ V_y=\lambda_5g_yt\pm\sqrt{2\lambda_7g_y\mathrm{y}}+\frac{\lambda_6g_yx}{V_x}+\frac{\lambda_8g_y\mathrm{z}}{V_z}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_z(V_z)}{V_z}. & (B)\\ V_z=\beta_5g_zt\pm\sqrt{2\beta_8g_zz}+\frac{\beta_6g_zx}{V_x}+\frac{\beta_7g_zy}{V_y}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y}\ . & (C)\end{cases}$$

Let $V_x=x$, $V_y=y$ and $V_z=z$. (x, y and z are being used for simplicity. They will be changed back to V_x , V_y, and V_z later, and they do not represent the variables x, y and z in the system of solutions)

Step 1 From the above system of solutions, let	**Step 2:** Then the solutions to the Euler system of equations become (ignoring the arbitrary functions)
$A = (fgt + \sqrt{2hg_x x})$; $D = (qg_x z)$; $E = (ng_x y)$ $B = (\lambda_5 g_y t + \sqrt{2\lambda_7 g_y y})$; $F = (\lambda_6 g_y x)$; $G = (\lambda_8 g_y z)$ $C = (\beta_5 g_z t + \sqrt{2\beta_8 g_z z})$; $J = (\beta_6 g_z x)$; $L = (\beta_7 g_z y)$	$\left.\begin{array}{l} x = A + \frac{D}{z} + \frac{E}{y} \\ y = B + \frac{F}{x} + \frac{G}{z} \\ z = C + \frac{J}{x} + \frac{L}{y} \end{array}\right\\}$ M

Step 3	**Step 4**
$\left.\begin{array}{ll} xyz = Ayz + Dy + Ez & (1) \\ xyz = Bxz + Fz + Gx & (2) \\ xyz = Cxy + Jy + Lx & (3)) \end{array}\right\\}$ N	$\left.\begin{array}{ll} 0 = Ayz + Dy + Ez - Bxz - Fz - Gx & (4) \\ 0 = Ayz + Dy + Ez - Cxy - Jy - Lx & (5) \\ 0 = Bxz + Fz + Gx - Cxy - Jy + -Lx & (6) \end{array}\right\\}$ P

Maples software was used to solve system P to obtain

Step 5	**Note**:
$x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ $V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ (back to V_x) $y = -\frac{L}{C}$; $V_y = -\frac{L}{C}$ (changing back to V_y as agreed to) $z = -\frac{L(D-J)}{LA - CE}$; $V_z = -\frac{L(D-J)}{LA - CE}$ (changing back to V_z as agrred to)	None of the popular academic programs could solve the system in M. Maples solved system P (step 4 above) for x, y, and z in terms of A, B, C, D. E. F, G. J. and L. Note also that x, y and z are not the same as the x, y and z in the system of equations.. They were used for convenience and simplicity .

Step 5: Apply and substitute from in steps 6-8 below

$A = (fgt \pm \sqrt{2hg_x x})$; $B = (\lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y})$; $C = (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})$; $D = (qg_x z)$;
$E = (ng_x y)$; $F = (\lambda_6 g_y x)$; $G = (\lambda_8 g_y z) J = (\beta_6 g_z x)$; $L = (\beta_7 g_z y)$,

Step 6	**Step 7**
$V_y = -\frac{L}{C} = -\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})}$ $\frac{ng_x y}{V_y} = ng_x y \div -(\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})})$ $\frac{ng_x y}{V_y} = -\frac{(ng_x y)[\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}]}{\beta_7 g_z y}$; $y \neq 0$	$\frac{ng_x y}{V_y} = -\frac{[\beta_5 g_z t](ng_x y) \pm [\sqrt{2\beta_8 g_z z}](ng_x y)}{\beta_7 g_z y}$ $\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}$

$$\boxed{\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z}(ng_x)}{\beta_7 g_z}}$$

Step 8:

$$V_z = -\frac{L(D-J)}{LA-CE}$$

$$V_z = -\frac{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}{(\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}$$

$$\frac{qg_x z}{V_z} = (qg_x z) \div -\left(\frac{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}{(\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}\right)$$

$$= -\frac{(qg_x z)\bullet(\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$= -\frac{(qg_x z)\bullet(\beta_7 g_z y fg_x t \pm \sqrt{2hg_x x}\beta_7 g_z y - \beta_5 g_z t ng_x y \pm \sqrt{2\beta_8 g_z z}\ ng_x y}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$= -\frac{qg_x g_x z\beta_7 fg_z yt \pm \sqrt{2hg_x x}\beta_7 g_x g_z qyz - \beta_5 g_x g_z nqtyz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqyz}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$\frac{qg_x z}{V_z} = -\frac{qg_x g_x z\beta_7 fg_z t \pm \sqrt{2hg_x x}\beta_7 g_x g_z qz - \beta_5 g_x g_z nqtz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqz}{(\beta_7 g_z)[qg_x z - \beta_6 g_z x]}$$

((Dividing out the "y" in the numerator and the denominator)

$$\frac{qg_x z}{V_z} = -\frac{qg_x g_x z\beta_7 fg_z t \pm \sqrt{2hg_x x}\beta_7 g_x g_z qz - \beta_5 g_x g_z nqtz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqz}{(\beta_7 g_z)[qg_x z - \beta_6 g_z x]} \quad \text{or}$$

$$\frac{qg_x z}{V_z} = \frac{qg_x g_x z\beta_7 fg_z t \pm \sqrt{2hg_x x}\beta_7 g_x g_z qz - \beta_5 g_x g_z nqtz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqz}{\beta_7\beta_6 g_z g_z x - \beta_7 qg_z g_x z}$$

$$\boxed{\frac{qg_x z}{V_z} = \frac{(\beta_7 fg_z g_x g_x q - \beta_5 g_x g_z nq)tz \pm \sqrt{2hg_x x}\beta_7 g_x g_z qz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqz}{\beta_7\beta_6 g_z g_z x - \beta_7 qg_z g_x z}}$$

Summary for the fractional terms of the *x*-direction solution

$\frac{ng_x y}{V_y}$ and $\frac{qg_x z}{V_z}$ in terms of x, y, z and t

$\left.\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}\right\}$ B	$\frac{ng_x y}{V_y} = -k_1 g_z t \pm \frac{\sqrt{2k_2 g_z z}\bullet g_x k_3}{g_z}$ $k_1 = \frac{n\beta_5}{\beta_7}$; $k_2 = \beta_8$; $k_3 = \frac{n}{\beta_7}$;

$$\left.\frac{qg_x z}{V_z} = \frac{(\beta_7 f_e g_z g_x g_x q - \beta_5 g_x g_z n_e q)tz \pm \sqrt{2hg_x x}\beta_7 g_x g_z qz \pm \sqrt{2\beta_8 g_z z}\ g_x g_x nqz}{\beta_7\beta_6 g_z g_z x - \beta_7 qg_z g_x z}\right\}\text{C}$$

$$\frac{qg_x z}{V_z} = \frac{(g_x^2 g_z k_4 - g_x g_z k_5)tz \pm \sqrt{2g_x k_6 x}\bullet g_x g_z k_7 z \pm \sqrt{2g_z k_8 z}\bullet g_x^2 k_9 z}{g_z^2 k_{10} x - g_x g_z k_{11} z}$$

$k_4 = \beta_7 fq$; $k_5 = \beta_5 nq$;; $k_6 = h$; $k_8 = \beta_8$; $k_9 = nq$ $k_{10} = \beta_7\beta_6$ $k_{11} = \beta_7 q$

Analysis of the Euler Solutions

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \dfrac{ng_x y}{V_y} + \dfrac{qg_x z}{V_z} + \underbrace{\dfrac{\psi_y(V_y)}{V_y} + \dfrac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};\ P(x) = d\rho g_x x \\ V_y \neq 0,\ V_z \neq 0;\ \ (d+f+h+n+q=1) \end{array}}\ x\text{-direction}$$

$$\left.\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}\right\}\text{B}$$

$$\left.\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_z g_x^2 q - \beta_5 g_x g_z nq)tz \pm \sqrt{2hg_x x}\,\beta_7 g_x g_z qz \pm \sqrt{2\beta_8 g_z z}\ g_x^2 nqz}{\beta_7\beta_6 g_z^2 x - \beta_7 q g_z g_x z}\right\}\text{C}$$

$$d+f+h+n+q=1;\quad \lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1;\quad \beta_4+\beta_5+\beta_6+\beta_7+\beta_8=1$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g_x were zero, the first four terms of the velocity solution and $P(x)$ would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

More Observations: Comparison of the Euler solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion under gravity equations: (B): $V = gt$; (C): $V = \sqrt{2gx}$;

Liquid Pressure, P at the bottom of a liquid of depth h units is given by $P = \rho gh$

Observe the following similarities above:

1. Observe the "gt" in $V = gt$ of (B) of the motion equations and the $fg_x t$ in the Euler solution.

2. Observe the "$\sqrt{2gx}$" in $V = \sqrt{2gx}$ of (C) and the $\sqrt{2hg_x x}$ in the Euler solution.

3. Observe the $P = \rho gh$ of the liquid pressure and the $P(x) = d\rho g_x x$ of the Euler solution.

There are five main terms (ignoring the arbitrary functions) in the Euler solution. Of these five terms, three terms, namely, $fg_x t$, $\sqrt{2hg_x x}$, $d\rho g_x x$ are the same (except for the constants involved) as the terms in the equations of motion under gravity. This similarity means that the approach used in solving the Euler equation is sound. One should also note that to obtain these three terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these three terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Euler equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

The **velocity profile** of the x–direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would be at right angles to that for laminar flow. The characteristic curve for the integral of the x–nonlinear term is such a parabola whose axis of symmetry is at right angles to that of laminar flow. The integral of the y–nonlinear term is similar parabolically to that of the x–nonlinear term. The characteristic curve for the integral of the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent

Standard form of the x-direction Euler equation for incompressible fluid flow
One will call the Euler equation with the gravity term as the subject of equation in (A) , the standard form of the Euler equation for the ratio method of solving these equations, since this form produces a solution on integration. None of the other forms in (B), (C), (D), (E), or (F), produces a solution. That is, the integration results of each of the other five equations do not satisfy the corresponding equation.

$$\boxed{\rho\frac{\partial v_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial v_x}{\partial y}+\rho V_z\frac{\partial v_x}{\partial z}+\frac{\partial p_x}{\partial x}=\rho g_x \qquad \text{(A)}} \quad < \text{standard form}$$

$$-\rho\frac{\partial v_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial v_x}{\partial y}-\rho V_z\frac{\partial v_x}{\partial z}+\rho g_x=\frac{\partial p_x}{\partial x} \qquad \text{(B)}$$

$$-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial v_x}{\partial y}-\rho V_z\frac{\partial v_x}{\partial z}+\frac{\partial p_x}{\partial x}+\rho g_x=\rho\frac{\partial v_x}{\partial t} \qquad \text{(C)}$$

$$-\rho\frac{\partial v_x}{\partial t}-\rho V_y\frac{\partial v_x}{\partial y}-\rho V_z\frac{\partial v_x}{\partial z}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_x\frac{\partial V_x}{\partial x} \qquad \text{(D)}$$

$$-\rho\frac{\partial v_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_z\frac{\partial v_x}{\partial z}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_y\frac{\partial v_x}{\partial y} \qquad \text{(E)}$$

$$-\rho\frac{\partial v_x}{\partial t}-\rho V_x\frac{\partial V_x}{\partial x}-\rho V_y\frac{\partial v_x}{\partial y}-\frac{\partial p_x}{\partial x}+\rho g_x=\rho V_z\frac{\partial v_x}{\partial z} \qquad \text{(F)}$$

Uniqueness of the solution of the Euler equation
When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. (vixra:1405.0251 of 2014). Similarly. the solution of the Euler solution is unique.

