

# Navier-Stokes Equations Solutions Completed

## Abstract

The simplest solution is usually the best solution---Albert Einstein

Over nearly a year and half ago, the Navier-Stokes equations in 3-D for incompressible fluid flow were analytically solved by the author. However, some of the solutions contained implicit terms. In this paper, the implicit terms have been expressed explicitly in terms of  $x$ ,  $y$ ,  $z$  and  $t$ . The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. It was mathematically shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known, and also, there would be no magnetohydrodynamics. In addition to the usual method of solving these equations, the N-S equations were also solved by a second method in which the three equations in the system were added to produce a single equation which was then integrated. The solutions by the two methods were identical, except for the constants involved. Ratios were used to split-up the equations; and the resulting sub-equations were readily integrable, and even, the nonlinear sub-equations were readily integrated. The examples in the preliminaries show everyday examples on using ratios to divide a quantity into parts, as well as possible applications of the solution method in mathematics, science, engineering, business, economics, finance, investment and personnel management decisions. The  $x$ -direction Navier-Stokes equation was linearized, solved, and the solution analyzed. This solution was followed by the solution of the Euler equation of fluid flow. The Euler equation represents the nonlinear part of the Navier-Stokes equation. Following the Euler solution, the Navier-Stokes equation was solved essentially by combining the solutions of the linearized equation and the Euler solution. For the Navier-Stokes equation, the linear part of the relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation; and the relation from the integration of the non-linear part satisfied the non-linear part of the equation. The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. The gravity term is the indispensable term in fluid flow, and it is involved in the parabolic and forward motion of fluids. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in the parabolic, periodic and decreasingly exponential motion. Periodicity increases with viscosity. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The fluid flow in the Navier-Stokes solution may be characterized as follows. The  $x$ -direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the  $x$ -nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The  $y$ -nonlinear term is similar parabolically to the  $x$ -nonlinear term. The characteristic curve for the  $z$ -nonlinear term is a combination of two similar parabolas and a hyperbola. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent.

For a spin-off, the smooth solutions from above are specialized and extended to satisfy the requirements of the CMI Millennium Prize Problems, and thus prove the existence of smooth solutions of the Navier-Stokes equations.

## Introduction

### Solutions of the Navier-Stokes Equations

#### Case 1: Solutions of the Linearized Navier-Stokes Equations ( $x$ -direction)

Equation 
$$-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho\left(\frac{\partial V_x}{\partial t}\right) = \rho g_x$$

Solutions 
$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x x$$

#### Case 2: Solutions of the Euler Equations for Incompressible Fluid Flow ()

Equation 
$$\rho\left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right) + \frac{\partial p_x}{\partial x} = \rho g_x \quad x\text{-direction}$$

Solutions

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}; \quad V_y \neq 0, V_z \neq 0; P(x) = d\rho g_x x;$$

#### Case 3: Solutions of the Navier-Stokes Equations (Original) : $x$ -direction

Equation: 
$$-\mu\frac{\partial^2 V_x}{\partial x^2} - \mu\frac{\partial^2 V_x}{\partial y^2} - \mu\frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho\frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x$$

Solutions

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + h + n + q = 1) \quad V_y \neq 0, V_z \neq 0$$

#### Summary for the fractional terms of the $x$ -direction

$\frac{ng_x y}{V_y}$  and  $\frac{qg_x z}{V_z}$  in terms of  $x, y, z$  and  $t$  (for Case 3)

$$\frac{ng_x y}{V_y} = \frac{-(ng_x)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1x + C_3y + C_5z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)}{\beta_7 g_z}$$

$$\frac{qg_x z}{V_z} = \frac{-(qg_x z)\{[(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x}\right) - [CE]]\}}{(\beta_7 g_z y)(qg_x z - \beta_6 g_z x)}$$

$$(CE = -(ng_x y)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1x + C_3y + C_5z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right))$$

One observes above that the most important insight of the above solutions is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity,  $g_x$ , were zero, for Case 1, the first three terms, the seventh, and  $P(x)$  would all be zero; for Case 2, the first four terms and  $P(x)$  would all be zero; and for Case 3, the first three terms, the seventh, the eighth, the ninth, the tenth terms and  $P(x)$  would all be zero. These results can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. It would not therefore be meaningful to write a Navier-Stokes equation for incompressible fluid flow without the gravity term, since there would be no fluid flow.

## More Observations Comparison of the N-S solutions with equations of motion under gravity and liquid pressure of elementary physics

### Motion equations of elementary physics:

$$(B): V_f = V_0 + gt; \quad (C): V_f^2 = V_0^2 + 2gx; \quad (D): V = \sqrt{2gx}; \quad (E): x = V_0t + \frac{1}{2}gt^2$$

The **liquid pressure**,  $P$  at the bottom of a liquid of depth  $h$  units is given by  $P = \rho gh$   
Observe the following about the Navier-Stokes Solutions (Case 3)

1. The first three terms are parabolic in  $x$ ,  $y$ , and  $z$ ; the minus sign shows the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; also note the " $gt$ " in  $V = gt$  of (B) of the motion equations and the  $fg_x t$  in the Navier-Stokes solution.
2. The pressure,  $P = \rho gh$  of the liquid pressure and the  $P(x) = d\rho g_x x$  of the Navier-Stokes solution. **Note** that, only the approach in this paper could yield  $P(x) = d\rho g_x x$  by integrating  $dp/dx = d\rho g_x$
3. Observe the " $\sqrt{2gx}$ " in  $V = \sqrt{2gx}$  of (D) and the  $\sqrt{2hg_x x}$  in the Navier-Stokes solution. There are eight main terms (ignoring the arbitrary functions) in the Navier-Stokes solution.

Of these eight terms, six terms, namely,  $-\frac{a\rho g_x}{2\mu}x^2$ ,  $-\frac{b\rho g_x}{2\mu}y^2$ ,  $-\frac{c\rho g_x}{2\mu}z^2$ ,  $fg_x t$ ,  $\sqrt{2hg_x x}$  and  $d\rho g_x x$  are similar (except for the constants involved) to the terms in the equations of motion and fluid pressure. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

4. With regards to the variables  $x$ ,  $y$ , and  $z$ , the parabolicity of the first three terms and the parabolicity of the eighth, ninth and tenth terms hint at inverse relations.. For examples,

$V_x = x^2$  and  $V_x = \pm\sqrt{x}$  are inverse relations of each other,  $V_x = y^2$  and  $V_x = \pm\sqrt{y}$  are inverse relations of each other,  $V_x = z^2$  and  $V_x = \pm\sqrt{z}$  are inverse relations of each other. The implications of knowing these relationships is that if one knows the steps, rules or formulas for designing for laminar flow, one can deduce the steps, rules or formulas for designing for turbulent flow by reversing the steps and using opposite operations in each step of the corresponding laminar flow design. Thus for every method, or formula for laminar flow, there is a corresponding method, formula for turbulent flow design (see also, "Power of Ratios" book by A. A. Frempong, p. 28).

### Velocity Profile

The  $x$ -direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize inverted parabolas. Flow distribution for laminar flow is parabolic with the axis of symmetry of the parabola in direction of the fluid flow. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the  $x$ -nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The  $y$ -nonlinear term is similar parabolically to the  $x$ -nonlinear term. The characteristic curve for the  $z$ -nonlinear term is a combination of two similar parabolas and a hyperbola. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent.

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The Navier-Stokes equations in three dimensions are three simultaneous equations in Cartesian coordinates for the flow of incompressible fluids. The equations are presented below:

$$\left\{ \begin{array}{l} \mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x = \rho\left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right) \quad (N_x) \\ \mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right) - \frac{\partial p}{\partial y} + \rho g_y = \rho\left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z}\right) \quad (N_y) \\ \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) - \frac{\partial p}{\partial z} + \rho g_z = \rho\left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z}\right) \quad (N_z) \end{array} \right.$$

Equation ( $N_x$ ) will be the first equation to be solved; and based on its solution, one will be able to write down the solutions for the other two equations, ( $N_y$ ), and ( $N_z$ ).

## Dimensional Consistency

The Navier-Stokes equations are dimensionally consistent as shown below:

$$\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x = \rho\left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right)$$

Using *MLT*

$$\boxed{M(L^{-2}T^{-2} + L^{-2}T^{-2} + L^{-2}T^{-2} - L^{-2}T^{-2} + L^{-2}T^{-2}) = M(L^{-2}T^{-2} + L^{-2}T^{-2} + L^{-2}T^{-2} + L^{-2}T^{-2})}$$

Using *kg-m-s*

$$\boxed{kg(m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2} - m^{-2}s^{-2} + m^{-2}s^{-2}) = kg(m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2} + m^{-2}s^{-2})}$$

$$\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) - \frac{\partial p_x}{\partial x} + \rho g_x = \rho\left(\underbrace{\frac{\partial V_x}{\partial t}}_{\text{variable acceleration (local rate of change of } V_x)} + \underbrace{V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}}_{\text{convective acceleration (rate of change in } V_x \text{ due to motion)}}\right)$$

inertia per volume

## Option 1

### Solution of 3-D Linearized Navier-Stokes Equation in the x-direction

The equation will be linearized by redefinition. The nine-term equation will be reduced to six terms.

$$\textbf{Given: } \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x = \rho\left(\frac{\partial v_x}{\partial t} + V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z}\right) \quad (\text{A})$$

$$-\mu \frac{\partial^2 v_x}{\partial x^2} - \mu \frac{\partial^2 v_x}{\partial y^2} - \mu \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial v_x}{\partial t} + \rho V_x \frac{\partial v_x}{\partial x} + \rho V_y \frac{\partial v_x}{\partial y} + \rho V_z \frac{\partial v_x}{\partial z} = \rho g_x \quad (\text{B})$$

$$-\mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho\left(\frac{\partial v_x}{\partial t}\right) = \rho g_x \quad (\text{C})$$

**Plan:** One will split-up equation (C) into five sub-equations, solve them, and combine the solutions. On splitting-up the equations and proceeding to solve them, the non linear terms could be redefined and made linear. This linearization is possible if the gravitational force term is the subject of the equation as in equation (B). After converting the non-linear terms to linear terms by redefinition, one will have only six terms as in equation (C). One will show logically how equation (C) was obtained from equation (B), using a ratio method.

Three main steps are covered.

In main Step 1, one shows how equation (C) was obtained from equation (B)

In main Step 2, equation (C) will be split-up into five equations.

In main Step 3, each equation will be solved.

In main Step 4, the solutions from the five equations will be combined.

In main Step 5, the combined relation will be checked in equation (C). for identity.

#### Preliminaries

Requirements and procedure for solving a partial differential equation

1. Integrate the partial differential equation.
2. Find the partial derivatives from the integration relation from Step 1
3. Substitute the derivatives from Step 2 in the original partial differential equation and simplify. both sides of the equation.
4. If the left-hand side of the equation is equal to the right-hand side of the equation, then the integration relation from Step 1 is a solution to the partial differential equation.  
( Steps 2-4 can be summarized as checking for identity, or determining if the integration relation satisfies the original partial differential equation.)

Note: If one does not successfully check for identity, one cannot claim a solution.

A ratio method will be used to split-up the partial differential equations into sub-equation which are then integrated.

**Example 1:** A grandmother left \$45,000 in her will to be divided between eight grandchildren, Betsy, Comfort, Elaine, Ingrid, Elizabeth, Maureen, Ramona, Marilyn, in the ratio  $\frac{1}{36} : \frac{1}{18} : \frac{1}{12} : \frac{1}{9} : \frac{5}{36} : \frac{1}{6} : \frac{7}{36} : \frac{2}{9}$ . (**Note:**  $\frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{6} + \frac{7}{36} + \frac{2}{9} = 1$ )

How much does each receive?