Extra:

Linearized Euler Equation: If one linearizes the Euler equation as was done in the linearization of the Navier-Stokes equation, one obtains $4\frac{\partial v_x}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=g_x$; whose solution is

$V_x=\frac{fg_x}{4}t+C; \quad P(x)=d\rho g_x x$.

Euler solutions in terms of x, y, z, and t.

$$\boxed{\begin{aligned}&V_x(x,y,z,t)=\\&fg_x t\pm\sqrt{2hg_x x}-\frac{n\beta_5 g_z t}{\beta_7}\pm\frac{\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}+\frac{(g_x^2 g_z k_4-g_x g_z k_5)tz\pm\sqrt{2g_x k_6 x}\bullet g_x g_z k_7 z\pm\sqrt{2g_z k_8 z}g_x^2 k_9 z}{g_z^2 k_{10}x-g_x g_z k_{11}z}+\\&\qquad\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}};\ P(x)=d\rho g_x x\\&V_y\neq 0,\ V_z\neq 0;\ \ (d+f+h+n+q=1)\end{aligned}}$$

Note: By comparison with Navier-Stokes equation and its relation, a relation to Euler equation can be found by deleting the Navier-Stokes relation resulting from the μ-terms.

Back to Options

Option 4
Solutions of 3-D Navier-Stokes Equations (Original)
Method 1

As in Option 1 for solving these equations, the first step here, is to split-up the equation into eight sub-equations using the ratio method. One will solve **only** the driver equation, based on the experience gained in solving the linearized equation. There are 8 supporter equations.

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\overbrace{\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}}=\rho g_x \qquad \text{(A)}$$

$$-K\frac{\partial^2 V_x}{\partial x^2}-K\frac{\partial^2 V_x}{\partial y^2}-K\frac{\partial^2 V_x}{\partial z^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=g_x \qquad (K=\frac{\mu}{\rho}) \quad \text{(B)}$$

Step 1: Apply the ratio method to equation (B) to obtain the following equations:

$$1.\ -K\frac{\partial^2 V_x}{\partial x^2}=ag_x;\ \ 2.\ -K\frac{\partial^2 V_x}{\partial y^2}=bg_x;\ \ 3.\ -K\frac{\partial^2 V_x}{\partial z^2}=cg_x;\ 4.\ \frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x;\ 5.\ \frac{\partial V_x}{\partial t}=fg_x$$

$$6.\ V_x\frac{\partial V_x}{\partial x}=hg_x;\ \ 7.\ V_y\frac{\partial V_x}{\partial y}=qg_x;\ \ 8.\ V_z\frac{\partial V_x}{\partial z}=ng_x$$

where $a,\ b,\ c,\ d,\ f,\ h,\ n,\ q$ are the ratio terms and $a+b+c+d+f+h+n+q=1$

Step 2: Solve the differential equations in Step 1.

Note that after splitting the equations, the equations can be solved using techniques of ordinary differential equations.

One can view each of the ratio terms $a,\ b,\ c,\ d,\ f,\ h,\ n,\ q$ as a fraction (a real number) of $\boxed{g_x}$ contributed by each expression on the left-hand side of equation (B) above.

Solutions of the eight sub-equations

$\boxed{1.\ -k\frac{\partial^2 V_x}{\partial x^2}=ag_x}$ $k\frac{\partial^2 V_x}{\partial x^2}=-ag_x$ $\frac{\partial^2 V_x}{\partial x^2}=-\frac{a}{k}g_x$ $\frac{\partial V_x}{\partial x}=-\frac{ag}{k}x+C_1$ $V_{x1}=-\frac{ag_x}{2k}x^2+C_1x+C_2$	$\boxed{2.\ -K\frac{\partial^2 V_x}{\partial y^2}=bg_x}$ $K\frac{\partial^2 V_x}{\partial y^2}=-bg_x$ $\frac{\partial^2 V_x}{\partial y^2}=-\frac{b}{k}g_x$ $\frac{\partial V_x}{\partial y}=-\frac{bg_x}{k}y+C_3$ $V_{x2}=-\frac{bg_x}{2k}y^2+C_3y+C_4$	$\boxed{3.\ -K\frac{\partial^2 V_x}{\partial z^2}=cg_x}$ $K\frac{\partial^2 V_x}{\partial z^2}=-cg_x$ $\frac{\partial^2 V_x}{\partial z^2}=-\frac{c}{k}g_x$ $\frac{\partial V_x}{\partial z}=-\frac{cg_x}{k}z+C_5$ $V_{x3}=-\frac{cg_x}{2k}z^2+C_5z+C_6$	$\boxed{4.\ \frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x}$ $\frac{1}{\rho}\frac{\partial p}{\partial x}=dg_x$ $\frac{\partial p}{\partial x}=d\rho g_x$ $p=d\rho g_x x+C_7$ $\boxed{5.\ \frac{\partial V_x}{\partial t}=fg_x}$ $V_{x4}=fgt$
$\boxed{6.\ V_x\frac{\partial V_x}{\partial x}=hg_x}$ $V_x\frac{dV_x}{dx}=hg_x$ $V_xdV_x=hg_xdx$ $\frac{V_x^{\ 2}}{2}=hg_xx$ $V_{x5}=\pm\sqrt{2hg_xx}+C_7$	$\boxed{7.\ V_y\frac{\partial V_x}{\partial y}=ng_x}$ $V_y\frac{dV_x}{dy}=ng_x$ $V_ydV_x=ng_x\mathrm{dy}$ $V_yV_x=ng_x\mathrm{y}\ +\psi_y(V_y)$ $V_{x6}=\frac{ng_x\mathrm{y}}{V_y}+\frac{\psi_y(V_y)}{V_y}$	$\boxed{8.\ V_z\frac{\partial V_x}{\partial z}=qg_x}$ $V_z\frac{dV_x}{dz}=qg_x$ $V_zdV_x=qg_x\mathrm{dz};$ $V_zV_x=qg_x\mathrm{z}\ +\psi_z(V_z)$ $V_{x7}=\frac{qg_x\mathrm{z}}{V_z}+\frac{\psi_z(V_z)}{V_z}$	**Note:** $\psi_y(V_y),\ \psi_z(V_z)$ are arbitrary functions, (integration constants) $V_y\neq 0$ $V_z\neq 0$

Step 3: One combines the above solutions

$$V_x(x,y,z,t) = V_{x1} + V_{x2} + V_{x3} + V_{x4} + V_{x5} + V_{x6} + V_{x7}$$

$$= -\frac{ag_x}{2k}x^2 + C_1x - \frac{bg_x}{2k}y^2 + C_3y - \frac{cg_x}{2k}z^2 + C_5z + fg_xt \pm \sqrt{2hg_xx} + \frac{ng_xy}{V_y} + \frac{qg_xz}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}$$

$$\overbrace{-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_xt}^{\text{relation for linear terms}} \pm \overbrace{\sqrt{2hg_xx} + \frac{ng_xy}{V_y} + \frac{qg_xz}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}}^{\text{relation for non - linear terms}} + C_9$$

$$P(x) = d\rho g_xx; \quad (a+b+c+d+f+h+n+q=1) \quad V_y \neq 0,\ V_z \neq 0$$

Step 4: Find the test derivatives

Test derivatives for the linear part					Test derivatives for the non-linear part		
$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}$	$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}$	$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu}$	$\frac{\partial p}{\partial x} = d\rho g_x$	$\frac{\partial V_x}{\partial t} = fg_x$	$V_x^2 = 2hg_xx$ $2V_x\frac{\partial V_x}{\partial x} = 2hg_x$ $\frac{\partial V_x}{\partial x} = \frac{hg_x}{V_x}, V_x \neq 0$	$\frac{\partial V_x}{\partial y} = \frac{ng_x}{V_y}$	$\frac{\partial V_x}{\partial z} = \frac{qg_x}{V_z}$

Step 5: **S**ubstitute the derivatives from Step 4 in equation (A) for the checking.

$$-\mu\frac{\partial^2 V_x}{\partial x^2} - \mu\frac{\partial^2 V_x}{\partial y^2} - \mu\frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x\frac{\partial V_x}{\partial x} + \rho V_y\frac{\partial V_x}{\partial y} + \rho V_z\frac{\partial V_x}{\partial z} = \rho g_x \quad \textbf{(A)}$$

$$-\mu\left(-\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu}\right) + d\rho g_x + f\rho g_x + \rho\left(V_x\frac{hg_x}{V_x}\right) + \rho V_y\left(\frac{ng_x}{V_y}\right) + \rho V_z\left(\frac{qg_x}{V_z}\right) \stackrel{?}{=} \rho g_x$$

$$a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + f\rho g_x + h\rho g_x + n\rho g_x + q\rho g_x \stackrel{?}{=} \rho g_x$$

$$ag_x + bg_x + cg_x + dg_x + fg_x + hg_x + ng_x + qg_x \stackrel{?}{=} g_x$$

$$g_x(a+b+c+d+f+h+n+q) \stackrel{?}{=} g_x$$

$$g_x(1) \stackrel{?}{=} g_x \quad \text{Yes} \quad (a+b+c+d+f+h+n+q=1)$$

Step 6: The linear part of the relation satisfies the linear part of the equation; and the non-linear part of the relation satisfies the non-linear part of the equation.(B) below is the solution.

Analogy for the Identity Checking Method: If one goes shopping with American dollars and Japanese yens (without any currency conversion) and after shopping, if one wants to check the cost of the items purchased, one would check the cost of the items purchased with dollars against the receipts for the dollars; and one would also check the cost of the items purchased with yens against the receipts for the yens purchase. However, if one converts one currency to the other, one would only have to check the receipts for only a single currency, dollars or yens. This conversion case is similar to the linearized equations, where there was no partitioning in identity checking. Note that for the Euler equations, there was no partitioning in taking derivatives for identity checking.

Note: After expressing $\frac{ng_xy}{V_y}$ and $\frac{q_eg_xz}{V_z}$ in terms of x, y, z, and t, there would be no partitioning in identity checking.

Summary of solutions for $V_x \quad V_y, V_z \quad (\; P(x) = d\rho g_x x; \; P(y) = \lambda_4 \rho g_y y \; , \; P(z) = \beta_4 \rho g_z z)$

$$\boxed{\begin{array}{l} V_x = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt \pm \sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+C_9 \\ P(x) = d\rho g_x x; \quad (a+b+c+d+h+n+q=1) \quad V_y \neq 0, \; V_z \neq 0 \end{array}}$$

$$\boxed{V_y = -\frac{\rho g_y}{2\mu}(\lambda_1x^2+\lambda_2y^2+\lambda_3z^2)+C_{10}x+C_{11}y+C_{12}z+\lambda_5g_yt \pm \sqrt{2\lambda_7g_y\mathrm{y}}+\frac{\lambda_6g_yx}{V_x}+\frac{\lambda_8g_y\mathrm{z}}{V_z}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_z(V_z)}{V_z}}$$

$$\boxed{V_z = -\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt \pm \sqrt{2\beta_8g_zz}+\frac{\beta_6g_zx}{V_x}+\frac{\beta_7g_zy}{V_y}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y}}$$

The above solutions are unique, because from the experience in Option 1, only the equations with the gravity terms as the subjects of the equations produced the solutions.

$$\boxed{\begin{cases} V_x = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt \pm \sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z} \\ V_y = -\frac{\rho g_y}{2\mu}(\lambda_1x^2+\lambda_2y^2+\lambda_3z^2)+C_{10}x+C_{11}y+C_{12}z+\lambda_5g_yt \pm \sqrt{2\lambda_7g_y\mathrm{y}}+\frac{\lambda_6g_yx}{V_x}+\frac{\lambda_8g_y\mathrm{z}}{V_z}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_z(V_z)}{V_z} \\ V_z = -\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt \pm \sqrt{2\beta_8g_zz}+\frac{\beta_6g_zx}{V_x}+\frac{\beta_7g_zy}{V_y}+\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y} \end{cases}}$$

One will next solve the above system of solutions for V_x , V_y, V_z in order to express $\frac{ng_xy}{V_y}$ and $\frac{q_eg_xz}{V_z}$ in terms of x, y, z, and t The author used the help of the Maples software for the simultaneous algebraic solutions for $V_y \quad V_z$. The basic expressions are of the forms $-\frac{\rho g_x}{2\mu}ax^2$, $-\frac{\rho g_x}{2\mu}by^2$,$-\frac{\rho g_x}{2\mu}cz^2$, fg_xt , $\sqrt{2hg_xx}$, and $d\rho g_x x$; These expressions are similar to the terms of the equations of motion under gravity and liquid pressure of elementary physics. Note that the explicit solutions will be the results of the basic operations (addition, subtraction, multiplication, division, power finding and root extraction) on the expressions in Step 1 below.

Solving for V_x , V_y, V_z $\frac{ng_xy}{V_y}$, and $\frac{q_eg_xz}{V_z}$

Let $V_x = x$, $V_y = y$ and $V_z = z$. (x, y and z are being used for simplicity. They will be changed back to V_x , V_y, and V_z later, and they do not represent the variables x, y and z in the solutions)

Step 1 From the above system of solutions, let	**Step 2** Then the solutions to the N-S system of equations become (ignoring the arbitrary functions)
$A = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt \pm \sqrt{2hg_xx}$ $B = -\frac{\rho g_y}{2\mu}(\lambda_1x^2+\lambda_2y^2+\lambda_3z^2)+C_{10}x+C_{11}y+C_{12}z+\lambda_5g_yt \pm \sqrt{2\lambda_7g_y\mathrm{y}}$ $C = -\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt \pm \sqrt{2\beta_8g_zz}$ $D = qg_xz$; $E = ng_xy$; $F = \lambda_6g_yx$ $G = \lambda_8g_y\mathrm{z}$; $J = \beta_6g_zx$; $L = \beta_7g_zy$	$\left.\begin{array}{l} x = A+\frac{D}{z}+\frac{E}{y} \\ y = B+\frac{F}{x}+\frac{G}{z} \\ z = C+\frac{J}{x}+\frac{L}{y} \end{array}\right\} \quad \mathrm{M}$

Step 3	**Step 4**
$\left.\begin{array}{ll} xyz = Ayz + Dy + Ez & (1) \\ xyz = Bxz + Fz + Gx & (2) \\ xyz = Cxy + Jy + Lx & (3)) \end{array}\right\}$ N	$\left.\begin{array}{ll} 0 = Ayz + Dy + Ez - Bxz - Fz - Gx & (4) \\ 0 = Ayz + Dy + Ez - Cxy - Jy - Lx & (5) \\ 0 = Bxz + Fz + Gx - Cxy - Jy + -Lx & (6) \end{array}\right\}$ P

Maples software was used to solve system P to obtain

Step 5	**Note**:
$x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ $V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ (back to V_x) $y = -\frac{L}{C}$; $V_y = -\frac{L}{C}$ (changing back to V_y as agreed to) $z = -\frac{L(D-J)}{LA - CE}$; $V_z = -\frac{L(D-J)}{LA - CE}$ (changing back to V_z as agrred to)	None of the popular academic programs could solve the system in M. Maples solved system P (step 4 above) for x, y,and z in terms of A, B, C, D. E. F, G. J. and L. Note also that x, y and z are not the same as the x, y and z in the system of equations.. They were used for convenience and simplicity .

Step 5: Apply the following and substitute for A, B, C, D. E. F, G., J. and L in steps 6-8 below

$$A = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_xt \pm \sqrt{2hg_xx}$$

$$B = -\frac{\rho g_y}{2\mu}(\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5g_yt \pm \sqrt{2\lambda_7g_y\mathrm{y}}$$

$$C = -\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5g_zt \pm \sqrt{2\beta_8g_zz}$$

$$D = qg_xz\ ;\ E = ng_xy;\ \ F = \lambda_6g_yx$$

$$G = \lambda_8g_y\mathrm{z};\ \ J = \beta_6g_zx;\ \ L = \beta_7g_zy$$

Step 6

$$V_y = -\frac{L}{C} = -\frac{(\beta_7g_zy)}{(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_zt \pm \sqrt{2\beta_8g_zz}}$$

$$\frac{ng_xy}{V_y} = ng_xy \div -(\frac{(\beta_7g_zy)}{(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_zt \pm \sqrt{2\beta_8g_zz}})$$

$$\frac{ng_xy}{V_y} = -\frac{(ng_xy)[(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_zt \pm \sqrt{2\beta_8g_zz})]}{\beta_7g_zy};$$

$$\frac{ng_xy}{V_y} = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_zt \pm \sqrt{2\beta_8g_zz})}{\beta_7g_z};\ \text{(cancel "y")}$$

Step 7: $V_z = -\frac{L(D-J)}{LA-CE} = \frac{JL-DL)}{LA-CE}$

$$\frac{qg_x z}{V_z} = (qg_x z)\bullet \frac{(\beta_7 g_z y)[-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-}{(\beta_7 g_z y)[\beta_6 g_z x - qg_x z]}$$

$$\frac{(ng_x y)[-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}]}{(\beta_7 g_z y)[\beta_6 g_z x - qg_x z]}$$

$$\frac{qg_x z}{V_z} = (qg_x z)\bullet \frac{(\beta_7 g_z)[-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-}{(\beta_7 g_z)[\beta_6 g_z x - qg_x z]}$$

$$\frac{(ng_x)[-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}]}{(\beta_7 g_z)[\beta_6 g_z x - qg_x z]}$$

Summary for the fractional terms of the x–direction solution

$\frac{ng_x y}{V_y}$ and $\frac{qg_x z}{V_z}$ in terms of x, y, z and t

$$\frac{ng_x y}{V_y} = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz})}{\beta_7 g_z}; \text{ (cancel "y")}$$

$$\frac{qg_x z}{V_z} = (qg_x z)\bullet \frac{(\beta_7 g_z)[-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-}{(\beta_7 g_z y)[\beta_6 g_z x - qg_x z]}$$

$$\frac{(ng_x)[-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}]}{(\beta_7 g_z)[\beta_6 g_z x - qg_x z]}$$

$$\left(CE = -(ng_x y)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}\right)$$

$$d+f+h+n+q=1; \quad \lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1; \quad \beta_4+\beta_5+\beta_6+\beta_7+\beta_8=1$$

Expanded N-S Solutions (in explicit solutions)

$$V_x(x,y,z,t) = A+B+F$$

$$A = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}+C_9$$

$$B = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz})}{\beta_7 g_z} + \frac{\psi_y(V_y)}{V_y}$$

$$F = (qg_x z)\bullet \frac{(\beta_7 g_z)[-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-}{(\beta_7 g_z y)[\beta_6 g_z x - qg_x z]}$$

$$\frac{(ng_x)[-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}]}{(\beta_7 g_z)[\beta_6 g_z x - qg_x z]} + \frac{\psi_z(V_z)}{V_z}$$

$$P(x) = d\rho g_x x; \quad (a+b+c+d+h+n+q=1) \quad \beta_1+\beta_2+\beta_3+\beta_5+\beta_6+\beta_7+\beta_8=1$$

Analysis of N-S Solutions

x–direction solution

$$\boxed{\begin{array}{l} V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_xt \pm \sqrt{2hg_xx} + \frac{ng_xy}{V_y} + \frac{qg_xz}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9 \\ P(x) = d\rho g_x x; \quad (a+b+c+d+h+n+q=1) \quad V_y \neq 0,\ V_z \neq 0 \end{array}}$$

$$\frac{ng_xy}{V_y} = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_zt \pm\sqrt{2\beta_8g_zz})}{\beta_7g_z}$$

$$\frac{qg_xz}{V_z} = \frac{-(qg_xz)\{[(\beta_7g_zy)(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_xt \pm \sqrt{2hg_xx}] - [CE]\}}{(\beta_7g_zy)(qg_xz - \beta_6g_zx)}$$

$$(CE = -(ng_xy)(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5g_zt \pm\sqrt{2\beta_8g_zz})$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g_x, were zero, the first three terms, the seventh, the eighth, the ninth, the tenth terms of the velocity solution and $P(x)$ would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

More Observations Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

Motion equations of elementary physics:

(B): $V_f = V_0 + gt$; (C): $V_f^2 = V_0^2 + 2gx$; (D): $V = \sqrt{2gx}$; (E): $x = V_0t + \frac{1}{2}gt^2$

Liquid Pressure,

The liquid pressure, P at the bottom of a liquid of depth h units is given by $P = \rho gh$

Observe the following above:

1. Observe that the first three terms are parabolic in x, y, and z; the minus sign showing the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal. Also note the " gt " in $V = gt$ of (B) of the motion equations and the fg_xt in the Navier-Stokes solution.

2. Observe the $P = \rho gh$ of the liquid pressure and the $P(x) = d\rho g_x x$ of the Navier-Stokes solution. Note that d is a ratio term.

3. Observe the " $\sqrt{2gx}$ " in $V = \sqrt{2gx}$ of (D) and the $\sqrt{2hg_xx}$ in the Navier-Stokes solution.

There are eight main terms (ignoring the arbitrary functions) in the Navier-Stokes solution. Of these eight terms, six terms, namely, $-\frac{a\rho g_x}{2\mu}x^2$, $-\frac{b\rho g_x}{2\mu}y^2$, $-\frac{c\rho g_x}{2\mu}z^2$, fg_xt, $\sqrt{2hg_xx}$ and $d\rho g_x x$ are similar (except for the constants involved) to the terms in the equations of motion. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.