**Solution:**

$$\text{Betsy's share of } \$45,000 = \frac{1}{36} \times \$45,000 = \$1,250$$

$$\text{Comfort's share of } \$45,000 = \frac{1}{18} \times \$45,000 = \$2,500$$

$$\text{Elaine's share of } \$45,000 = \frac{1}{12} \times \$45,000 = \$3,750$$

$$\text{Ingrid's share of } \$45,000 = \frac{1}{9} \times \$45,000 = \$5,000$$

$$\text{Elizabeth's share of } \$45,000 = \frac{5}{36} \times \$45,000 = \$6,250$$

$$\text{Maureen's share of } \$45,000 = \frac{1}{6} \times \$45,000 = \$7,500$$

$$\text{Ramona's share of } \$45,000 = \frac{7}{36} \times \$45,000 = \$8,750$$

$$\text{Marilyn's share of } \$45,000 = \frac{2}{9} \times \$45,000 = \$10,000$$

$$\text{Check; Sum of shares } = \boxed{\$45,000}$$

$$\text{Sum of the fractions} = 1$$

**Example 2:** Sir Isaac Newton left  $\rho g_x$  units in his will to be divided between  $-\mu \frac{\partial^2 v_x}{\partial x^2}$ ,  $-\mu \frac{\partial^2 v_x}{\partial y^2}$ ,

$$-\mu \frac{\partial^2 v_x}{\partial z^2}, \frac{\partial p}{\partial x}, \rho \frac{\partial v_x}{\partial t}, \rho v_x \frac{\partial v_x}{\partial x}, \rho v_y \frac{\partial v_x}{\partial y}, \rho v_z \frac{\partial v_x}{\partial z} \text{ in the ratio } a : b : c : d : f : h : m : n.$$

where  $a + b + c + d + f + h + m + n = 1$ . How much does each receive?

**Solution**  $-\mu \frac{\partial^2 v_x}{\partial x^2}$ 's share of  $\rho g_x$  units =  $a\rho g_x$  units

$$-\mu \frac{\partial^2 v_x}{\partial y^2}$$
's share of  $\rho g_x$  units =  $b\rho g_x$  units

$$-\mu \frac{\partial^2 v_x}{\partial z^2}$$
's share of  $\rho g_x$  units =  $c\rho g_x$  units

$$\frac{\partial p}{\partial x}$$
's share of  $\rho g_x$  units =  $d\rho g_x$  units

$$\rho \frac{\partial v_x}{\partial t}$$
's share of  $\rho g_x$  units =  $f\rho g_x$  units

$$\rho v_x \frac{\partial v_x}{\partial x}$$
's share of  $\rho g_x$  units =  $h\rho g_x$  units

$$\rho v_y \frac{\partial v_x}{\partial y}$$
's share of  $\rho g_x$  units =  $m\rho g_x$  units

$$\rho v_z \frac{\partial v_x}{\partial z}$$
's share of  $\rho g_x$  units =  $n\rho g_x$  units

$$\text{Sum of shares} = \boxed{\rho g_x \text{ units}} \quad \text{Note: } a + b + c + d + f + h + m + n = 1$$

**Example 3:**

The returns on investments  $A, B, C, D$  are in the ratio  $a : b : c : d$ . If the total return on these four investments is  $P$  dollars, what is the return on each of these investments?

$$(a + b + c + d = 1)$$

**Solution** Return on investment  $A = aP$  dollars  
Return on investment  $B = bP$  dollars  
Return on investment  $C = cP$  dollars  
Return on investment  $D = dP$  dollars

**Check**

$$aP + bP + cP + dP = P$$

$$P(a + b + c + d) = P$$

$$a + b + c + d = 1 \text{ (dividing both sides by } P)$$

**Example 4:** Solve the quadratic equation;

$$6x^2 + 11x - 10 = 0$$

**Method 1** (a common method)

By factoring,

$$6x^2 + 11x - 10 = 0$$

$$(3x - 2)(2x + 5) = 0 \text{ and solving,}$$

$$(3x - 2) = 0 \text{ or } (2x + 5) = 0$$

$$x = \frac{2}{3}, x = -\frac{5}{2}.$$

$$\text{Solution set: } \left\{-\frac{5}{2}, \frac{2}{3}\right\}$$

**Example 4, Method 2:**

One applies the discussion in Example 2

One will call this method the **multiplier method**.

**Step 1:** From  $6x^2 + 11x - 10 = 0$  (1)

$$6x^2 + 11x = 10$$

$$6x^2 = 10a; \text{ (Here, } a \text{ is a multiplier)}$$

$$3x^2 = 5a \text{ (2)}$$

$$11x = 10b \text{ (Here, } b \text{ is a multiplier)}$$

$$11x = 10(1 - a) \text{ (} a + b = 1)$$

$$11x = 10 - 10a$$

$$x = \frac{10 - 10a}{11}$$

$$3\left(\frac{10 - 10a}{11}\right)^2 = 5a \text{ (Substituting for } x \text{ in (2))}$$

$$3\left(\frac{100 - 200a + 100a^2}{121}\right) = 5a$$

**Step 2:**  $300a^2 - 1205a + 300 = 0$

$$60a^2 - 241a + 60 = 0$$

$$a = \frac{241 \pm \sqrt{241^2 - 4(60)(60)}}{120}$$

$$a = \frac{241 \pm \sqrt{43681}}{120}$$

$$a = \frac{241 \pm 209}{120}$$

$$a = \frac{241 \pm 209}{120} = \frac{241 + 209}{120} \text{ or } \frac{241 - 209}{120}$$

$$= \frac{450}{120} \text{ or } \frac{32}{120}$$

$$= \frac{15}{4} \text{ or } \frac{4}{15}$$

**Step 3:** Since  $a + b = 1$ , when  $a = \frac{15}{4}$  or  $3\frac{3}{4}$

$$b = 1 - 3\frac{3}{4} = -2\frac{3}{4} \text{ or } -\frac{11}{4}$$

$$\text{when } a = \frac{4}{15}, b = 1 - \frac{4}{15} = \frac{11}{15}$$

**Step 4:** When  $b = -\frac{11}{4}$ ,  $11x = 10(-\frac{11}{4})$

$$x = -\frac{5}{2}$$

$$\text{When } b = \frac{11}{15}, 11x = 10\left(\frac{11}{15}\right)$$

$$x = \frac{10}{11}\left(\frac{11}{15}\right); x = \frac{2}{3}$$

Again, one obtains the same solution set  $\left\{-\frac{5}{2}, \frac{2}{3}\right\}$  as by the factoring method.

The objective of presenting examples 1, 2, 3, and 4 was to convince the reader that the principles to be used in splitting the Navier-Stokes equations are valid. In Examples 4, one could have used the quadratic formula directly to solve for  $x$ , without finding  $a$  and  $b$  first. The objective was to show that the introduction of  $a$  and  $b$  did not change the solution set of the original equation.

For the rest of the coverage in this paper, a multiplier is the same as a ratio term

The multiplier method is the same as the ratio method.

## Main Step 1

### Linearization of the Non-Linear Terms

**Step 1:** The main principle is to multiply the right side of the equation by the ratio terms

This step is critical to the removal of the non-linearity of the equation.

$\rho g_x$  is to be divided by the terms on the left-hand-side of the equation in the ratio

$$a : b : c : d : f : h : m : n \quad (a + b + c + d + f + h + m + n = 1)$$

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \underbrace{\rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}}_{\text{nonlinear terms}} = \rho g_x \quad (1)$$

all acceleration terms

Apply the principles involved in the ratio method covered in the preliminaries, to the nonlinear terms (the last three terms.)

Then  $\rho V_z \frac{\partial V_x}{\partial z} = n \rho g_x$ , where  $n$  is the ratio term corresponding to  $\rho V_z \frac{\partial V_x}{\partial z}$ .

$$V_z \frac{\partial V_x}{\partial z} = n g_x \quad (2)$$

$V_z \frac{dV_x}{dz} = n g_x$  (One drops the partials symbol, since a single independent variable is involved)

$$\frac{dz}{dt} \frac{dV_x}{dz} = n g_x \quad (V_z = \frac{dz}{dt}, \text{ by definition})$$

$$\frac{dV_x}{dt} = n g_x \quad (3)$$

Therefore,  $\boxed{V_z \frac{\partial V_x}{\partial z} = \frac{dV_x}{dt} = n g_x}$  (4)

**Step 2:** Similarly, Let  $\rho V_y \frac{\partial V_x}{\partial y} = m \rho g_x$  ( $m$  is the ratio term corresponding to  $\rho V_y \frac{\partial V_x}{\partial y}$ ) (5)

$V_y \frac{dV_x}{dy} = m g_x$  (One drops the partials symbol, since a single independent variable is involved)

$$\frac{dy}{dt} \frac{dV_x}{dy} = m g_x \quad (V_y = \frac{dy}{dt})$$

$$\frac{dV_x}{dt} = m g_x \quad (6)$$

Therefore,  $\boxed{V_y \frac{\partial V_x}{\partial y} = \frac{dV_x}{dt} = m g_x}$  (7)

**Step 3:** Let  $\rho V_x \frac{\partial V_x}{\partial x} = h \rho g_x$  where  $h$  is the ratio term corresponding to  $\rho V_x \frac{\partial V_x}{\partial x}$ .

$$V_x \frac{\partial V_x}{\partial x} = h g_x \quad (8)$$

$V_x \frac{dV_x}{dx} = h g_x$  (One drops the partials symbol, since a single independent variable is involved)

$$\frac{dx}{dt} \frac{dV_x}{dx} = h g_x \quad (V_x = \frac{dx}{dt})$$

$$\frac{dV_x}{dt} = h g_x \quad (9) \quad \text{Therefore, } \boxed{V_x \frac{\partial V_x}{\partial x} = \frac{dV_x}{dt} = h g_x} \quad (10)$$



From equations (4), (7), (10),  $V_x \frac{\partial v_x}{\partial x} = V_y \frac{\partial v_x}{\partial y} = V_z \frac{\partial v_x}{\partial z} = \frac{dv_x}{dt}$  and

$$V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z} = \boxed{3 \frac{dv_x}{dt}} \quad (11)$$

Thus, the ratio of the linear term  $\frac{\partial v_x}{\partial t}$  to the nonlinear sum  $V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} + V_z \frac{\partial v_x}{\partial z}$  in equation (1) is 1 to 3. Unquestionably, there is a ratio between the sum of the nonlinear terms and the linear term  $\frac{\partial v_x}{\partial t}$ . This ratio must be verified experimentally.

**Note:** One could have obtained equation (C) from equation (A) by redefining the nonlinear terms by **carelessly** disregarding the partial derivatives of the nonlinear terms in equation (1). However, the author did not do that, but logically, the terms became linearized.

**Note** also that the above linearization is possible only if  $\rho g_x$  is the subject of the equation, and it will later be learned that a solution to the logically linearized Navier-Stokes equation is obtained only if  $\rho g_x$  is the subject of the equation.

**Step 4:** Substitute the right side of equation (11) for the nonlinear terms on the left- side of

$$-\mu \frac{\partial^2 v_x}{\partial x^2} - \mu \frac{\partial^2 v_x}{\partial y^2} - \mu \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + \underbrace{\rho \frac{\partial v_x}{\partial t} + \rho V_x \frac{\partial v_x}{\partial x} + \rho V_y \frac{\partial v_x}{\partial y} + \rho V_z \frac{\partial v_x}{\partial z}}_{\text{all acceleration terms}} = \rho g_x \quad (12)$$

Then one obtains 
$$-\mu \frac{\partial^2 v_x}{\partial x^2} - \mu \frac{\partial^2 v_x}{\partial y^2} - \mu \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + \underbrace{\rho \frac{\partial v_x}{\partial t} + 3\rho \frac{\partial v_x}{\partial x}}_{\text{all acceleration terms}} = \rho g_x$$

$$\boxed{-\mu \frac{\partial^2 v_x}{\partial x^2} - \mu \frac{\partial^2 v_x}{\partial y^2} - \mu \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial v_x}{\partial t} = \rho g_x} \quad (\text{simplifying}) \quad (13)$$

Now, instead of solving equation (1), previous page, one will solve the following equation

$$\boxed{-K \frac{\partial^2 v_x}{\partial x^2} - K \frac{\partial^2 v_x}{\partial y^2} - K \frac{\partial^2 v_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + 4 \frac{\partial v_x}{\partial t} = g_x} \quad (k = \frac{\mu}{\rho}) \quad (14)$$

## Main Step 2

**Step 5:** In equation (14) divide  $g_x$  by the terms on the left side in the ratio  $a : b : c : d : f$ .

$$\boxed{-K \frac{\partial^2 v_x}{\partial x^2} = a g_x; \quad -K \frac{\partial^2 v_x}{\partial y^2} = b g_x; \quad -K \frac{\partial^2 v_x}{\partial z^2} = c g_x; \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = d g_x; \quad 4 \frac{\partial v_x}{\partial t} = f g_x}$$

(  $a, b, c, d, f$  are the ratio terms and  $a + b + c + d + f = 1$ ).

As proportions: 
$$\frac{-K \frac{\partial^2 v_x}{\partial x^2}}{a} = \frac{g_x}{1}; \quad \frac{-K \frac{\partial^2 v_x}{\partial y^2}}{b} = \frac{g_x}{1}; \quad \frac{-K \frac{\partial^2 v_x}{\partial z^2}}{c} = \frac{g_x}{1}; \quad \frac{\frac{1}{\rho} \frac{\partial p}{\partial x}}{d} = \frac{g_x}{1}; \quad \frac{4 \frac{\partial v_x}{\partial t}}{f} = \frac{g_x}{1}$$

One can view each of the ratio terms  $a, b, c, d, f$  as a fraction (a real number) of  $\boxed{g_x}$  contributed by each expression on the left-hand side of equation (14) above.

## Main Step 3

**Step 6:** Solve the differential equations in Step 5.

### Solutions of the five sub-equations

$$\boxed{-K \frac{\partial^2 V_x}{\partial x^2} = ag_x}$$

$$k \frac{\partial^2 V_x}{\partial x^2} = -ag$$

$$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{k} g$$

$$\frac{\partial V_x}{\partial x} = -\frac{ag}{k} x + C_1$$

$$V_{x1} = -\frac{ag}{2k} x^2 + C_1 x + C_2$$

$$\boxed{-K \frac{\partial^2 V_x}{\partial y^2} = bg_x}$$

$$K \frac{\partial^2 V_x}{\partial y^2} = -bg$$

$$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{k} g$$

$$\frac{\partial V_x}{\partial y} = -\frac{bg}{k} y + C_3$$

$$V_{x2} = -\frac{bg}{2k} y^2 + C_3 y + C_4$$

$$\boxed{-K \frac{\partial^2 V_x}{\partial z^2} = cg_x}$$

$$K \frac{\partial^2 V_x}{\partial z^2} = -cg$$

$$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{k} g$$

$$\frac{\partial V_x}{\partial z} = -\frac{cg}{k} z + C_5$$

$$V_{x3} = -\frac{cg}{2k} z^2 + C_5 z + C_6$$

$$\boxed{\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = dg$$

$$\frac{\partial p}{\partial x} = d\rho g$$

$$p = d\rho g x + C_7$$

$$\boxed{4 \frac{\partial V_x}{\partial t} = fg_x}$$

$$\frac{\partial V_x}{\partial t} = \frac{f}{4} g_x$$

$$V_{x4} = \frac{fg_x}{4} t$$

## Main Step 4

**Step 7:** One combines the above solutions

$$V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}$$

$$= -\frac{ag_x}{2k} x^2 + C_1 x + C_2 - \frac{bg_x}{2k} y^2 + C_3 y + C_4 - \frac{cg_x}{2k} z^2 + C_5 z + C_6 + \frac{fg_x}{4} t + C_7$$

$$= -\frac{ag_x}{2k} x^2 + C_1 x - \frac{bg_x}{2k} y^2 + C_3 y - \frac{cg_x}{2k} z^2 + C_5 z + \frac{fg_x}{4} t + C_9$$

$$= -\frac{ag_x}{2k} x^2 - \frac{bg_x}{2k} y^2 - \frac{cg_x}{2k} z^2 + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$= -\frac{ag_x}{2k} x^2 - \frac{bg_x}{2k} y^2 - \frac{cg_x}{2k} z^2 + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$= -\frac{g_x}{2k} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$V_x = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$P(x) = d\rho g_x x$$

$$V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}$$

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$P(x) = d\rho g_x x$$

## Main Step 5

### Checking in equation (C)

**Step 8:** Find the derivatives, using

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9$$

$$P(x) = d\rho g_x x$$

$\frac{\partial V_x}{\partial x} = -\frac{\rho g_x}{2\mu}(2ax) + C_1$ $1. \quad \frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}$ $4. \quad \frac{\partial p}{\partial x} = d\rho g_x;$	$\frac{\partial V_x}{\partial y} = -\frac{\rho g_x}{\mu}(by) + C_3$ $2. \quad \frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}$ $5. \quad \frac{\partial V_x}{\partial t} = \frac{fg_x}{4}$	$\frac{\partial V_x}{\partial z} = -\frac{\rho g_x}{\mu}(cz)$ $3. \quad \frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu};$
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**Step 9:** Substitute the derivatives from Step 8 in  $-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x$  to check for identity (to determine if the relation obtained satisfies the original equation).

$-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x$ $-\mu\left(-\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu}\right) + d\rho g_x + 4\rho \frac{f}{4} g_x = \rho g_x$ $a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + \rho f g_x = \rho g_x$ $a g_x + b g_x + c g_x + d g_x + f g_x = g_x$ $g_x(a + b + c + d + f) = g_x$ $g_x(1) = g_x \quad (a + b + c + d + f = 1)$ $g_x = g_x \quad \text{Yes}$	<p style="text-align: center;">Scrapwork</p> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;"> <math display="block">\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu};</math> </div> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;"> <math display="block">\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu};</math> </div> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;"> <math display="block">\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu};</math> </div> <div style="display: flex; justify-content: space-between;"> <div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;"> <math display="block">\frac{\partial p}{\partial x} = d\rho g_x;</math> </div> <div style="border: 1px solid black; padding: 5px;"> <math display="block">\frac{\partial V_x}{\partial t} = \frac{fg_x}{4}</math> </div> </div>
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An identity is obtained and therefore, the solution of equation (C), p.96, is given by

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x x$$

The above solution is unique, because all possible equations were integrated but only a single equation, the equation with the gravity term as the subject of the equation produced the solution.

## Solution Summary for $V_x$ , $V_y$ and $V_z$

**For  $V_x$**   $a + b + c + d + f = 1$

$$\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x = \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right)$$

$$-K \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - K \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + 4 \frac{\partial V_x}{\partial t} = g_x$$

$$V_x = V_{x1} + V_{x2} + V_{x3} + V_{x4}$$

$$= -\frac{ag_x}{2k} x^2 + C_1 x + C_2 - \frac{bg_x}{2k} y^2 + C_3 y + C_4 - \frac{cg_x}{2k} z^2 + C_5 z + C_6 + \frac{fg_x}{4} t + C_7 + C_8$$

$$= -\frac{ag_x}{2k} x^2 + C_1 x - \frac{bg_x}{2k} y^2 + C_3 y - \frac{cg_x}{2k} z^2 + C_5 z + \frac{fg_x}{4} t + C_9$$

$$= -\frac{ag_x}{2k} x^2 - \frac{bg_x}{2k} y^2 - \frac{cg_x}{2k} z^2 + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$V_y(x, y, z, t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x}{4} t + C_9$$

$$P(x) = d\rho g_x x$$

**For  $V_y$**   $h + j + m + n + q = 1$

$$\mu \left( \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right) - \frac{\partial p}{\partial y} + \rho g_y = \rho \left( \frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} \right)$$

$$-K \frac{\partial^2 V_y}{\partial x^2} - K \frac{\partial^2 V_y}{\partial y^2} - K \frac{\partial^2 V_y}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial y} + 4 \frac{\partial V_y}{\partial t} = g_y$$

$$V_y = -\frac{hg_y}{2k} x^2 + C_1 x - \frac{jg_y}{2k} y^2 + C_3 y - \frac{mg_y}{2k} z^2 + C_5 z + \frac{ng_y}{4} t$$

$$V_y(x, y, z, t) = -\frac{\rho g_y}{2\mu} (hx^2 + jy^2 + mz^2) + C_1 x + C_3 y + C_5 z + \frac{qg_y}{4} t + C$$

$$P(y) = n\rho g_y y$$

**For  $V_z$**   $r + s + u + v + w = 1$

$$\mu \left( \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial p}{\partial z} + \rho g_z = \rho \left( \frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right)$$

$$-k \frac{\partial^2 V_z}{\partial x^2} - k \frac{\partial^2 V_z}{\partial y^2} - k \frac{\partial^2 V_z}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial z} + 4 \frac{\partial V_z}{\partial t} = g_z$$

$$V_z = -\frac{rg_z}{2k} x^2 + C_1 x - \frac{sg_z}{2k} y^2 + C_3 y - \frac{ug_z}{2k} z^2 + C_5 z + \frac{wg_z}{4} t$$

$$V_z(x, y, z, t) = -\frac{\rho g_z}{2\mu} (rx^2 + sy^2 + uz^2) + C_1 x + C_3 y + C_5 z + \frac{wg_z}{4} t + C$$

$$P(z) = v\rho g_z z$$

## Discussion About Linearized N-S Solutions

A solution to equation  $-\mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho\left(\frac{\partial v_x}{\partial t}\right) = \rho g_x$  (C) is

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + f = 1)$$

This relation gives an identity when checked in Equation (C) above.

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity,  $g$ , were zero, the first three terms, the seventh term, and  $P(x)$  would all be zero.. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The above result will be the same when one covers the general case, Option 4.

The above parabolic solution is also encouraging. It reminds one of the parabolic curve obtained when a stone is projected upwards at an acute angle to the horizontal..

### More Observations Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

#### Motion equations of elementary physics:

$$(B): V_f = V_0 + gt; \quad (C): V_f^2 = V_0^2 + 2gx; \quad (D): V = \sqrt{2gx}; \quad (E): x = V_0t + \frac{1}{2}gt^2$$

#### Liquid Pressure,

The liquid pressure,  $P$  at the bottom of a liquid of depth  $h$  units is given by  $P = \rho gh$

$x$ - direction linearized Navier–Stokes equation:

$$-\mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \frac{\partial p_x}{\partial x} + 4\rho\frac{\partial v_x}{\partial t} = \rho g_x$$

$x$ - direction Navier–Stokes solution :

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{1}{4}fg_x t + C_9; \quad P(x) = d\rho g_x x$$

Observe the following above:

1. Observe that the first three terms of the solution are parabolic in  $x$ ,  $y$ , and  $z$ ; the minus sign showing the inverted parabola when a projectile is fired upwards at an acute angle to the horizontal; Also note the " $gt$ " in  $V = gt$  of (B) of the motion equations and the  $fg_x t$  in the Navier-Stokes solution.

2. Observe the  $P = \rho gh$  of the liquid pressure and the  $P(x) = d\rho g_x x$  of the Navier-Stokes solution.

Note that  $d$  is a ratio term.

There are five main terms in the solution of the linearized Navier-Stokes equation. All of these five

terms, namely,  $-\frac{a\rho g_x}{2\mu}x^2$ ,  $-\frac{b\rho g_x}{2\mu}y^2$ ,  $-\frac{c\rho g_x}{2\mu}z^2$ ,  $fg_x t$ , and  $d\rho g_x x$  are similar (except for the

constants involved) to the terms in the equations of motion and fluid pressure of elementary physics. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these five terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the linearized Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.

The author also tried the following possible approaches: (D), (E) and (F), but none of the possible solutions completely satisfied the corresponding original equations (D), (E) or (F) .