Velocity Profile, Polynomial and Radical Parabolas, Laminar and Turbulent flow

For communication purposes, each of the terms containing the even powers x^2, y^2 and z^2 will be called a polynomial parabola, and each of the terms containing the square roots $\pm\sqrt{x}$, $\pm\sqrt{y}$ and $\pm\sqrt{z}$ will be called a radical parabola. For each polynomial parabola, the axis of symmetry is in the direction of fluid flow; but for each radical parabola, the axis of symmetry is at right angles to the direction of fluid flow.

The fluid flow in the Navier-Stokes solution may be characterized as follows. The x–direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize polynomial parabolas. The characteristic curve for the integral of the x–nonlinear term is a radical parabola. The integral of the y–nonlinear term is similar parabolically to that of the x–nonlinear term. The integral of the z–nonlinear term is a combination of two radical parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.

In the N-S solution, during fluid flow, both the polynomial and radical parabolas are present at any speed. The polynomial parabolas are prominent and dominate flow while the radical parabolas are dormant at low speeds. At a low speed, a radical parabola (or a polynomial parabola susceptible to radicalization).is not active, since the radicand of the parabola is small and consequently, the root is small. When the speed becomes large, the "x" in $\sqrt{2hg_x x}$ becomes large and therefore the radical parabola becomes active. One can also observe how gravity interacts with the "x" of the radicand. By "g" and "x" being factors of the radicand (instead of "g" being outside the radical), "g" is closely aligned with x. Note that the radical parabola will be moving at right angles to the direction of fluid flow, the direction of which is also that of the axis of symmetry of the dominating polynomial parabola. Consequently, the flow profile becomes relatively more uniform or flattened due to the radical parabola moving at right angles to the direction of fluid flow. When viscosity increases, speed decreases, and the radicand (the factor x in $\sqrt{2hg_x x}$ decreases) of the radical parabola decreases. Consequently, the disruptive behavior of the radical parabola diminishes. When the fluid flows over an obstacle, the radical parabolas temporarily become significant resulting in turbulence. For a low value of x (i.e., from low fluid velocity), the viscous term dominates and the inertial term is not significant. At high fluid velocity, the factor "x" of the radicand is large. Also when density increases, velocity increases and the radicand increases, adding to the effect of the radical parabola.

Analogy:

Imagine a crowded marathon race involving one thousand runners at various positions on the race route, all running in the same direction. Imagine also that at certain points on the route, during the race, some of the runners at various positions suddenly begin to run to the left or to the right in directions at right angles to the direction of the race route; and imagine the resulting collisions and chaos. The polynomial parabolas are those runners following the route of the race, and the radical parabolas are those runners making ninety-degree turns from various positions. Literally, the radical parabolas disrupt the laminar flow.

Uniqueness of the solution of the Navier-Stokes equation

When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. Similarly, the solution of the Navier-Stokes equation solution is unique.

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Option 5
Solutions of 4-D Navier-Stokes Equations

In the above method, the solution can easily be extended to any number of dimensions..

Adding $\mu\frac{\partial^2 V_x}{\partial s^2}$ and $\rho V_s\frac{\partial V_x}{\partial s}$ to the 3-D x–direction equation yields the 4-D N-S equation

$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial^2 V_x}{\partial s^2}\right)+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho V_s\frac{\partial V_x}{\partial s}=\rho g_x$$

whose solution is given by

$$\boxed{\begin{array}{l} V_x(x,y,z,s,t)= \\ -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2+es^2)+C_1x+C_3y+C_5z+C_6s+fg_xt\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{rg_xs}{V_s}+ \\ \qquad\qquad\qquad\qquad \underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}+\frac{\psi_s(V_s)}{V_s}}_{\text{arbitrary functions}}+C_9 \\ P(x)=d\rho g_xx \qquad (a+b+c+d+e+f+h+n+q+r=1) \qquad V_x\neq 0,\ V_y\neq 0,\ V_s\neq 0, \end{array}}$$

For n–dimensions one can repeat the above as many times as one wishes.

Option 5b
Two-term Linearized Navier-Stokes Equation (one nonlinear term)

By linearization as in Option 1, if one replaces $\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}$ by $2\rho\frac{\partial V_x}{\partial t}$ in

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x$$ one obtains

$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right)+\frac{\partial p}{\partial x}+3\rho\left(\frac{\partial V_x}{\partial t}\right)+\rho V_x\frac{\partial V_x}{\partial x}=\rho g_x$$, whose solution is

$$V_x(x,y,z,t)=-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\frac{fg_xt}{3}\pm\sqrt{2hg_xx}+C_6$$

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Conclusion (for Option 4)

One will begin from the general case and end with the special cases.

Solutions of the Navier--Stokes equations (general case)

x–direction **Navier-Stokes Equation** (also driver equation)

$$\boxed{-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+\rho\frac{\partial V_x}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}=\rho g_x} \quad x\text{–direction}$$

$$\boxed{\begin{aligned}&V_x(x,y,z,t)=\\&\overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt}^{\text{solution for linear terms}}\pm\overbrace{\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}}^{\text{solution for non - linear terms}}+C_9\\&P(x)=d\rho g_xx;\quad (a+b+c+d+h+n+q=1)\quad V_y\neq 0,\ V_z\neq 0\end{aligned}}$$

$$\frac{ng_xy}{V_y}=\frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_1x+C_3y+C_5z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz})}{\beta_7g_z}$$

$$\frac{qg_xz}{V_z}=\frac{-(qg_xz)\{[(\beta_7g_zy)(-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt\pm\sqrt{2hg_xx}]-[CE]\}}{(\beta_7g_zy)(qg_xz-\beta_6g_zx)}$$

$$CE=-(ng_xy)(-\frac{\rho g_z}{2\mu}(\beta_1x^2+\beta_2y^2+\beta_3z^2)+C_{14}x+C_{15}y+C_{16}z+\beta_5g_zt\pm\sqrt{2\beta_8g_zz}$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity, g_x, were zero, the first three terms, the 7th term, the 8th term, the 9th term, the 10th term and $P(x)$ would all be zero.
This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and also each term utilizes gravity to function. This law was applied in splitting-up the Navier-Stokes equations. The resulting sub-equations were readily integrable, and even the nonlinear sub-equations were readily integrated.
The x–direction Navier-Stokes equation was split-up into sub-equations using ratios. The sub-equations were solved and combined. The relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation and the relation obtained from integrating the nonlinear part of the equation satisfied the nonlinear part of the equation. By solving algebraically and simultaneously for V_x, V_y and V_z, the (ng_xy/V_y) and (qg_xz/V_z) terms woe expressed explicitly in terms of x, y, z and t. The above x–direction solution is the solution everyone has been waiting for, for nearly 150 years. It was obtained in two simple steps, namely, splitting the equation using ratios and integrating.

Special Cases of the Navier-Stokes Equations

1. Linearized Navier-Stokes equations

One may note that there are six linear terms and three nonlinear terms in the Navier-Stokes equation. The linearized case was covered before the general case, and the experience gained in the linearized case guided one to solve the general case efficiently. In particular, the gravity term must be the subject of the equation for a solution. When the gravity term was the subject of the equation, the equation was called the driver equation. A splitting technique was applied to the linearized Navier-Stokes equations (Option 1). Twenty sub-equations were solved. (Four sets of equations with different equation subjects). The integration relations of one of the sets satisfied the linearized Navier-Stokes equation; and this set was from the equation with g_x as the subject of the equation. In addition to finding a solution, the results of the integration revealed the roles of the terms of the Navier-Stokes equations in fluid flow. In particular, the gravity forces and $\partial p/\partial x$ are involved mainly in the parabolic as well as the forward motion of fluids; $\partial V_x/\partial t$ and $\partial^2 V_x/\partial x^2$ are involved in the periodic motion of fluids, and one may infer that as μ increases, the periodicity increases. One should determine experimentally, if the ratio of the linear term $\partial V_x/\partial t$ to the nonlinear sum $V_x(\partial V_x/\partial x) + V_y(\partial V_x/\partial y) + V_z(\partial V_x/\partial z)$ is 1 to 3.

$$V_x(x,y,z,t) = \overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4}t + C_9}^{\textbf{Solution to the linearized Navier–Stokes equation}} \; ; \; P(x) = d\rho g_x x$$

$$\boxed{\overbrace{-\mu\frac{\partial^2 V_x}{\partial x^2} - \mu\frac{\partial^2 V_x}{\partial y^2} - \mu\frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho\frac{\partial V_x}{\partial t} = \rho g_x}^{\textbf{Linearized Equation}}}$$

2. Solutions of the Euler equation

Since one has solved the Navier-Stokes equation, one has also solved the Euler equation.

$$\text{Euler equation } (\mu = 0): \; \frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = g_x$$

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \dfrac{ng_x y}{V_y} + \dfrac{qg_x z}{V_z} + \underbrace{\dfrac{\psi_y(V_y)}{V_y} + \dfrac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C \\ P(x) = d\rho g_x x \qquad (f+h+n+q+d=1) \; V_y \neq 0, \; V_z \neq 0 \end{array}} \; x\text{-direction}$$

$$\left.\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}\right\} \text{B}$$

$$\left.\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_z g_x^2 q - \beta_5 g_x g_z nq)tz \pm \sqrt{2hg_x x}\,\beta_7 g_x g_z qz \pm \sqrt{2\beta_8 g_z z}\; g_x^2 nqz}{\beta_7\beta_6 g_z^2 x - \beta_7 q g_z g_x z}\right\} \text{C}$$