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x} \quad (\text{D}) \quad (\text{One uses the subject } \boxed{\frac{\partial p}{\partial x}})$$

$$\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t} \quad (\text{E}), \quad (\text{One uses the subject } \boxed{\frac{\partial V_x}{\partial t}})$$

$$-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2} \quad (\text{F}) \quad (\text{One uses subject } \boxed{\frac{\partial^2 V_x}{\partial x^2}})$$

### Integration Results Summary

$$\text{Case 1: } -\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 4\rho \left( \frac{\partial V_x}{\partial t} \right) = \rho g_x \quad (\text{C})$$

$$\boxed{V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9} \quad \leftarrow \text{---Solution}$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + f = 1)$$

$$\text{Case 2: } \mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x} \quad (\text{D}). \quad (\text{One uses the subject } \boxed{\frac{\partial p}{\partial x}})$$

$$\boxed{V_x(x,y,z,t) = \frac{\lambda_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + \lambda_p x + C_3y + C_5z - \frac{f\lambda}{4\rho}t + C}$$

$$P(x) = \frac{1}{d} \rho g_x x$$

$$\text{Case 3: } \frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t} \quad (\text{E}). \quad (\text{One uses the subject } \boxed{\frac{\partial V_x}{\partial t}})$$

$$\boxed{V_x(x,y,z,t) = (C_1 \cos \lambda_x x + C_2 \sin \lambda_x x) e^{-(\lambda^2/\beta)t} + (C_3 \cos \lambda_y y + C_4 \sin \lambda_y y) e^{-(\lambda_y^2/\omega)t} + (C_5 \cos \lambda_z z + C_6 \sin \lambda_z z) e^{-(\lambda_z^2/\epsilon)t} + \frac{g}{4f}t + \lambda x + C_8}$$

$$P(x) = \lambda x = d\rho g_x x$$

$$\text{Case 4: } -\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2} \quad (\text{F}). \quad (\text{One uses the subject } \boxed{\frac{\partial^2 V_x}{\partial x^2}})$$

$$\boxed{V_x(x,y,z,t) = (A \cos \lambda y + B \sin \lambda y) \left( C e^{(\frac{\lambda\sqrt{a}}{a})x} + D e^{-(\frac{\lambda\sqrt{a}}{a})x} \right) + (E \cos \lambda z + F \sin \lambda z) \left( H e^{(\frac{\lambda\sqrt{b}}{b})x} + L e^{-(\frac{\lambda\sqrt{b}}{b})x} \right) - \frac{\rho g_x x^2}{2c\mu} + Ax + B + (A_1 \cos \lambda x + B_1 \sin \lambda x) e^{-(\lambda^2/\alpha)t} + \frac{\lambda}{2\mu f} x^2 + C_2 x + C_3; \quad P(x) = d\rho g_x x}$$

**Note:** Relations for equations with subjects  $g_x$  and  $\frac{\partial p}{\partial x}$  are almost identical.

By comparing possible solutions for equations (C) and (D),  $\lambda_x = -\rho g_x$  in relation for (D).

$$\boxed{V_x(x,y,z,t) = \frac{\lambda_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1x + \lambda_p x + C_3y + C_5z - \frac{f\lambda}{4\rho}t + C; \quad P(x) = \frac{1}{d} \rho g_x x}$$

**Comparative analysis of the possible solutions when checked in each corresponding equation**

Equation	Equation Subject	Number of terms of possible solutions <b>not</b> satisfying original equation
Case 1: $-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho\left(\frac{\partial V_x}{\partial t}\right) = \rho g_x$	$\rho g_x$	None Case 1 yields the solution
Case 2: $\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	One term
Case 3: $\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	At least 2 terms
Case 4: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	At least 2 terms
Case 5: $-\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	At least 2 terms
Case 6: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial x^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	At least 2 terms

Note above that only Case 1 is the solution, and this may imply that the solution to the Navier-Stokes equation is unique. Out of six possible subjects, only one subject produced a solution. The above results show that a relation obtained by the integration of a partial differential equation must be checked in the corresponding equation for identity before the relation becomes a solution, Cases 2, 3, 4, 5 and 6, are not solutions but integration relations. For example, it would be incorrect to say that the equation in Case 3 has a periodic solution; but it would be correct to say that the equation in Case 3 has a periodic relation, since the relation obtained by integration does not satisfy its corresponding equation. It would be correct to say that the equation in Case 1 has a parabolic solution or a parabolic relation.

Below are detailed explanation of results of the identity checking process.

**Outcome 1:** With  $g_x$  included and with  $g_x$  as the subject of the equation. The solution is straightforward and the possible solution checks well in the original equation (C). Also, if  $g_x$  or  $\rho g_x$  is not the subject of the equation, the linearization of the nonlinear terms could not be justified.

**Outcome 2:** With  $g_x$  included but with  $\frac{\partial V_x}{\partial t}$  as the subject of the equation.

There are two problems when checking . **1.** For  $\frac{\partial V_x}{\partial t} = -\frac{1}{4\rho} \frac{\partial p}{\partial x} \rightarrow -\frac{\lambda t}{4\rho d}$ ; **2.**  $\frac{g_x}{4} = \frac{\partial V_x}{\partial t} \rightarrow \frac{g_x t}{4f}$

With  $d$  and  $f$  in the denominators, the multipliers sum  $a + b + c + d + f = 1$  is false.

**Outcome 3 :** With  $g_x$  excluded, and  $\frac{\partial V_x}{\partial t}$  as the subject of the equation, there is one problem:

$-\frac{1}{4\rho} \frac{\partial p}{\partial x} = \frac{\partial V_x}{\partial t} \rightarrow -\frac{\lambda t}{4\rho d}$ . With  $d$  in the denominator  $a + b + c + d + f = 1$  is false

**Outcome 4 :** With  $g_x$  included, and  $\frac{\partial^2 V_x}{\partial x^2}$  as the subject of the equation, there are at least, two problems in the checking with the multipliers  $c$  and  $f$  in the denominators.

Checking for  $a + b + c + d + f = 1$  is impossible.

**Outcomes 5 and 6** are similar to Outcome 4.

**Characteristic curves of the integration results**

Equations	Equation Subject	Curve characteristics
<b>Case 1:</b> $-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho\left(\frac{\partial V_x}{\partial t}\right) = \rho g_x$	$\rho g_x$	Parabolic and Inverted
<b>Case 2:</b> $\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	Parabolic
<b>Case 3:</b> $\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	Periodic and decreasingly exponential
<b>Case 4:</b> $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	Periodic, parabolic, and decreasingly exponential
<b>Case 5:</b> $-\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	Periodic, parabolic, and decreasingly exponential
<b>Case 6:</b> $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial x^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	Periodic, parabolic, and decreasingly exponential

The following are possible interpretations of the roles of the terms based on the types of curves produced when using the terms as subjects of the equations.

1.  $g_x$  and  $\frac{\partial p}{\partial x}$  are involved in the parabolic motion;  $g_x$  is responsible for the forward motion.
2.  $\frac{\partial V_x}{\partial t}$  is involved in the periodic and decreasingly exponential behavior.
3.  $\frac{\partial^2 V_x}{\partial x^2}$ ,  $\frac{\partial^2 V_x}{\partial y^2}$  and  $\frac{\partial^2 V_x}{\partial z^2}$  are involved in the parabolic, periodic and decreasingly exponential motion. As  $\mu$  increases, the periodicity increases



### Definitions and Classification of Equations

$$\boxed{-K \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - K \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + 4 \frac{\partial V_x}{\partial t} = g_x} \quad (k = \frac{\mu}{\rho})$$

One may classify the equations involved in Option 1 according to the following:

**Driver Equation:** A differential equation whose integration relation satisfies its corresponding equation.

**Supporter equation:** A differential equation which contains the same terms as the driver equation but whose integration relation does not satisfy its corresponding equation but provides useful information about the driver equation.

Note that the driver equation and a supporter equation differ only in the subject of the equation.

Equation	Equation Subject	Type of equation	# of terms of relation not satisfying original equation
Case 1: $-\mu(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \frac{\partial p}{\partial x} + 4\rho(\frac{\partial V_x}{\partial t}) = \rho g_x$	$\rho g_x$	Driver Equation	None
Case 2: $\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x - 4\rho \frac{\partial V_x}{\partial t} = \frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial x}$	Supporter equation	One term
Case 3: $\frac{K}{4} \frac{\partial^2 V_x}{\partial x^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial y^2} + \frac{K}{4} \frac{\partial^2 V_x}{\partial z^2} - \frac{1}{4\rho} \frac{\partial p}{\partial x} + \frac{g_x}{4} = \frac{\partial V_x}{\partial t}$	$\frac{\partial V_x}{\partial t}$	Supporter equation	At least 2 terms
Case 4: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial x^2}$	$\frac{\partial^2 V_x}{\partial x^2}$	Supporter equation	At least 2 terms
Case 5: $-\frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial z^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial y^2}$	$\frac{\partial^2 V_x}{\partial y^2}$	Supporter equation	At least 2 terms
Case 6: $-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial x^2} - \frac{\rho g_x}{\mu} + \frac{4\rho}{\mu} \frac{\partial V_x}{\partial t} + \frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial z^2}$	$\frac{\partial^2 V_x}{\partial z^2}$	Supporter equation	At least 2 terms

The uniqueness of the above solution will guide one to save time and not try to solve some forms of Euler or Navier-Stokes equation which do not produce solutions. That is, one will solve only the equations with the gravity term as the subject. This uniqueness will also guide one to solve the magnetohydrodynamic equations.

## Applications of the splitting technique in science, engineering, business fields

The approach used in solving the equations allows for how the terms interact with each other. The author has not seen this technique anywhere, but the results are revealing and promising.

### Fluid flow design considerations:

1. Maximize the role of  $g_x$  forces, followed by;
2.  $\frac{\partial p}{\partial x}$  forces; then
3.  $\frac{\partial V_x}{\partial t}$

Make  $g_x$  happy by always providing a workable slope.

For long distance flow design such as for water pipelines, water channels, oil pipelines. whenever possible, the design should facilitate and maximize the role of gravity forces, and if design is

impossible to facilitate the role of gravity forces, design for  $\frac{\partial p}{\partial x}$  to take over flow.

The performance of  $\frac{\partial^2 V_x}{\partial x^2}$  should be studied further, since its role is the most complicated: periodic, parabolic, and decreasingly exponential.

### Tornado Effect Relief

Perhaps, machines can be designed and built to chase and neutralize or minimize tornadoes during touch-downs. The energy in the tornado at touch-down can be harnessed for useful purposes.

### Business and economics applications.

1. Figuratively, if  $g_x$  is the president of a company, it will have good working relationships with all the members of the board of directors, according to the solution of the Navier-Stokes equation. If  $g_x$  is present at a meeting  $g_x$  must preside over the meeting for the best outcome.

2. If  $g_x$  is absent from a meeting, let  $\frac{\partial p}{\partial x}$  preside over the meeting, and everything will workout well. However, if  $g_x$  is present,  $g_x$  must preside over the meeting.

To apply the results of the solutions of the Navier-Stokes equations in other areas or fields, the properties, characteristics and functions of  $g_x$ ,  $\frac{\partial p}{\partial x}$ ,  $\frac{\partial v_x}{\partial t}$  must be studied to determine analogous terms in those areas of possible applications. Other areas of applications include investments choice decisions, financial decisions, personnel management and family relationships.

## Option 2

### Solutions of 4-D Linearized Navier-Stokes Equations

One advantage of the pairing approach is that the above solution can easily be extended to any number of dimensions.