Comparison of Linearized N-S Solutions, Euler Solutions. and N-S Solutions

x- direction linearized N–S Solution

$$V_x(x,y,z,t) = -\overbrace{\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)}^{\text{axis in direction of fluid flow}} + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9\ ;\ P(x) = d\rho g_x x \quad \mathbf{L}$$

x-direction **Euler Solution**

$$\left.\begin{aligned} &V_x(x,y,z,t) = fg_x t \pm \overbrace{\sqrt{2hg_x x}}^{\text{axis rotated } 90^0} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C \\ &P(x) = d\rho g_x x \qquad (f+h+n+q+d=1)\ V_y \neq 0,\ V_z \neq 0 \end{aligned}\right\}\mathbf{E}$$

$$\left.\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}\right\}\text{B}$$

$$\left.\frac{qg_x z}{V_z} = \frac{(\beta_7 fg_z g_x^2 q - \beta_5 g_x g_z nq)tz \pm \overbrace{\sqrt{2hg_x x}}^{\text{rotated axis}}\ \beta_7 g_x g_z qz \pm \overbrace{\sqrt{2\beta_8 g_z z}}^{\text{rotated axis}}\ g_x^2 nqz}{\beta_7\beta_6 g_z^2 x - \beta_7 qg_z g_x z}\right\}\mathbf{C}$$

x- diretion N-S Solution

$$\left.\begin{aligned} &V_x = -\frac{\rho g_x}{2\mu}\overbrace{(ax^2+by^2+cz^2)}^{\text{fluid flow direction axis}} + C_1x + C_3y + C_5z + fg_x t \pm \overbrace{\sqrt{2hg_x x}}^{\text{rotated axis}} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9 \\ &P(x) = d\rho g_x x;\quad (a+b+c+d+h+n+q=1)\quad V_y \neq 0,\ V_z \neq 0 \end{aligned}\right\}\mathbf{S}$$

$$\frac{ng_x y}{V_y} = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}\overbrace{(\beta_1x^2+\beta_2y^2+\beta_3z^2)}^{\text{axis in direction of fluid flow}} + C_1x + C_3y + C_5z + \beta_5 g_z t \pm \overbrace{\sqrt{2\beta_8 g_z z}}^{\text{rotated axis}})}{\beta_7 g_z} \quad \mathbf{P}$$

$$\frac{qg_x z}{V_z} = \quad \mathbf{T}$$

$$\frac{-(qg_x z)\{-\frac{\rho g_x}{2\mu}[(\beta_7 g_z a - ng_x\beta_1)x^2 - (\beta_7 g_z b - ng_x\beta_2)y^2 - (\beta_7 g_z cz^2 - ng_x\beta_3)z^2] + (\beta_7 g_z C_1 - ng_x C_{14})x + (\beta_7 g_z C_3 - ng_x C_{15})y + (\beta_7 g_z C_5 - ng_x C_{16})z + (\beta_7 g_z fg_x - ng_x\beta_5 g_z)t \pm \sqrt{2hg_x x}\,\beta_7 g_z \mp \sqrt{2\beta_8 g_z z}\,ng_x\}}{(\beta_7 g_z)(qg_x z - \beta_6 g_z x)}$$

Back to Options

Option 6
Solutions of 3-D Navier-Stokes Equations
(Method 2)

Here, the three equations below, will be added together; and a single equation will be integrated

$$\begin{cases} -\mu(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2})+\frac{\partial p}{\partial x}+\rho(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z})=\rho g_x \quad (1) \\ -\mu(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2})+\frac{\partial p}{\partial y}+\rho(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z})=\rho g_y \quad (2) \\ -\mu(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2})+\frac{\partial p}{\partial z}+\rho(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z})=\rho g_z \quad (3) \end{cases}$$

Step 1: Apply the axiom, if $a=b$ and $c=d$, then $a+c=b+d$; and therefore, add the left sides and add the right sides of the above equations . That is, $(1)+(2)+(3)=\rho g_x+\rho g_y+\rho g_z$

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}-\mu\frac{\partial^2 V_y}{\partial x^2}-\mu\frac{\partial^2 V_y}{\partial y^2}-\mu\frac{\partial^2 V_y}{\partial z^2}-\mu\frac{\partial^2 V_z}{\partial x^2}-\mu\frac{\partial^2 V_z}{\partial y^2}-\mu\frac{\partial^2 V_z}{\partial z^2}+\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}$$
$$+\frac{\partial p}{\partial z}+\rho\frac{\partial V_x}{\partial t}+\rho\frac{\partial V_y}{\partial t}+\rho\frac{\partial V_z}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}+\rho V_x\frac{\partial V_y}{\partial x}+\rho V_y\frac{\partial V_y}{\partial y}+\rho V_z\frac{\partial V_y}{\partial z}$$
$$+\rho V_x\frac{\partial V_z}{\partial x}+\rho V_y\frac{\partial V_z}{\partial y}+\rho V_z\frac{\partial V_z}{\partial z}=(\rho g_x+\rho g_y+\rho g_z)$$

(Three lines per equation)

Let $\rho g_x+\rho g_y+\rho g_z=\rho G$ **,** where $G=\left|g_x+g_y+g_z\right|$ to obtain

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}-\mu\frac{\partial^2 V_y}{\partial x^2}-\mu\frac{\partial^2 V_y}{\partial y^2}-\mu\frac{\partial^2 V_y}{\partial z^2}-\mu\frac{\partial^2 V_z}{\partial x^2}-\mu\frac{\partial^2 V_z}{\partial y^2}-\mu\frac{\partial^2 V_z}{\partial z^2}$$
$$+\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}+\frac{\partial p}{\partial z}++\rho\frac{\partial V_x}{\partial t}+\rho\frac{\partial V_y}{\partial t}+\rho\frac{\partial V_z}{\partial t}+\rho V_x\frac{\partial V_x}{\partial x}+\rho V_y\frac{\partial V_x}{\partial y}+\rho V_z\frac{\partial V_x}{\partial z}$$
$$+\rho V_x\frac{\partial V_y}{\partial x}+\rho V_y\frac{\partial V_y}{\partial y}+\rho V_z\frac{\partial V_y}{\partial z}+\rho V_x\frac{\partial V_z}{\partial x}+\rho V_y\frac{\partial V_z}{\partial y}+\rho V_z\frac{\partial V_z}{\partial z}=\rho G$$

Step 2: Solve the above 25-term equation using the ratio method. (24 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

a ,b, c, d, f, m, n, q, r, β_1, β_2, β_3, β_4, β_5, β_6, λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8, λ_9

$$-\mu\frac{\partial^2 V_x}{\partial x^2}=a\rho G\ ; \qquad -\mu\frac{\partial^2 V_x}{\partial y^2}=b\rho G\ ; \qquad -\mu\frac{\partial^2 V_x}{\partial z^2}=c\rho G\ ; \qquad -\mu\frac{\partial^2 V_y}{\partial x^2}=d\rho G\ ;$$

$$-\mu\frac{\partial^2 V_y}{\partial y^2}=f\rho G\ ; \qquad -\mu\frac{\partial^2 V_y}{\partial z^2}=h\rho G\ ; \qquad -\mu\frac{\partial^2 V_z}{\partial x^2}=m\rho G\ ; \qquad -\mu\frac{\partial^2 V_z}{\partial y^2}=n\rho G\ ;$$

$$-\mu\frac{\partial^2 V_z}{\partial z^2}=r\rho G\ ; \qquad \frac{\partial p}{\partial x}=\beta_1\rho G; \qquad \frac{\partial p}{\partial y}=\beta_2\rho G\ ; \qquad \frac{\partial p}{\partial z}=\beta_3\rho G\ ;$$

$$\rho\frac{\partial V_x}{\partial t}=\beta_4\rho G; \qquad \rho\frac{\partial V_y}{\partial t}=\beta_5\rho G\ ; \qquad \rho\frac{\partial V_z}{\partial t}=\beta_6\rho G\ ; \qquad \rho V_x\frac{\partial V_x}{\partial x}=\lambda_1\rho G\ ;$$

$$\rho V_y\frac{\partial V_x}{\partial y}=\lambda_2\rho G\ ; \qquad \rho V_z\frac{\partial V_x}{\partial z}=\lambda_3\rho G\ ; \qquad \rho V_x\frac{\partial V_y}{\partial x}=\lambda_4\rho G\ ; \qquad \rho V_y\frac{\partial V_y}{\partial y}=\lambda_5\rho G\ ;$$

$$\rho V_z\frac{\partial V_y}{\partial z}=\lambda_6\rho G\ ; \qquad \rho V_x\frac{\partial V_z}{\partial x}=\lambda_7\rho G\ ; \qquad \rho V_y\frac{\partial V_z}{\partial y}=\lambda_8\rho G\ ; \qquad \rho V_z\frac{\partial V_z}{\partial z}=\lambda_9\rho G$$

1

$$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{\mu}\rho G$$
$$\frac{\partial V_x}{\partial x} = -\frac{a}{\mu}\rho G x + C_1$$
$$V_x = -\frac{a}{\mu}\rho G\frac{x^2}{2} + C_1 x + C_2$$

2

$$-\mu\frac{\partial^2 V_x}{\partial y^2} = b\rho G$$
$$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{\mu}\rho G$$
$$\frac{\partial V_x}{\partial y} = -\frac{b}{\mu}\rho G y + C_3$$
$$V_x = -\frac{b}{\mu}\rho G\frac{y^2}{2} + C_3 y + C_4$$

3

$$-\mu\frac{\partial^2 V_x}{\partial z^2} = c\rho G$$
$$-\mu\frac{\partial^2 V_x}{\partial z^2} = c\rho G$$
$$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{\mu}\rho G$$
$$\frac{\partial V_x}{\partial z} = -\frac{c}{\mu}\rho G z + C_5$$
$$V_x = -\frac{c}{\mu}\rho G\frac{z^2}{2} + C_5 z + C_6$$

4

$$-\mu\frac{\partial^2 V_y}{\partial x^2} = d\rho G$$
$$-\mu\frac{\partial^2 V_y}{\partial x^2} = d\rho G$$
$$\frac{\partial^2 V_y}{\partial x^2} = -\frac{d}{\mu}\rho G$$
$$\frac{\partial V_y}{\partial x} = -\frac{d}{\mu}\rho G x + C_7$$
$$V_y = -\frac{d}{\mu}\rho G\frac{x^2}{2} + C_7 x + C_8$$

5

$$-\mu\frac{\partial^2 V_y}{\partial y^2} = f\rho G$$
$$\frac{\partial^2 V_y}{\partial y^2} = -\frac{f}{\mu}\rho G$$
$$\frac{\partial V_y}{\partial y} = -\frac{f}{\mu}\rho G y + C_9$$
$$V_y = -\frac{f}{\mu}\rho G\frac{y^2}{2} + C_9 y + C_{10}$$

6

$$-\mu\frac{\partial^2 V_y}{\partial z^2} = h\rho G$$
$$\frac{\partial^2 V_y}{\partial z^2} = -\frac{h}{\mu}\rho G$$
$$\frac{\partial V_y}{\partial z} = -\frac{h}{\mu}\rho G z + C_{11}$$
$$V_y = -\frac{h}{\mu}\rho G\frac{z^2}{2} + C_{11} z + C_{12}$$

7

$$-\mu\frac{\partial^2 V_z}{\partial x^2} = m\rho G$$
$$\frac{\partial^2 V_z}{\partial x^2} = -\frac{m}{\mu}\rho G$$
$$\frac{\partial V_z}{\partial x} = -\frac{m}{\mu}\rho G x + C_{13}$$
$$V_z = -\frac{m}{\mu}\rho G\frac{x^2}{2} + C_{13} x + C_{14}$$

8

$$-\mu\frac{\partial^2 V_z}{\partial y^2} = n\rho G$$
$$\frac{\partial^2 V_z}{\partial y^2} = -\frac{n}{\mu}\rho G$$
$$\frac{\partial V_z}{\partial y} = -\frac{n}{\mu}\rho G y + C_{15}$$
$$V_z = -\frac{m}{\mu}\rho G\frac{y^2}{2} + C_{15} y + C_{16}$$

9

$$-\mu\frac{\partial^2 V_z}{\partial z^2} = r\rho G$$
$$\frac{\partial^2 V_z}{\partial z^2} = -\frac{r}{\mu}\rho G$$
$$\frac{\partial V_z}{\partial z} = -\frac{r}{\mu}\rho G z + C_{17}$$
$$V_z = -\frac{r}{\mu}\rho G\frac{z^2}{2} + C_{17} z + C_{18}$$