If one adds  $\mu \frac{\partial^2 V_x}{\partial s^2}$  and  $\rho V_s \frac{\partial V_x}{\partial s}$  to the 3-D  $x$ -direction equation, one obtains the 4-D Navier-

Stokes equation  $-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial s^2} \right) + \frac{\partial p}{\partial x} + 4\rho \left( \frac{\partial V_x}{\partial t} \right) + \rho V_s \frac{\partial V_x}{\partial s} = \rho g_x$

After linearization,  $-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial s^2} \right) + \frac{\partial p}{\partial x} + 5\rho \left( \frac{\partial V_x}{\partial t} \right) = \rho g_x$  and its solution is

$$\boxed{V_x(x,y,z,s,t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2 + es^2) + C_1x + C_3y + C_5z + C_7s + \frac{fg_x}{5}t + C_9}$$

$$P(x) = d\rho g_x x \quad (a + b + c + d + e + f = 1)$$

For  $n$ -dimensions one can repeat the above as many times as one wishes.

## Option 3

### Solutions of the Euler Equations of Fluid flow

In the Navier-Stokes equation, if  $\mu = 0$ , one obtains the Euler equation. From

$$\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x = \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right), \text{ one obtains}$$

Euler equation : ( $\mu = 0$ )  $-\frac{\partial p}{\partial x} + \rho g_x = \rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right)$  or

$$\rho \left( \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) + \frac{\partial p_x}{\partial x} = \rho g_x \quad \leftarrow \text{---driver equation.}$$

Euler equation ( $\mu = 0$ ):  $\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x$   $\leftarrow$  driver equation

Split the equation using the ratio terms  $f, h, n, q, d,$  and solve. ( $f + h + n + q + d = 1$ )

<b>1.</b> $\frac{\partial V_x}{\partial t} = fg_x$ $V_{x4} = fg_x t$ $V_{x4} = fg_x t$	<b>2.</b> $V_x \frac{\partial V_x}{\partial x} = hg_x$ $V_x \frac{dV_x}{dx} = hg_x$ $V_x dV_x = hg_x dx$ $\frac{V_x^2}{2} = hg_x x$ or $V_x^2 = 2hg_x x$ $V_x = \pm \sqrt{2hg_x x}$	<b>3.</b> $V_y \frac{\partial V_x}{\partial y} = ng_x$ $V_y \frac{dV_x}{dy} = ng_x$ $V_y dV_x = ng_x dy$ $V_y V_x = ng_x y + \psi_y(V_y)$ $V_{x6} = \frac{ng_x y}{V_y} + \frac{\psi_y(V_y)}{V_y}$ $V_y \neq 0$	<b>4.</b> $V_z \frac{\partial V_x}{\partial z} = qg_x$ $V_z \frac{dV_x}{dz} = qg_x$ $V_z dV_x = qg_x dz$ ; $V_z V_x = qg_x z + \psi_z(V_z)$ $V_{x7} = \frac{qg_x z}{V_z} + \frac{\psi_z(V_z)}{V_z}$ $V_z \neq 0$	<b>5.</b> $\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x$ $\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x$ $\frac{\partial p}{\partial x} = d\rho g_x$ $p = d\rho g_x x + C_7$
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$$V_x(x, y, z, t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C$$

$$P(x) = d\rho g_x x \quad (f + h + n + q + d = 1) \quad V_y \neq 0, V_z \neq 0$$

Find the test derivatives to check in the original equation.

<b>1.</b> $\frac{\partial V_x}{\partial t} = fg_x$	<b>2.</b> $V_x^2 = 2hg_x x$ ; $2V_x \frac{\partial V_x}{\partial x} = 2hg_x$ ; $\frac{\partial V_x}{\partial x} = \frac{hg_x}{V_x}, V_x \neq 0$	<b>3.</b> $\frac{\partial V_x}{\partial y} = \frac{ng_x}{V_y}$ $V_y \neq 0$	<b>4.</b> $\frac{\partial V_x}{\partial z} = \frac{qg_x}{V_z}$ $V_z \neq 0$	<b>5.</b> $\frac{\partial p}{\partial x} = d\rho g_x$
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$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x \quad (\text{Above, } \psi_y(V_y) \text{ and } \psi_z(V_z) \text{ are arbitrary functions})$$

$$fg_x + V_x \frac{hg_x}{V_x} + V_y \frac{ng_x}{V_y} + V_z \frac{qg_x}{V_z} + \frac{1}{\rho} d\rho g_x = g_x$$

$$fg_x + hg_x + ng_x + qg_x + dg_x = g_x$$

$$g_x (f + h + n + q + d) = g_x$$

$$g_x (1) = g_x \quad (f + h + n + q + d = 1)$$

$$g_x = g_x \quad \text{Yes}$$

The relation obtained satisfies the Euler equation. Therefore the solution to the Euler equation

$$: \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x \text{ is}$$

$$\boxed{V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}; P(x) = d\rho g_x x} \quad \text{x-direction}$$

$$V_y \neq 0, V_z \neq 0; (d + f + h + n + q = 1)$$

Similarly, the equations and solutions for the other two directions are respectively

$$\text{For } V_y, \frac{\partial p}{\partial y} + \rho \frac{\partial v_y}{\partial t} + \rho V_x \frac{\partial v_y}{\partial x} + \rho V_y \frac{\partial v_y}{\partial y} + \rho V_z \frac{\partial v_y}{\partial z} = \rho g_y$$

$$\boxed{V_y(x,y,z,t) = \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_y x}{V_x} + \frac{\lambda_8 g_y z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}; P(y) = \lambda_4 \rho g_y y} \quad \text{y-direction}$$

$$V_x \neq 0, V_z \neq 0; (\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1)$$

$$\text{For } V_z, \frac{\partial p}{\partial z} + \rho \frac{\partial v_z}{\partial t} + \rho V_x \frac{\partial v_z}{\partial x} + \rho V_y \frac{\partial v_z}{\partial y} + \rho V_z \frac{\partial v_z}{\partial z} = \rho g_z$$

$$\boxed{V_z(x,y,z,t) = \beta_3 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_z x}{V_x} + \frac{\beta_7 g_z y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}; P(z) = \beta_4 \rho g_z z} \quad \text{z-direction}$$

$$V_x \neq 0, V_y \neq 0; (\beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1)$$

One will next solve the above system of solutions for  $V_x, V_y, V_z$  in order to express

$$\frac{ng_x y}{V_y} \text{ and } \frac{q_e g_x z}{V_z} \text{ in terms of } x, y, z, \text{ and } t.$$

**Solving for**  $V_x, V_y, V_z, \frac{ng_x y}{V_y}$ , and  $\frac{q_e g_x z}{V_z}$

$$\left\{ \begin{array}{l} V_x = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}. \quad (A) \\ V_y = \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_y x}{V_x} + \frac{\lambda_8 g_y z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}. \quad (B) \\ V_z = \beta_3 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_z x}{V_x} + \frac{\beta_7 g_z y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}. \quad (C) \end{array} \right.$$

Let  $V_x = x, V_y = y$  and  $V_z = z$ . ( $x, y$  and  $z$  are being used for simplicity. They will be changed back to  $V_x, V_y$ , and  $V_z$  later, and they do not represent the variables  $x, y$  and  $z$  in the system of solutions)

<p><b>Step 1</b> From the above system of solutions, let  <math>A = (fgt + \sqrt{2hg_x x})</math>; <math>D = (qg_x z)</math>; <math>E = (ng_x y)</math>  <math>B = (\lambda_5 g_y t + \sqrt{2\lambda_7 g_y y})</math>; <math>F = (\lambda_6 g_y x)</math>; <math>G = (\lambda_8 g_y z)</math>  <math>C = (\beta_5 g_z t + \sqrt{2\beta_8 g_z z})</math>; <math>J = (\beta_6 g_z x)</math>; <math>L = (\beta_7 g_z y)</math></p>	<p><b>Step 2:</b> Then the solutions to the Euler system of equations become (ignoring the arbitrary functions)</p> $\left. \begin{aligned} x &= A + \frac{D}{z} + \frac{E}{y} \\ y &= B + \frac{F}{x} + \frac{G}{z} \\ z &= C + \frac{J}{x} + \frac{L}{y} \end{aligned} \right\} M$
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<p><b>Step 3</b></p> $\left. \begin{aligned} xyz &= Ayz + Dy + Ez & (1) \\ xyz &= Bxz + Fz + Gx & (2) \\ xyz &= Cxy + Jy + Lx & (3) \end{aligned} \right\} N$	<p><b>Step 4</b></p> $\left. \begin{aligned} 0 &= Ayz + Dy + Ez - Bxz - Fz - Gx & (4) \\ 0 &= Ayz + Dy + Ez - Cxy - Jy - Lx & (5) \\ 0 &= Bxz + Fz + Gx - Cxy - Jy + -Lx & (6) \end{aligned} \right\} P$
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Maples software was used to solve system P to obtain

<p><b>Step 5</b></p> $x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ $V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \quad (\text{back to } V_x)$ $y = -\frac{L}{C};$ $V_y = -\frac{L}{C} \quad (\text{changing back to } V_y \text{ as agreed to})$ $z = -\frac{L(D - J)}{LA - CE};$ $V_z = -\frac{L(D - J)}{LA - CE} \quad (\text{changing back to } V_z \text{ as agreed to})$	<p><b>Note:</b>                  None of the popular academic programs could solve the system in M.                  Maples solved system P (step 4 above) for <math>x</math>, <math>y</math>, and <math>z</math> in terms of <math>A, B, C, D, E, F, G, J</math>, and <math>L</math>.                  Note also that <math>x, y</math> and <math>z</math> are not the same as the <math>x, y</math> and <math>z</math> in the system of equations..                  They were used for convenience and simplicity .</p>
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Step 5: Apply and substitute from in steps 6-8 below

$$A = (fgt \pm \sqrt{2hg_x x}) ; B = (\lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y}) ; C = (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}) ; D = (qg_x z) ; E = (ng_x y) ; F = (\lambda_6 g_y x) ; G = (\lambda_8 g_y z) ; J = (\beta_6 g_z x) ; L = (\beta_7 g_z y)$$

<p><b>Step 6</b></p> $V_y = -\frac{L}{C} = -\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})}$ $\frac{ng_x y}{V_y} = ng_x y \div -\left(\frac{(\beta_7 g_z y)}{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})}\right)$ $\frac{ng_x y}{V_y} = -\frac{(ng_x y)[\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}]}{\beta_7 g_z y}; \quad y \neq 0$	<p><b>Step 7</b></p> $\frac{ng_x y}{V_y} = -\frac{[\beta_5 g_z t](ng_x y) \pm \sqrt{2\beta_8 g_z z}(ng_x y)}{\beta_7 g_z y}$ $\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}$
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$\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z}$
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**Step 8:**

$$V_z = -\frac{L(D-J)}{LA-CE}$$

$$V_z = -\frac{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}{(\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}$$

$$\frac{qg_x z}{V_z} = (qg_x z) \div -\left(\frac{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}{(\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}\right)$$

$$= -\frac{(qg_x z) \cdot (\beta_7 g_z y)(fg_x t \pm \sqrt{2hg_x x}) - (\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})(ng_x y)}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$= -\frac{(qg_x z) \cdot (\beta_7 g_z y)fg_x t \pm \sqrt{2hg_x x}\beta_7 g_z y - \beta_5 g_z tng_x y \pm \sqrt{2\beta_8 g_z z}ng_x y}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$= -\frac{qg_x g_x z \beta_7 f g_z y t \pm \sqrt{2hg_x x} \beta_7 g_x g_z q y z - \beta_5 g_x g_z n q t y z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q y z}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

$$\frac{qg_x z}{V_z} = -\frac{qg_x g_x z \beta_7 f g_z t \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q z}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]}$$

((Dividing out the "y" in the numerator and the denominator)

$$\frac{qg_x z}{V_z} = -\frac{qg_x g_x z \beta_7 f g_z t \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q z}{(\beta_7 g_z y)[qg_x z - \beta_6 g_z x]} \quad \text{or}$$

$$\frac{qg_x z}{V_z} = \frac{qg_x g_x z \beta_7 f g_z t \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z - \beta_5 g_x g_z n q t z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z}$$

$$\boxed{\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_z g_x g_x q - \beta_5 g_x g_z n q) t z \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z}}$$