10

$$\frac{\partial p}{\partial x} = \beta_1\rho G$$
$$\frac{dp}{dx} = \beta_1\rho G$$
$$P(x) = \beta_1\rho G x + C_{19}$$

11

$$\frac{\partial p}{\partial y} = \beta_2\rho G$$
$$\frac{dp}{dy} = \beta_2\rho G$$
$$P(y) = \beta_2\rho G y + C_{20}$$

12

$$\frac{\partial p}{\partial z} = \beta_3\rho G$$
$$\frac{dp}{dz} = \beta_3\rho G$$
$$P(z) = \beta_3\rho G z + C_{21}$$

13 $\rho\frac{\partial V_x}{\partial t}=\beta_4\rho G$ $\frac{dV_x}{dt}=\beta_4 G$ $V_x=\beta_4 Gt+C_{22}$	**14** $\rho\frac{\partial V_y}{\partial t}=\beta_5\rho G$ $\frac{dV_y}{dt}=\beta_5 G$ $V_y=\beta_5 Gt+C_{23}$	**15** $\rho\frac{\partial V_z}{\partial t}=\beta_6\rho G$ $\frac{dV_z}{dt}=\beta_6 G$ $V_z=\beta_6 Gt+C_{24}$	**16** $\rho V_x\frac{\partial V_x}{\partial x}=\lambda_1\rho G$ $V_x\frac{\partial V_x}{\partial x}=\lambda_1 G$ $V_x\frac{dV_x}{dx}=\lambda_1 G$ $V_x dV_x=\lambda_1 G\,dx$ $\frac{V^2{}_x}{2}=\lambda_1 G\,x$ $V^2{}_x=2\lambda_1 G\,x$ $V_x=\pm\sqrt{2\lambda_1 G\,x}+C_{25}$
17 $\rho V_y\frac{\partial V_x}{\partial y}=\lambda_2\rho G$ $V_y\frac{dV_x}{dy}=\lambda_2 G$ $V_y dV_x=\lambda_2 G\,dy$ $V_y V_x=\lambda_2 G\,y+\psi_y(V_y)$ $V_x=\frac{\lambda_2 G\,y}{V_y}+\frac{\psi_y(V_y)}{V_y}$	**18** $\rho V_z\frac{\partial V_x}{\partial z}=\lambda_3\rho G$ $V_z\frac{dV_x}{dz}=\lambda_3 G$ $V_z dV_x=\lambda_3 G\,dz$ $V_z V_x=\lambda_3 G\,z+\psi_z(V_z)$ $V_x=\frac{\lambda_3 G\,z}{V_z}+\frac{\psi_z(V_z)}{V_z}$	**19** $\rho V_x\frac{\partial V_y}{\partial x}=\lambda_4\rho G$ $V_x\frac{dV_y}{dx}=\lambda_4 G$ $V_x dV_y=\lambda_4 G\,dx$ $V_x V_y=\lambda_4 G\,x+\psi_x(V_x)$ $V_y=\frac{\lambda_4 G\,x}{V_x}+\frac{\psi_x(V_x)}{V_x}$	**20** $\rho V_y\frac{\partial V_y}{\partial y}=\lambda_5\rho G$ $V_y\frac{dV_y}{dy}=\lambda_5 G$ $V_y dV_y=\lambda_5 G\,dy$ $\frac{V_y^{\,2}}{2}=\lambda_5 G\,y$ $V_y^{\,2}=2\lambda_5 G\,y$ $V_y=\pm\sqrt{2\lambda_5 G\,y}+C_{26}$
21 $\rho V_z\frac{\partial V_y}{\partial z}=\lambda_6\rho G$ $V_z\frac{dV_y}{dz}=\lambda_6 G$ $V_z dV_y=\lambda_6 G\,dz$ $V_z V_y=\lambda_6 G\,z+\psi_z(V_z)$ $V_y=\frac{\lambda_6 G\,z}{V_z}+\frac{\psi_z(V_z)}{V_z}$	**22** $\rho V_x\frac{\partial V_z}{\partial x}=\lambda_7\rho G$ $V_x\frac{dV_z}{dx}=\lambda_7 G$ $V_x dV_z=\lambda_7 G\,dx$ $V_x V_z=\lambda_7 G\,x+\psi_x(V_x)$ $V_z=\frac{\lambda_7 G\,x}{V_x}+\frac{\psi_x(V_x)}{V_x}$	**23** $\rho V_y\frac{\partial V_z}{\partial y}=\lambda_8\rho G$ $V_y\frac{dV_z}{dy}=\lambda_8 G$ $V_y dV_z=\lambda_8 G\,dy$ $V_y V_z=\lambda_8 G\,y+\psi_y(V_y)$ $V_z=\frac{\lambda_8 G\,y}{V_y}+\frac{\psi_y(V_y)}{V_y}$	**24** $\rho V_z\frac{\partial V_z}{\partial z}=\lambda_9\rho G$ $V_z\frac{dV_z}{dz}=\lambda_9 G$ $V_z dV_z=\lambda_9 G\,dz$ $\frac{V_z^{\,2}}{2}=\lambda_9 G\,z$ $V_z^{\,2}=2\lambda_9 G\,z$ $V_z=\pm\sqrt{2\lambda_9 G\,z}+C_{27}$

Step 3 : One Collects the integrals of the sub-equations, above, for $V_x, V_y, V_z, P(x), P(y), P(z)$

For V_x, $P(x)$	For V_y, $P(y)$	For V_z, $P(z)$
Sum of integrals from sub-equations #1, #2, #3, #13, #16, #17, #18, #10	**S**um of integrals from sub-equations #4, #5, #6, #14, #19, #20, #21,#11	**S**um of integrals from sub-equations #7, #8, #9, #15, #22, #23, #24, #12,
$V_x=-\frac{a}{\mu}\rho G\frac{x^2}{2}+C_1x+C_2$	$V_y=-\frac{d}{\mu}\rho G\frac{x^2}{2}+C_7x+C_8$	$V_z=-\frac{m}{\mu}\rho G\frac{x^2}{2}+C_{13}x+C_{14}$
$V_x=-\frac{b}{\mu}\rho G\frac{y^2}{2}+C_3y+C_4$	$V_y=-\frac{f}{\mu}\rho G\frac{y^2}{2}+C_9y+C_{10}$	$V_z=-\frac{m}{\mu}\rho G\frac{y^2}{2}+C_{15}y+C_{16}$
$V_x=-\frac{c}{\mu}\rho G\frac{z^2}{2}+C_5z+C_6$	$V_y=-\frac{h}{\mu}\rho G\frac{z^2}{2}+C_{11}z+C_{12}$	$V_z=-\frac{r}{\mu}\rho G\frac{z^2}{2}+C_{17}z+C_{18}$
$V_x=\beta_4Gt+C_{22}$	$V_y=\beta_5Gt+C_{21}$	$V_z=\beta_6Gt+C_{24}$
$V_x=\pm\sqrt{2\lambda_1G\,x}+C_{25}$	$V_y=\frac{\lambda_4G\,x}{V_x}+\frac{\psi_x(V_x)}{V_x}$	$V_z=\frac{\lambda_7G\,x}{V_x}+\frac{\psi_x(V_x)}{V_x}$
$V_x=\frac{\lambda_2G\,y}{V_y}+\frac{\psi_y(V_y)}{V_y}$	$V_y=\pm\sqrt{2\lambda_5G\,y}+C_{26}$	# $V_z=\frac{\lambda_8G\,y}{V_y}+\frac{\psi_y(V_y)}{V_y}$
$V_x=\frac{\lambda_3G\,z}{V_z}+\frac{\psi_z(V_z)}{V_z}$	$V_y=\frac{\lambda_6G\,z}{V_z}+\frac{\psi_z(V_z)}{V_z}$	$V_z=\pm\sqrt{2\lambda_9G\,z}+C_{27}$
$P(x)=\beta_1\rho Gx+C_{19}$	$P(y)=\beta_2\rho Gy+C_{20}$	$P(z)=\beta_3\rho Gz+C_{21}$

From above**,**

For V_x, Sum of integrals from sub-equations #1, #2, #3, #13, #16, #17, #18, #10

$V_x(x,y,z,t)$

$$=-\frac{a}{\mu}\rho G\frac{x^2}{2}+C_1x-\frac{b}{\mu}\rho G\frac{y^2}{2}+C_3y-\frac{c}{\mu}\rho G\frac{z^2}{2}+C_5z+\beta_4Gt\pm\sqrt{2\lambda_1G\,x}+\frac{\lambda_2G\,y}{V_y}+\frac{\lambda_3G\,z}{V_z}$$

$$\underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$P(x)=\beta_1\rho Gx+C_{19}$$

For V_y : Sum of integrals from sub-equations #4, #5, #6,#14, #19, #20, #21,#11

$V_y(x,y,z,t)$

$$=-\frac{d}{\mu}\rho G\frac{x^2}{2}+C_7x-\frac{f}{\mu}\rho G\frac{y^2}{2}+C_9y-\frac{h}{\mu}\rho G\frac{z^2}{2}+C_{11}z+\beta_5Gt+\pm\sqrt{2\lambda_5G\,y}+\frac{\lambda_4G\,x}{V_x}+\frac{\lambda_6G\,z}{V_z}$$

$$+\underbrace{\frac{\psi_x(V_x)}{V_x}\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$P(y)=\beta_2\rho Gy+C_{20}$$

For V_z: Sum of integrals from sub-equations #7, #8, #9,#15, #22, #23, #24, #12,

$$V_z=-\frac{m}{\mu}\rho G\frac{x^2}{2}+C_{13}x-\frac{n}{\mu}\rho G\frac{y^2}{2}+C_{15}y-\frac{r}{\mu}\rho G\frac{z^2}{2}+C_{17}z+\beta_6Gt\pm\sqrt{2\lambda_9G\,z}+\frac{\lambda_7G\,x}{V_x}$$

$$+\frac{\lambda_8G\,y}{V_y}+\underbrace{\frac{\psi_x(V_x)}{V_x}+\frac{\psi_y(V_y)}{V_y}}_{\text{arbitrary functions}}$$

$$P(z)=\beta_3\rho Gz+C_{21}$$

Step 4: Simplify the sums of the integrals from above..(**Method 2 solutions of N-S equations**