**Summary for the fractional terms of the x-direction solution**

$$\frac{ng_x y}{V_y} \text{ and } \frac{qg_x z}{V_z} \text{ in terms of } x, y, z \text{ and } t$$

$$\left. \frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z} \right\} \text{B}$$

$$\frac{ng_x y}{V_y} = -k_1 g_z t \pm \frac{\sqrt{2k_2 g_z z} \cdot g_x k_3}{g_z}$$

$$k_1 = \frac{n\beta_5}{\beta_7}; \quad k_2 = \beta_8; \quad k_3 = \frac{n}{\beta_7}$$

;

$$\left. \frac{qg_x z}{V_z} = \frac{(\beta_7 f_e g_z g_x g_x q - \beta_5 g_x g_z n_e q) t z \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z \pm \sqrt{2\beta_8 g_z z} g_x g_x n q z}{\beta_7 \beta_6 g_z g_z x - \beta_7 q g_z g_x z} \right\} \text{C}$$

$$\frac{qg_x z}{V_z} = \frac{(g_x^2 g_z k_4 - g_x g_z k_5) t z \pm \sqrt{2g_x k_6 x} \cdot g_x g_z k_7 z \pm \sqrt{2g_z k_8 z} \cdot g_x^2 k_9 z}{g_z^2 k_{10} x - g_x g_z k_{11} z}$$

$$k_4 = \beta_7 f q; \quad k_5 = \beta_5 n q; \quad k_6 = h; \quad k_8 = \beta_8; \quad k_9 = n q \quad k_{10} = \beta_7 \beta_6 \quad k_{11} = \beta_7 q$$

## Analysis of the Euler Solutions

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}; P(x) = d\rho g_x x$$

x-direction

$$V_y \neq 0, V_z \neq 0; (d + f + h + n + q = 1)$$

$$\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_x t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_x z})(ng_x)}{\beta_7 g_x} \Big\} B$$

$$\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_x g_x^2 q - \beta_5 g_x g_x n q) t z \pm \sqrt{2hg_x x} \beta_7 g_x g_x q z \pm \sqrt{2\beta_8 g_x z} g_x^2 n q z}{\beta_7 \beta_6 g_x^2 x - \beta_7 q g_x g_x z} \Big\} C$$

$$d + f + h + n + q = 1; \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1; \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity,  $g_x$  were zero, the first four terms of the velocity solution and  $P(x)$  would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

**More Observations: Comparison of the Euler solutions with equations of motion under gravity and liquid pressure of elementary physics**

**Motion under gravity equations:** (B):  $V = gt$ ; (C):  $V = \sqrt{2gx}$  ;

**Liquid Pressure,  $P$**  at the bottom of a liquid of depth  $h$  units is given by  $P = \rho gh$

Observe the following similarities above:

1. Observe the " $gt$ " in  $V = gt$  of (B) of the motion equations and the  $fg_x t$  in the Euler solution.
2. Observe the " $\sqrt{2gx}$ " in  $V = \sqrt{2gx}$  of (C) and the  $\sqrt{2hg_x x}$  in the Euler solution.
3. Observe the  $P = \rho gh$  of the liquid pressure and the  $P(x) = d\rho g_x x$  of the Euler solution.

There are five main terms (ignoring the arbitrary functions) in the Euler solution. Of these five terms, three terms, namely,  $fg_x t$ ,  $\sqrt{2hg_x x}$ ,  $d\rho g_x x$  are the same (except for the constants involved) as the terms in the equations of motion under gravity. This similarity means that the approach used in solving the Euler equation is sound. One should also note that to obtain these three terms simultaneously, only the equation with the gravity term as the subject of the equation will yield these three terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Euler equation, since in this form, one can immediately split-up the equations using ratios, and integrate.

### Velocity profile

The  $x$ -direction solution consists of linear, parabolic, and hyperbolic terms. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the  $x$ -nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The  $y$ -nonlinear term is similar parabolically to the  $x$ -nonlinear term. The characteristic curve for the  $z$ -nonlinear term is a combination of two similar parabolas and a hyperbola. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent

**Standard form of the x-direction Euler equation for incompressible fluid flow**

One will call the Euler equation with the gravity term as the subject of equation in (A), the standard form of the Euler equation for the ratio method of solving these equations, since this form produces a solution on integration. None of the other forms in (B), (C), (D), (E), or (F), produces a solution. That is, the integration results of each of the other five equations do not satisfy the corresponding equation.

$$\rho \frac{\partial v_x}{\partial t} + \rho V_x \frac{\partial v_x}{\partial x} + \rho V_y \frac{\partial v_x}{\partial y} + \rho V_z \frac{\partial v_x}{\partial z} + \frac{\partial p_x}{\partial x} = \rho g_x \quad (\text{A}) < \text{standard form}$$

$$-\rho \frac{\partial v_x}{\partial t} - \rho V_x \frac{\partial v_x}{\partial x} - \rho V_y \frac{\partial v_x}{\partial y} - \rho V_z \frac{\partial v_x}{\partial z} + \rho g_x = \frac{\partial p_x}{\partial x} \quad (\text{B})$$

$$-\rho V_x \frac{\partial v_x}{\partial x} - \rho V_y \frac{\partial v_x}{\partial y} - \rho V_z \frac{\partial v_x}{\partial z} + \frac{\partial p_x}{\partial x} + \rho g_x = \rho \frac{\partial v_x}{\partial t} \quad (\text{C})$$

$$-\rho \frac{\partial v_x}{\partial t} - \rho V_y \frac{\partial v_x}{\partial y} - \rho V_z \frac{\partial v_x}{\partial z} - \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_x \frac{\partial v_x}{\partial x} \quad (\text{D})$$

$$-\rho \frac{\partial v_x}{\partial t} - \rho V_x \frac{\partial v_x}{\partial x} - \rho V_z \frac{\partial v_x}{\partial z} - \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_y \frac{\partial v_x}{\partial y} \quad (\text{E})$$

$$-\rho \frac{\partial v_x}{\partial t} - \rho V_x \frac{\partial v_x}{\partial x} - \rho V_y \frac{\partial v_x}{\partial y} - \frac{\partial p_x}{\partial x} + \rho g_x = \rho V_z \frac{\partial v_x}{\partial z} \quad (\text{F})$$

**Uniqueness of the solution of the Euler equation**

When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution.

(vixra:1405.0251 of 2014). Similarly, the solution of the Euler solution is unique.

**Extra:**

**Linearized Euler Equation:** If one linearizes the Euler equation as was done in the linearization of

the Navier-Stokes equation, one obtains  $4 \frac{\partial v_x}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x$ ; whose solution is

$$V_x = \frac{fg_x}{4} t + C; \quad P(x) = dp g_x x.$$

One May also solve the following for  $V_x$ ,  $V_y$ ,  $V_z$  in terms of  $x$ ,  $y$  and  $z$ .

<p><b>A Euler solution system to solve for <math>V_x, V_y, V_z</math></b></p> $V_x = \frac{(fg_x t \pm \sqrt{2hg_x x})V_y V_z + [qg_x z + \psi_z(V_z)]V_y + [ng_x y + \psi_y(V_y)]V_z}{V_y V_z}$ $V_y = \frac{(\lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y})V_x V_z + [\lambda_8 g_y z + \psi_z(V_z)]V_x + [\lambda_6 g_y x + \psi_x(V_x)]V_z}{V_x V_z}$ $V_z = \frac{(\beta_5 g_z t \pm \sqrt{2\beta_8 g_z z})V_x V_y + [\beta_6 g_z x + \psi_x(V_x)]V_y + [\beta_7 g_z y + \psi_y(V_y)]V_x}{V_x V_y}$
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**Note:** By comparison with Navier-Stokes equation and its relation, a relation to Euler equation can be found by deleting the Navier-Stokes relation resulting from the  $\mu$ -terms.



## Option 4 Solutions of 3-D Navier-Stokes Equations (Original)

### Mehod 1

As in Option 1 for solving these equations, the first step here, is to split-up the equation into eight sub-equations using the ratio method. One will solve **only** the driver equation, based on the experience gained in solving the linearized equation. There are 8 supporter equations.

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \overbrace{\rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}}^{\text{nonlinear terms}} = \rho g_x \quad (\text{A})$$

$$-K \frac{\partial^2 V_x}{\partial x^2} - K \frac{\partial^2 V_x}{\partial y^2} - K \frac{\partial^2 V_x}{\partial z^2} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} = g_x \quad (K = \frac{\mu}{\rho}) \quad (\text{B})$$

**Step 1:** Apply the ratio method to equation (B) to obtain the following equations:

$$1. -K \frac{\partial^2 V_x}{\partial x^2} = ag_x; \quad 2. -K \frac{\partial^2 V_x}{\partial y^2} = bg_x; \quad 3. -K \frac{\partial^2 V_x}{\partial z^2} = cg_x; \quad 4. \frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x; \quad 5. \frac{\partial V_x}{\partial t} = fg_x$$

$$6. V_x \frac{\partial V_x}{\partial x} = hg_x; \quad 7. V_y \frac{\partial V_x}{\partial y} = qg_x; \quad 8. V_z \frac{\partial V_x}{\partial z} = ng_x$$

where  $a, b, c, d, f, h, n, q$  are the ratio terms and  $a + b + c + d + f + h + n + q = 1$

**Step 2:** Solve the differential equations in Step 1.

**Note** that after splitting the equations, the equations can be solved using techniques of ordinary differential equations.

One can view each of the ratio terms  $a, b, c, d, f, h, n, q$  as a fraction (a real number) of  $\boxed{g_x}$  contributed by each expression on the left-hand side of equation (B) above.

### Solutions of the eight sub-equations

<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">1. <math>-k \frac{\partial^2 V_x}{\partial x^2} = ag_x</math></div> $k \frac{\partial^2 V_x}{\partial x^2} = -ag_x$ $\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{k} g_x$ $\frac{\partial V_x}{\partial x} = -\frac{ag}{k} x + C_1$ $V_{x1} = -\frac{ag_x}{2k} x^2 + C_1 x + C_2$	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">2. <math>-K \frac{\partial^2 V_x}{\partial y^2} = bg_x</math></div> $K \frac{\partial^2 V_x}{\partial y^2} = -bg_x$ $\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{k} g_x$ $\frac{\partial V_x}{\partial y} = -\frac{bg_x}{k} y + C_3$ $V_{x2} = -\frac{bg_x}{2k} y^2 + C_3 y + C_4$	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">3. <math>-K \frac{\partial^2 V_x}{\partial z^2} = cg_x</math></div> $K \frac{\partial^2 V_x}{\partial z^2} = -cg_x$ $\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{k} g_x$ $\frac{\partial V_x}{\partial z} = -\frac{cg_x}{k} z + C_5$ $V_{x3} = -\frac{cg_x}{2k} z^2 + C_5 z + C_6$	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">4. <math>\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x</math></div> $\frac{1}{\rho} \frac{\partial p}{\partial x} = dg_x$ $\frac{\partial p}{\partial x} = d\rho g_x$ $p = d\rho g_x x + C_7$ <div style="border: 1px solid black; padding: 5px; margin-top: 5px;">5. <math>\frac{\partial V_x}{\partial t} = fg_x</math></div> $V_{x4} = fgt$
<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">6. <math>V_x \frac{\partial V_x}{\partial x} = hg_x</math></div> $V_x \frac{dV_x}{dx} = hg_x$ $V_x dV_x = hg_x dx$ $\frac{V_x^2}{2} = hg_x x$ $V_{x5} = \pm \sqrt{2hg_x x} + C_7$	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">7. <math>V_y \frac{\partial V_x}{\partial y} = ng_x</math></div> $V_y \frac{dV_x}{dy} = ng_x$ $V_y dV_x = ng_x dy$ $V_y V_x = ng_x y + \psi_y(V_y)$ $V_{x6} = \frac{ng_x y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	<div style="border: 1px solid black; padding: 5px; margin-bottom: 5px;">8. <math>V_z \frac{\partial V_x}{\partial z} = qg_x</math></div> $V_z \frac{dV_x}{dz} = qg_x$ $V_z dV_x = qg_x dz$ $V_z V_x = qg_x z + \psi_z(V_z)$ $V_{x7} = \frac{qg_x z}{V_z} + \frac{\psi_z(V_z)}{V_z}$	<p><b>Note:</b>  <math>\psi_y(V_y), \psi_z(V_z)</math>                      are arbitrary functions,                      (integration constants)  <math>V_y \neq 0</math>  <math>V_z \neq 0</math></p>

**Step 3:** One combines the above solutions

$$\begin{aligned}
 V_x(x,y,z,t) &= V_{x1} + V_{x2} + V_{x3} + V_{x4} + V_{x5} + V_{x6} + V_{x7} \\
 &= -\frac{ag_x}{2k}x^2 + C_1x - \frac{bg_x}{2k}y^2 + C_3y - \frac{cg_x}{2k}z^2 + C_5z + fg_xt \pm \sqrt{2hg_xx} + \frac{ng_xy}{V_y} + \frac{qg_xz}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} \\
 &\quad \underbrace{\hspace{10em}}_{\text{relation for linear terms}} \quad \underbrace{\hspace{10em}}_{\text{relation for non-linear terms}} \\
 &= -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_xt \pm \sqrt{2hg_xx} + \frac{ng_xy}{V_y} + \frac{qg_xz}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C_9 \\
 P(x) &= d\rho g_x x; \quad (a + b + c + d + f + h + n + q = 1) \quad v_y \neq 0, v_z \neq 0
 \end{aligned}$$

**Step 4:** Find the test derivatives

Test derivatives for the linear part				Test derivatives for the non-linear part			
$\frac{\partial^2 V_x}{\partial x^2} = -\frac{a\rho g_x}{\mu}$	$\frac{\partial^2 V_x}{\partial y^2} = -\frac{b\rho g_x}{\mu}$	$\frac{\partial^2 V_x}{\partial z^2} = -\frac{c\rho g_x}{\mu}$	$\frac{\partial p}{\partial x} = d\rho g_x$	$\frac{\partial V_x}{\partial t} = fg_x$	$V_x^2 = 2hg_x x$	$\frac{\partial V_x}{\partial y} = \frac{ng_x}{V_y}$	$\frac{\partial V_x}{\partial z} = \frac{qg_x}{V_z}$
					$2V_x \frac{\partial V_x}{\partial x} = 2hg_x$		
					$\frac{\partial V_x}{\partial x} = \frac{hg_x}{V_x}, V_x \neq 0$		

**Step 5:** Substitute the derivatives from Step 4 in equation (A) for the checking.

$$\begin{aligned}
 -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} &= \rho g_x \quad \text{(A)} \\
 -\mu \left( -\frac{a\rho g_x}{\mu} - \frac{b\rho g_x}{\mu} - \frac{c\rho g_x}{\mu} \right) + d\rho g_x + f\rho g_x + \rho \left( V_x \frac{hg_x}{V_x} \right) + \rho V_y \left( \frac{ng_x}{V_y} \right) + \rho V_z \left( \frac{qg_x}{V_z} \right) &= \rho g_x \\
 a\rho g_x + b\rho g_x + c\rho g_x + d\rho g_x + f\rho g_x + h\rho g_x + n\rho g_x + q\rho g_x &= \rho g_x \\
 ag_x + bg_x + cg_x + dg_x + fg_x + hg_x + ng_x + qg_x &= g_x \\
 g_x(a + b + c + d + f + h + n + q) &= g_x \\
 g_x(1) &= g_x \quad \text{Yes} \quad (a + b + c + d + f + h + n + q = 1)
 \end{aligned}$$

**Step 6:** The linear part of the relation satisfies the linear part of the equation; and the non-linear part of the relation satisfies the non-linear part of the equation.(B) below is the solution.

**Analogy for the Identity Checking Method:** If one goes shopping with American dollars and Japanese yens (without any currency conversion) and after shopping, if one wants to check the cost of the items purchased, one would check the cost of the items purchased with dollars against the receipts for the dollars; and one would also check the cost of the items purchased with yens against the receipts for the yens purchase. However, if one converts one currency to the other, one would only have to check the receipts for only a single currency, dollars or yens. This conversion case is similar to the linearized equations, where there was no partitioning in identity checking. Note that for the Euler equations, there was no partitioning in taking derivatives for identity checking.

**Summary of solutions for**  $V_x, V_y, V_z$  ( $P(x) = d\rho g_x x$ ;  $P(y) = \lambda_4 \rho g_y y$ ,  $P(z) = \beta_4 \rho g_z z$ )

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + h + n + q = 1) \quad V_y \neq 0, V_z \neq 0$$

$$V_y = -\frac{\rho g_y}{2\mu}(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_y x}{V_x} + \frac{\lambda_8 g_y z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}$$

$$V_z = -\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_z x}{V_x} + \frac{\beta_7 g_z y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}$$

The above solutions are unique, because from the experience in Option 1, only the equations with the gravity terms as the subjects of the equations produced the solutions.

$$\left\{ \begin{array}{l} V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} \\ V_y = -\frac{\rho g_y}{2\mu}(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y} + \frac{\lambda_6 g_y x}{V_x} + \frac{\lambda_8 g_y z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z} \\ V_z = -\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z} + \frac{\beta_6 g_z x}{V_x} + \frac{\beta_7 g_z y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y} \end{array} \right.$$

One will next solve the above system of solutions for  $V_x, V_y, V_z$  in order to express

$$\frac{ng_x y}{V_y} \text{ and } \frac{qg_x z}{V_z} \text{ in terms of } x, y, z, \text{ and } t.$$

### Solving for $V_x, V_y, V_z, \frac{ng_x y}{V_y}$ , and $\frac{qg_x z}{V_z}$

Let  $V_x = x, V_y = y$  and  $V_z = z$ . ( $x, y$  and  $z$  are being used for simplicity. They will be changed back to  $V_x, V_y$ , and  $V_z$  later, and they do not represent the variables  $x, y$  and  $z$  in the system of solutions)

**Step 1** From the above system of solutions, let

$$A = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x}$$

$$B = -\frac{\rho g_y}{2\mu}(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5 g_y t \pm \sqrt{2\lambda_7 g_y y}$$

$$C = -\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}$$

$$D = qg_x z; \quad E = ng_x y; \quad F = \lambda_6 g_y x$$

$$G = \lambda_8 g_y z; \quad J = \beta_6 g_z x; \quad L = \beta_7 g_z y$$

### Step 2

Then the solutions to the N-S system of equations become (ignoring the arbitrary functions)

$$\left. \begin{array}{l} x = A + \frac{D}{z} + \frac{E}{y} \\ y = B + \frac{F}{x} + \frac{G}{z} \\ z = C + \frac{J}{x} + \frac{L}{y} \end{array} \right\} M$$

<p><b>Step 3</b></p> $\left. \begin{aligned} xyz &= Ayz + Dy + Ez & (1) \\ xyz &= Bxz + Fz + Gx & (2) \\ xyz &= Cxy + Jy + Lx & (3) \end{aligned} \right\} \text{N}$	<p><b>Step 4</b></p> $\left. \begin{aligned} 0 &= Ayz + Dy + Ez - Bxz - Fz - Gx & (4) \\ 0 &= Ayz + Dy + Ez - Cxy - Jy - Lx & (5) \\ 0 &= Bxz + Fz + Gx - Cxy - Jy + -Lx & (6) \end{aligned} \right\} \text{P}$
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Maples software was used to solve system P to obtain

<p><b>Step 5</b></p> $x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)}$ $V_x = \frac{L(FCD - FCJ - JLA + JCE)}{C(-BLD + BLJ + GLA - GCE)} \quad (\text{back to } V_x)$ $y = -\frac{L}{C};$ $V_y = -\frac{L}{C} \quad (\text{changing back to } V_y \text{ as agreed to})$ $z = -\frac{L(D - J)}{LA - CE};$ $V_z = -\frac{L(D - J)}{LA - CE} \quad (\text{changing back to } V_z \text{ as agreed to})$	<p><b>Note:</b></p> <p>None of the popular academic programs could solve the system in M. Maples solved system P (step 4 above) for <math>x</math>, <math>y</math>, and <math>z</math> in terms of <math>A, B, C, D, E, F, G, J</math>, and <math>L</math>.</p> <p>Note also that <math>x</math>, <math>y</math> and <math>z</math> are not the same as the <math>x</math>, <math>y</math> and <math>z</math> in the system of equations.. They were used for convenience and simplicity .</p>
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**Step 5:** Apply the following and substitute for  $A, B, C, D, E, F, G, J$ , and  $L$  in steps 6-8 below

$$A = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x}$$

$$B = -\frac{\rho g_y}{2\mu}(\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2) + C_{10}x + C_{11}y + C_{12}z + \lambda_5g_y t \pm \sqrt{2\lambda_7g_y y}$$

$$C = -\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_{14}x + C_{15}y + C_{16}z + \beta_5g_z t \pm \sqrt{2\beta_8g_z z}$$

$$D = qg_x z; \quad E = ng_x y; \quad F = \lambda_6g_y x$$

$$G = \lambda_8g_y z; \quad J = \beta_6g_z x; \quad L = \beta_7g_z y$$

<p><b>Step 6</b></p> $V_y = -\frac{L}{C} = -\frac{(\beta_7g_z y)}{(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z t \pm \sqrt{2\beta_8g_z z})}$ $\frac{ng_x y}{V_y} = ng_x y \div -\left(\frac{(\beta_7g_z y)}{(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z t \pm \sqrt{2\beta_8g_z z})}\right)$ $\frac{ng_x y}{V_y} = \frac{(ng_x y)[(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z t \pm \sqrt{2\beta_8g_z z})]}{\beta_7g_z y};$ $\frac{ng_x y}{V_y} = \frac{-(ng_x)(-\frac{\rho g_z}{2\mu}(\beta_1x^2 + \beta_2y^2 + \beta_3z^2) + C_1x + C_3y + C_5z + \beta_5g_z t \pm \sqrt{2\beta_8g_z z})}{\beta_7g_z}; \quad (\text{cancel "y"})$
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**Step 7**

$$V_z = -\frac{L(D-J)}{LA-CE}$$

$$V_z = -\frac{(\beta_7 g_z y)(q g_x z - \beta_6 g_z x)}{(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g_x t \pm \sqrt{2h g_x x}\right) - CE}$$

$$CE = -(ng_x y)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14} x + C_{15} y + C_{16} z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)$$

$$\frac{q g_x z}{V_z} = \frac{-(q g_x z)\{[(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g_x t \pm \sqrt{2h g_x x}\right)] - [CE]\}}{(\beta_7 g_z y)(q g_x z - \beta_6 g_z x)}$$

(Note : y can be cancelled from numerator and denominator)

**Summary for the fractional terms of the  $x$ -direction solution**

$\frac{ng_x y}{V_y}$  and  $\frac{q g_x z}{V_z}$  in terms of  $x, y, z$  and  $t$

$$\frac{ng_x y}{V_y} = \frac{-(ng_x)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1 x + C_3 y + C_5 z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)}{\beta_7 g_z}; \text{ (cancel "y")}$$

$$\frac{q g_x z}{V_z} = \frac{-(q g_x z)\{[(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + f g_x t \pm \sqrt{2h g_x x}\right)] - [CE]\}}{(\beta_7 g_z y)(q g_x z - \beta_6 g_z x)}$$

$$\left( CE = -(ng_x y)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14} x + C_{15} y + C_{16} z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right) \right)$$

$$d + f + h + n + q = 1; \quad \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1; \quad \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1$$