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = -\dfrac{\rho G}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\beta_4 Gt \pm \sqrt{2\lambda_1 G\,x}+\dfrac{\lambda_2 G\,y}{V_y}+\dfrac{\lambda_3 G\,z}{V_z} \\ P(x)=\beta_1\rho Gx+C_{19} \qquad (V_y\neq 0,\ V_z\neq 0) \qquad +\underbrace{\dfrac{\psi_y(V_y)}{V_y}+\dfrac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \end{array}}$$

$$\boxed{\begin{array}{l} V_y(x,y,z,t) = -\dfrac{\rho G}{2\mu}(dx^2+fy^2+hz^2)+C_7x+C_9y+C_{11}z+C_{10}\beta_5 Gt \pm \sqrt{2\lambda_5 G\,y}+\dfrac{\lambda_4 G\,x}{V_x}+\dfrac{\lambda_6 G\,z}{V_z} \\ P(y)=\beta_2\rho Gy+C_{20} \qquad (V_x\neq 0,\ V_z\neq 0) \qquad +\underbrace{\dfrac{\psi_x(V_x)}{V_x}+\dfrac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \end{array}}$$

$$\boxed{\begin{array}{l} V_z(x,y,z,t) = -\dfrac{\rho G}{2\mu}(mx^2+ny^2+rz^2)+C_{13}x+C_{15}y+C_{17}z+\beta_6 Gt \pm \sqrt{2\lambda_9 G\,z}+\dfrac{\lambda_7 G\,x}{V_x}+\dfrac{\lambda_8 G\,y}{V_y} \\ P(z)=\beta_3\rho Gz+C_{21} \qquad (V_y\neq 0,\ V_y\neq 0) \qquad +\underbrace{\dfrac{\psi_x(V_x)}{V_x}+\dfrac{\psi_y(V_y)}{V_y}}_{\text{arbitrary functions}} \end{array}}$$

The above are solutions for $V_x\ V_y,\ V_z\quad P(x),\ P(y),\ P(z)$ .of the Navier-Stokes Equations

Comparison of Method 1 (Option 4) and Method 2 (Option 6) of Solutions of Navier-Stokes Equations

Method 1: x–direction solution of Navier-Stokes equation

$$\boxed{\begin{aligned} &V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+fg_xt \pm \sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+ \\ &P(x) = d\rho g_xx;\ (a+b+c+d+h+n+q=1)\quad (V_y\neq 0,\ V_z\neq 0)\qquad \underbrace{+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}+C_9 \end{aligned}} \quad \text{(A)}$$

Method 2: x–direction solution of Navier-Stokes equation

$$\boxed{\begin{aligned} &V_x(x,y,z,t) = -\frac{\rho G}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\beta_4Gt \pm \sqrt{2\lambda_1 G\,x}+\frac{\lambda_2 G\,y}{V_y}+\frac{\lambda_3 G\,z}{V_z} \\ &P(x) = \beta_1\rho Gx + C_{19}\qquad (V_y\neq 0,\ V_z\neq 0)\qquad + \underbrace{\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \end{aligned}} \quad \textbf{(B)}$$

It is pleasantly surprising that the above solutions (A) and (B) are almost identical (except for the constants), even though they were obtained by different approaches as in Option 4 and Option 6. Such an agreement confirms the validity of the solution method for the system of magnetohydrodynamic equations (see viXra:1405.0251.. For the system of magnetohydrodynamic equations, there is only a single "driver" equation. For the system of N-S equations, there are three driver equations, since each equation contains the gravity term. Therefore, one was able to solve each of the three simultaneous equations separately (as in Method 1); but in addition, one obtained an identical solution (except for the constants) in solving the simultaneous N-S system by adding the three equations in the system and integrating a single driver equation. In Method 1, the gravity term was ρg . In Method 2, the gravity term was ρG, where G is the magnitude of the vector sum of the gravity terms. Note that in Method 1, the sum of the ratio terms (8 ratio terms for each equation) equals unity, but in Method 2, the sum of the ratio terms (24 ratio terms) for the single driver equation solved equals unity. Note that in Method 2, only a single "driver" equation was solved, but in Method 1, three "driver" equations were solved. In Method 2, one could say that the system of N-S equations was "more simultaneously" solved than in Method 1.

To summarize, solving the Navier-Stokes equations by the first method helped one to solve the magnetohydrodynamic equations (not presented in this paper.. See viXra:1405.0251) and solving the magnetohydrodynamic equations encouraged one to solve the Navier-Stokes equations by the second method.
(" Navier-Stokes equations "scratched the back" of magnetohydrodynamic equations; and in return, magnetohydrodynamic equations "scratched the back" of Navier-Stokes equations")

About integrating only a single equation
If one asked for help in solving the N-S equations, and one was told to add the three equations together and then solve them, one would think that one was being given a nonsensical advice; but now, after studying the above Option 6 method, one would appreciate such a suggestion.

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Option 7
Solutions of 3-D Linearized Navier-Stokes Equations
Method 2

Here, the three equations below, will be added together; and a single equation will be integrated.

$$\begin{cases} -\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+4\rho\frac{\partial V_x}{\partial t}=\rho g_x \quad (1)\\ -\mu\frac{\partial^2 V_y}{\partial x^2}-\mu\frac{\partial^2 V_y}{\partial y^2}-\mu\frac{\partial^2 V_y}{\partial z^2}+\frac{\partial p}{\partial y}+4\rho\frac{\partial V_y}{\partial t}=\rho g_y \quad (2)\\ -\mu\frac{\partial^2 V_z}{\partial x^2}-\mu\frac{\partial^2 V_z}{\partial y^2}-\mu\frac{\partial^2 V_z}{\partial z^2}+\frac{\partial p}{\partial z}+4\rho\frac{\partial V_z}{\partial t}=\rho g_z \quad (3) \end{cases}$$

Step 1: Apply the axiom, if $a=b$ and $c=d$, then $a+c=b+d$; and therefore, add the left sides and add the right sides of the above equations . That is, $(1)+(2)+(3)=\rho g_x+\rho g_y+\rho g_z$

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+4\rho\frac{\partial V_x}{\partial t}-\mu\frac{\partial^2 V_y}{\partial x^2}-\mu\frac{\partial^2 V_y}{\partial y^2}-\mu\frac{\partial^2 V_y}{\partial z^2}+\frac{\partial p}{\partial y}+4\rho\frac{\partial V_y}{\partial t}$$
$$-\mu\frac{\partial^2 V_z}{\partial x^2}-\mu\frac{\partial^2 V_z}{\partial y^2}-\mu\frac{\partial^2 V_z}{\partial z^2}+\frac{\partial p}{\partial z}4\rho\frac{\partial V_z}{\partial t}=\rho g_x+\rho g_y+\rho g_z \quad \text{(Two lines per equation)}$$

Let $\rho g_x+\rho g_y+\rho g_z=\rho G$ **,** where $G=\left|g_x+g_y+g_z\right|$ to obtain

$$-\mu\frac{\partial^2 V_x}{\partial x^2}-\mu\frac{\partial^2 V_x}{\partial y^2}-\mu\frac{\partial^2 V_x}{\partial z^2}+\frac{\partial p}{\partial x}+4\rho\frac{\partial V_x}{\partial t}-\mu\frac{\partial^2 V_y}{\partial x^2}-\mu\frac{\partial^2 V_y}{\partial y^2}-\mu\frac{\partial^2 V_y}{\partial z^2}+\frac{\partial p}{\partial y}+4\rho\frac{\partial V_y}{\partial t}$$
$$-\mu\frac{\partial^2 V_z}{\partial x^2}-\mu\frac{\partial^2 V_z}{\partial y^2}-\mu\frac{\partial^2 V_z}{\partial z^2}+\frac{\partial p}{\partial z}+4\rho\frac{\partial V_z}{\partial t}=\rho G \qquad \text{(Two lines per equation)}$$

Step 2: Solve the above 15-term equation using the ratio method. (14 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

$a, b, c, d, f, h, j, m, n, q, r, s, u, v, w.$ (Sum of the ratio terms = 1)

$$-\mu\frac{\partial^2 V_x}{\partial x^2}=a\rho G;\quad -\mu\frac{\partial^2 V_x}{\partial y^2}=b\rho G;\quad -\mu\frac{\partial^2 V_x}{\partial z^2}=c\rho G;\quad \frac{\partial p}{\partial x}=d\rho G$$
$$4\rho\frac{\partial V_x}{\partial t}=f\rho G;\quad -\mu\frac{\partial^2 V_y}{\partial x^2}=h\rho G;\quad -\mu\frac{\partial^2 V_y}{\partial y^2}=j\rho G;\quad -\mu\frac{\partial^2 V_y}{\partial z^2}=m\rho G;$$
$$\frac{\partial p}{\partial y}=n\rho G;\quad 4\rho\frac{\partial V_y}{\partial t}=q\rho G;\quad -\mu\frac{\partial^2 V_z}{\partial x^2}=r\rho G;\quad -\mu\frac{\partial^2 V_z}{\partial y^2}=s\rho G;$$
$$-\mu\frac{\partial^2 V_z}{\partial z^2}=u\rho G;\quad \frac{\partial p}{\partial z}=v\rho G;\quad 4\rho\frac{\partial V_z}{\partial t}=w\rho G$$

1. $-\mu\frac{\partial^2 V_x}{\partial x^2}=a\rho G$ $\frac{\partial^2 V_x}{\partial x^2}=-\frac{a}{\mu}\rho G$ $\frac{\partial V_x}{\partial x}=-\frac{a}{\mu}\rho Gx+C_1$ $V_x=-\frac{\rho Ga}{2\mu}x^2+C_1x+C_2$	**2.** $-\mu\frac{\partial^2 V_x}{\partial y^2}=b\rho G$ $\frac{\partial^2 V_x}{\partial y^2}=-\frac{b}{\mu}\rho G$ $\frac{\partial V_x}{\partial y}=-\frac{b}{\mu}\rho Gy+C_3$ $V_x=-\frac{\rho Gb}{2\mu}y^2+C_3y+C_4$	**3** $-\mu\frac{\partial^2 V_x}{\partial z^2}=c\rho G$ $\frac{\partial^2 V_x}{\partial z^2}=-\frac{c}{\mu}\rho G$ $\frac{\partial V_x}{\partial z}=-\frac{c}{\mu}\rho Gz+C_5$ $V_x=-\frac{\rho Gc}{2\mu}z^2+C_5z+C_6$

4 $\frac{\partial p}{\partial x} = d\rho G$ $P(x) = d\rho Gx + C_7$	**5** $4\rho \frac{\partial V_x}{\partial t} = f\rho G$ $\frac{\partial V_x}{\partial t} = \frac{fG}{4}$ $V_x = \frac{fG}{4} t + C_8$	**6** $-\mu \frac{\partial^2 V_y}{\partial x^2} = h\rho G$ $\frac{\partial^2 V_y}{\partial x^2} = -\frac{h}{\mu}\rho G$ $\frac{\partial V_y}{\partial x} = -\frac{h}{\mu}\rho Gx + C_9$ $V_y = -\frac{\rho Gh}{2\mu} x^2 + C_9 x + C_{10}$
7 $-\mu \frac{\partial^2 V_y}{\partial y^2} = j\rho G$ $\frac{\partial^2 V_y}{\partial y^2} = -\frac{j}{\mu}\rho G$ $\frac{\partial V_y}{\partial y} = -\frac{j}{\mu}\rho Gy + C_{11}$ $V_y = -\frac{\rho Gj}{2\mu} y^2 + C_{11} y + C_{12}$	**8** $-\mu \frac{\partial^2 V_y}{\partial z^2} = m\rho G$ $\frac{\partial^2 V_y}{\partial z^2} = -\frac{m}{\mu}\rho G$ $\frac{\partial V_y}{\partial z} = -\frac{m}{\mu}\rho Gz + C_{13}$ $V_y = -\frac{\rho Gm}{2\mu} z^2 + C_{13} z + C_{14}$	**9** $\frac{\partial p}{\partial y} = n\rho G$ $P(y) = n\rho Gy + C_{15}$
10 $4\rho \frac{\partial V_y}{\partial t} = q\rho G$ $\frac{\partial V_y}{\partial t} = \frac{qG}{4}$ $V_y = \frac{qG}{4} t + C_{16}$	**11** $-\mu \frac{\partial^2 V_z}{\partial x^2} = r\rho G$ $\frac{\partial^2 V_z}{\partial x^2} = -\frac{r}{\mu}\rho G$ $\frac{\partial V_z}{\partial x} = -\frac{r}{\mu}\rho Gx + C_{17}$ $V_z = -\frac{\rho Gr}{2\mu} x^2 + C_{17} x + C_{18}$	**12** $-\mu \frac{\partial^2 V_z}{\partial y^2} = s\rho G$ $\frac{\partial^2 V_z}{\partial y^2} = -\frac{s}{\mu}\rho G$ $\frac{\partial V_z}{\partial y} = -\frac{s}{\mu}\rho Gy + C_{19}$ $V_z = -\frac{\rho Gs}{2\mu} y^2 + C_{19} y + C_{20}$

13 $-\mu \frac{\partial^2 V_z}{\partial z^2} = u\rho G$ $\frac{\partial^2 V_z}{\partial z^2} = -\frac{u}{\mu}\rho G$ $\frac{\partial V_z}{\partial z} = -\frac{u}{\mu}\rho Gz + C_{21}$ $V_z = -\frac{\rho Gu}{2\mu} z^2 + C_{21} z + C_{22}$	**14** $\frac{\partial p}{\partial z} = v\rho G$ $P(z) = v\rho Gz + C_{23}$	**15** $4\rho \frac{\partial V_z}{\partial t} = w\rho G$ $\frac{\partial V_z}{\partial t} = \frac{wG}{4}$ $V_z = \frac{wG}{4} t + C_{24}$

Step 3: One collect the solutions from Step 2 for (V_x, V_y, V_z, $P(x)$, $P(y)$, $P(z)$)

For V_x, **S**um of integrals from sub-equations #1, #2, #3, #5, and for $P(x)$, from #4

$$V_x(x,y,z,t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = d\rho Gx + C_7$$

For V_y **S**um of integrals from sub-equations #6, #7, #8, #10, and for $P(y)$, from #9.

$$V_y(x,y,z,t) = -\frac{\rho G}{2\mu}(hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = n\rho Gy + C_{15}$$

For V_z: **S**um of integrals from sub-equations #11, #12, #13, and for $P(z)$, from #14

$$V_z(x,y,z,t) = -\frac{\rho G}{2\mu}(rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}$$

Comparison of the above methods for the solutions of Linearized Navier-Stokes Equations

Note below that the solutions by the two different methods are the same except for the constants involved. Now, one has two different methods for solving the system of Navier-Stokes equations. Such an agreement and consistency confirm the validity of the method used in solving the magnetohydrodynamic equations.

Solutions by Method 1

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho gx$$

$$V_y(x,y,z,t) = -\frac{\rho g_y}{2\mu}(hx^2 + jy^2 + mz^2) + C_1x + C_3y + C_5z + \frac{qg_y}{4}t + C; \quad P(y) = n\rho g_y y$$

$$V_z(x,y,z,t) = -\frac{\rho g_z}{2\mu}(rx^2 + sy^2 + uz^2) + C_1x + C_3y + C_5z + \frac{wg_z}{4}t + C; \quad P(z) = v\rho g_z z$$

Solutions by Method 2

$$V_x(x,y,z,t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = d\rho Gx + C_7$$

$$V_y(x,y,z,t) = -\frac{\rho G}{2\mu}(hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = n\rho Gy + C_{15}$$

$$V_z(x,y,z,t) = -\frac{\rho G}{2\mu}(rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}$$

Overall Conclusion

The Navier-Stokes (N-S) equations in 3-D and 4-D have been solved analytically for the first time by two different methods. In Method 1, the three equations were separately integrated.
In Method 2, the three equations were first added together and a single equation was integrated. The solutions from these two methods were the same, except for the constants involved. The N-S solution is unique. The experience gained in solving the linearized equation helped the author to propose a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. The application of this law helped speed-up the solutions of the non-linearized N-S equations, since there was no more experimentation as to the subject of the equation. It was also shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.
The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. Most importantly, the gravity term is the indispensable term in fluid flow, and it is involved in the parabolic as well as the forward motion. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in parabolic, periodic and decreasingly exponential motion. As the viscosity increases, periodicity increases. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The convective acceleration terms produce square root function behavior and behavior of fractional terms containing square root functions with variables in the denominator. In terms of the velocity profile, the first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the x–nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The y–nonlinear term is similar parabolically to the x–nonlinear term. The characteristic curve for the z–nonlinear term is a combination of two similar parabolas and a hyperbola. If the above x–direction flow is repeated simultaneously in the y– and z– directions, the flow is chaotic and consequently turbulent.
The following statements can be made:
(a) The N-S equations have unique solutions; (b) The N-S equations have parabolic solutions;
3. The N-S equations have square root function solutions. 4. The N-S equations do not have periodic solutions but have periodic relations. **5.**. The N-S equations do not have decreasingly exponential solutions but have decreasingly exponential relations.
In applications, the ratio terms a, b, c, d, f, h, n, q and others may perhaps be determined using information such as initial and boundary conditions or may have to be determined experimentally. The author came to the experimental determination conclusion after referring to preliminaries...The question is how did the grandmother determine the terms of the ratio for her grandchildren? Note that so far as the general solutions of the N-S equations are concerned, one needs not find the specific values of the ratio terms.
Finally, for any fluid flow design, one should always maximize the role of gravity for cost-effectiveness, durability, and dependability. Perhaps, Newton's law for fluid flow should read "Sum of everything else equals ρg" ; and this would imply the proposed new law that the other terms divide the gravity term in a definite ratio, and each term utilizes gravity to function.

P.S.

Maples software was used to help express the implicit terms in terms of x, y, z, and t, by solving System P (p.27, 28). None of the academic programs could solve the system of solutions M. The author would like to find a software that can solve the original system, System M, for comparison purposes.

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Option 8
Spin-off: CMI Millennium Prize Problem Requirements

Proof 1
For the linearized Navier-Stokes equations

Proof of the existence of solutions of the Navier-Stokes equations

Since from page 11, it has been shown that the smooth equations given by

$\boxed{V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4}t + C_9 \ ; \ P(x) = d\rho g_x x}$ are solutions

of the linearized equation, $-\mu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho\frac{\partial v_x}{\partial t} = \rho g_x$, it has been shown that smooth solutions to the above differential equation exist. and the proof is complete.

From, above, if $y = 0,\ z = 0$, $\boxed{V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1 x + \frac{fg_x}{4}t + C_9}$; $P(x) = d\rho gx + C_{10}$

Therefore, $V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_{10}x + C_9$

Finding $P(x,t)$

1. $V_x(x,t) = -\frac{\rho g_x}{2\mu}(ax^2) + C_1 x + \frac{fg_x}{4}t + C_9$; $P(x) = d\rho g_x x$ **2**. $\frac{\partial p}{\partial x} = d\rho g$;

Required: To find $P(x,t)$ (that is, find a formula for P in terms of x and t)

$$\frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx}V_x \qquad (\frac{dx}{dt} = V_x)$$

$$\frac{dp}{dt} = d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1 x + \frac{fg_x}{4}t + C_9\right) \qquad (\frac{dp}{dx} = d\rho g_x)$$

$$P(x,t) = \int d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1 x + \frac{fg_x}{4}t + C_9\right)dt$$

$$P(x,t) = d\rho g_x\left(-\frac{a\rho g_x}{2\mu}x^2 t + C_1 xt + \frac{fg_x}{8}t^2 + C_9 \mathrm{t}\right) + C_{10}$$

For the corresponding coverage for the original Navier-Stokes equation, see the next page

Proof 2

For the Non-linearized Navier-Stokes equations (Original Equations)

Proof of the existence of solutions of the Navier-Stokes equations

From page 30, if $y=0,\ z=0$ in

$$\boxed{\begin{array}{l} V_x(x,y,z,t) = \overbrace{-\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2)+C_1x+C_3y+C_5z+\underbrace{fg_xt}_{\text{continued}}}^{\textbf{Solution to Linear part}} \underbrace{\pm\sqrt{2hg_xx}+\frac{ng_xy}{V_y}+\frac{qg_xz}{V_z}+\frac{\psi_y(V_y)}{V_y}+\frac{\psi_z(V_z)}{V_z}}_{\textbf{solution of Euler equation}} \\ P(x)=d\rho g_xx \end{array}}$$

one obtains

$$\boxed{V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9\ ; \quad P(x) = d\rho g_xx;}$$

$$V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x \pm \sqrt{2hg_xx} + C_9\ ; \quad P(x) = d\rho g_xx;$$

Since previously, from p.21, it has been shown that the smooth equations given by

$V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9\ ;\ P(x) = d\rho g_xx;$ are solutions of

$-\mu\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x\frac{\partial V_x}{\partial x} = \rho g_x$ (deleting the $y-$ and $z-$terms of (A)), p.25, one has shown that smooth solutions to the above differential equation exist, and the proof is complete.

Finding $P(x,t)$:

1. $V_x(x,t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + fg_xt \pm \sqrt{2hg_xx} + C_9;\ P(x) = d\rho g_xx;$ **2.** $\frac{\partial p}{\partial x} = d\rho g;$

$$\frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx}V_x \qquad \left(\frac{dx}{dt} = V_x\right)$$

$$\frac{dp}{dt} = d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x \pm \sqrt{2hg_xx} + fg_xt + C_9\right) \qquad \left(\frac{dp}{dx} = d\rho g_x\right)$$

$$P(x,t) = \int d\rho g_x\left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x \pm \sqrt{2hg_xx} + fg_xt + C_9\right)dt$$

$$P(x,t) = d\rho g_x\left(-\frac{a\rho g_x}{2\mu}x^2t + C_1xt \pm t\sqrt{2hg_xx} + \frac{fg_xt^2}{2} + C_9t\right) + C_{10}$$

References:

For paper edition of the above paper, see Chapter 11 & Appendix 7 of the book entitled "Power of Ratios "by A. A. Frempong, published by Yellowtextbooks.com.Without using ratios or proportion, the author would never be able to split-up the Navier-Stokes equations into sub-equations which were readily integrable. The impediment to solving the Navier-Stokes equations for over 150 years (whether linearized or non-linearized) has been due to finding a way to split-up the equations. Since ratios were the key to splitting the Navier-Stokes equations, and solving them, the solutions have also been published in the " Power of Ratios" book which covers definition of ratio and applications of ratio in mathematics, science, engineering, economics and business fields.

Adonten

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