## Analysis of N-S Solutions

$x$ -direction solution

$$V_x = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_{xy}}{V_y} + \frac{qg_{xz}}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + C_9$$

$$P(x) = d\rho g_x x; \quad (a + b + c + d + h + n + q = 1) \quad V_y \neq 0, V_z \neq 0$$

$$\frac{ng_{xy}}{V_y} = \frac{-(ng_x)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1x + C_3y + C_5z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)}{\beta_7 g_z}$$

$$\frac{qg_{xz}}{V_z} = \frac{-(qg_x z)\left\{(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x}\right) - [CE]\right\}}{(\beta_7 g_z y)(qg_x z - \beta_6 g_z x)}$$

$$(CE = -(ng_{xy})\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1x + C_3y + C_5z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right))$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity,  $g_x$ , were zero, the first three terms, the seventh, the eighth, the ninth, the tenth terms of the velocity solution and  $P(x)$  would all be zero. This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

### More Observations    Comparison of the Navier-Stokes solutions with equations of motion under gravity and liquid pressure of elementary physics

#### Motion equations of elementary physics:

$$(B): V_f = V_0 + gt; \quad (C): V_f^2 = V_0^2 + 2gx; \quad (D): V = \sqrt{2gx}; \quad (E): x = V_0 t + \frac{1}{2}gt^2$$

#### Liquid Pressure,

The liquid pressure,  $P$  at the bottom of a liquid of depth  $h$  units is given by  $P = \rho gh$

Observe the following above:

1. Observe that the first three terms are parabolic in  $x$ ,  $y$ , and  $z$ ; the minus sign showing the usual inverted parabola when a projectile is fired upwards at an acute angle to the horizontal. Also note the " $gt$ " in  $V = gt$  of (B) of the motion equations and the  $fg_x t$  in the Navier-Stokes solution.
2. Observe the  $P = \rho gh$  of the liquid pressure and the  $P(x) = d\rho g_x x$  of the Navier-Stokes solution. Note that  $d$  is a ratio term.
3. Observe the " $\sqrt{2gx}$ " in  $V = \sqrt{2gx}$  of (D) and the  $\sqrt{2hg_x x}$  in the Navier-Stokes solution. There are eight main terms (ignoring the arbitrary functions) in the Navier-Stokes solution. Of these eight terms, six terms, namely,  $-\frac{a\rho g_x}{2\mu}x^2$ ,  $-\frac{b\rho g_x}{2\mu}y^2$ ,  $-\frac{c\rho g_x}{2\mu}z^2$ ,  $fg_x t$ ,  $\sqrt{2hg_x x}$  and  $d\rho g_x x$  are similar (except for the constants involved) to the terms in the equations of motion. This similarity means that the approach used in solving the Navier-Stokes equation is sound. One should also note that to obtain these six terms simultaneously on integration, only the equation with the gravity term as the subject of the equation will yield these six terms. The author suggests that this form of the equation with the gravity term as the subject of the equation be called the standard form of the Navier-Stokes equation, since in this form, one can immediately split-up the equation using ratios, and integrate.

### Velocity Profile

The fluid flow in the Navier-Stokes solution may be characterized as follows. The  $x$ -direction solution consists of linear, parabolic, and hyperbolic terms. The first three terms characterize inverted parabolas. Flow distribution for laminar flow is parabolic with the axis of symmetry of the parabola in the direction of fluid flow. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the  $x$ -nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The  $y$ -nonlinear term is similar parabolically to the  $x$ -nonlinear term. The characteristic curve for the  $z$ -nonlinear term is a combination of two similar parabolas and a hyperbola. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent.

### Standard form of the $x$ -direction Navier-Stokes equation for fluid flow

One will call the Navier-Stokes equation with the gravity term as the subject of equation in (A), the standard form of the Navier-Stokes equation for the ratio method of solving these equations, since this form produces a solution on integration. None of the other forms in (B), (C), (D), (E), (F), (G), (G), or (J) produces a solution. That is, the integration results of each of the other eight equations do not satisfy the corresponding equation.

$$\boxed{-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x} \quad \text{(A) } < \text{ standard form}$$

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} - \frac{\partial p}{\partial x} - \rho \frac{\partial V_x}{\partial t} - \rho V_x \frac{\partial V_x}{\partial x} - \rho V_y \frac{\partial V_x}{\partial y} + \rho g_x = \rho V_z \frac{\partial V_x}{\partial z} \quad \text{(B)}$$

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} - \frac{\partial p}{\partial x} - \rho \frac{\partial V_x}{\partial t} - \rho V_x \frac{\partial V_x}{\partial x} - \rho V_z \frac{\partial V_x}{\partial z} + \rho g_x = \rho V_y \frac{\partial V_x}{\partial y} \quad \text{(C)}$$

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} - \frac{\partial p}{\partial x} - \rho \frac{\partial V_x}{\partial t} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \rho g_x = \rho V_x \frac{\partial V_x}{\partial x} \quad \text{(D)}$$

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} - \frac{\partial p}{\partial x} - \rho V_x \frac{\partial V_x}{\partial x} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \rho g_x = \rho \frac{\partial V_x}{\partial t} \quad \text{(E)}$$

$$\mu \frac{\partial^2 V_x}{\partial x^2} + \mu \frac{\partial^2 V_x}{\partial y^2} + \mu \frac{\partial^2 V_x}{\partial z^2} - \rho \frac{\partial V_x}{\partial t} - \rho V_x \frac{\partial V_x}{\partial x} - \rho V_y \frac{\partial V_x}{\partial y} - \rho V_z \frac{\partial V_x}{\partial z} + \rho g_x = \frac{\partial p}{\partial x} \quad \text{(F)}$$

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} - \rho g_x = \mu \frac{\partial^2 V_x}{\partial z^2} \quad \text{(G)}$$

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} - \rho g_x = \mu \frac{\partial^2 V_x}{\partial y^2} \quad \text{(H)}$$

$$-\mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} - \rho g_x = \mu \frac{\partial^2 V_x}{\partial x^2} \quad \text{(J)}$$

### Uniqueness of the solution of the Navier-Stokes equation

When each term of the linearized Navier-Stokes equation was made subject of the N-S equation, only the equation with the gravity term as the subject of the equation produced a solution. Similarly, the solution of the Navier-Stokes equation solution is unique.

**Also note:** The liquid pressure,  $P$  at the bottom of a liquid of depth  $h$  units is given by  $\boxed{P = \rho gh}$ .

From the N-S solution in this paper,  $\boxed{P(x) = d\rho gx}$  from integrating  $\frac{dp}{dx} = d\rho g$  where  $d$  is ratio term. Each of the other terms in the N-S equation must also be set equal to the product of a ratio term and  $\rho g$ . This result implies that the approach used in solving the N-S equations is sound.

## Option 5

### Solutions of 4-D Navier-Stokes Equations

In the above method, the solution can easily be extended to any number of dimensions..

Adding  $\mu \frac{\partial^2 V_x}{\partial s^2}$  and  $\rho V_s \frac{\partial V_x}{\partial s}$  to the 3-D  $x$ -direction equation yields the 4-D N-S equation

$$-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial s^2} \right) + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_s \frac{\partial V_x}{\partial s} = \rho g_x$$

whose solution is given by

$$V_x(x, y, z, s, t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2 + es^2) + C_1 x + C_3 y + C_5 z + C_6 s + fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{rg_x s}{V_s} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z} + \frac{\psi_s(V_s)}{V_s}}_{\text{arbitrary functions}} + C_9$$

$$P(x) = d\rho g_x x \quad (a + b + c + d + e + f + h + n + q + r = 1) \quad V_x \neq 0, V_y \neq 0, V_s \neq 0,$$

For  $n$ -dimensions one can repeat the above as many times as one wishes.

## Option 5b

### Two-term Linearized Navier-Stokes Equation (one nonlinear term)

By linearization as in Option 1, if one replaces  $\rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z}$  by  $2\rho \frac{\partial V_x}{\partial t}$  in

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x \text{ one obtains}$$

$$-\mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \frac{\partial p}{\partial x} + 3\rho \left( \frac{\partial V_x}{\partial t} \right) + \rho V_x \frac{\partial V_x}{\partial x} = \rho g_x, \text{ whose solution is}$$

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu} (ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + \frac{fg_x t}{3} \pm \sqrt{2hg_x x} + C_6$$



## Conclusion (for Option 4)

One will begin from the general case and end with the special cases.

### Solutions of the Navier--Stokes equations (general case)

$x$ -direction **Navier-Stokes Equation** (also driver equation)

$$\boxed{-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} = \rho g_x} \quad x\text{-direction}$$

$$\boxed{V_x(x,y,z,t) = \underbrace{-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + fg_x t}_{\text{solution for linear terms}} \pm \underbrace{\sqrt{2hg_x x} + \frac{ng_{xy}}{V_y} + \frac{qg_{xz}}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{solution for non-linear terms}} + C_9}$$

arbitrary functions

$$P(x) = d\rho g_x x; \quad (a + b + c + d + h + n + q = 1) \quad V_y \neq 0, V_z \neq 0$$

$$\frac{ng_{xy}}{V_y} = \frac{-(ng_x)\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_1 x + C_3 y + C_5 z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)}{\beta_7 g_z}$$

$$\frac{qg_{xz}}{V_z} = \frac{-(qg_x z)\left\{[(\beta_7 g_z y)\left(-\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1 x + C_3 y + C_5 z + fg_x t \pm \sqrt{2hg_x x}\right)] - [CE]\right\}}{(\beta_7 g_z y)(qg_{xz} - \beta_6 g_z x)}$$

$$CE = -(ng_{xy})\left(-\frac{\rho g_z}{2\mu}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) + C_{14} x + C_{15} y + C_{16} z + \beta_5 g_z t \pm \sqrt{2\beta_8 g_z z}\right)$$

One observes above that the most important insight of the above solution is the indispensability of the gravity term in incompressible fluid flow. Observe that if gravity,  $g_x$ , were zero, the first three terms, the 7th term, the 8th term, the 9th term, the 10th term and  $P(x)$  would all be zero.

This result can be stated emphatically that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known. The author proposed and applied a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and also each term utilizes gravity to function. This law was applied in splitting-up the Navier-Stokes equations. The resulting sub-equations were readily integrable, and even the nonlinear sub-equations were readily integrated.

The  $x$ -direction Navier-Stokes equation was split-up into sub-equations using ratios. The sub-equations were solved and combined. The relation obtained from the integration of the linear part of the equation satisfied the linear part of the equation and the relation obtained from integrating the nonlinear part of the equation satisfied the nonlinear part of the equation. By solving algebraically and simultaneously for  $V_x$ ,  $V_y$  and  $V_z$ , the  $(ng_{xy}/V_y)$  and  $(qg_{xz}/V_z)$  terms would be expressed explicitly in terms of  $x$ ,  $y$ ,  $z$  and  $t$ . The above  $x$ -direction solution is the solution everyone has been waiting for, for nearly 150 years. It was obtained in two simple steps, namely, splitting the equation using ratios and integrating.

## Special Cases of the Navier-Stokes Equations

### 1. Linearized Navier-Stokes equations

One may note that there are six linear terms and three nonlinear terms in the Navier-Stokes equation. The linearized case was covered before the general case, and the experience gained in the linearized case guided one to solve the general case efficiently. In particular, the gravity term must be the subject of the equation for a solution. When the gravity term was the subject of the equation, the equation was called the driver equation. A splitting technique was applied to the linearized Navier-Stokes equations (Option 1). Twenty sub-equations were solved. (Four sets of equations with different equation subjects). The integration relations of one of the sets satisfied the linearized Navier-Stokes equation; and this set was from the equation with  $g_x$  as the subject of the equation. In addition to finding a solution, the results of the integration revealed the roles of the terms of the Navier-Stokes equations in fluid flow. In particular, the gravity forces and  $\partial p/\partial x$  are involved mainly in the parabolic as well as the forward motion of fluids;  $\partial V_x/\partial t$  and  $\partial^2 V_x/\partial x^2$  are involved in the periodic motion of fluids, and one may infer that as  $\mu$  increases, the periodicity increases. One should determine experimentally, if the ratio of the linear term  $\partial V_x/\partial t$  to the nonlinear sum  $V_x(\partial V_x/\partial x) + V_y(\partial V_x/\partial y) + V_z(\partial V_x/\partial z)$  is 1 to 3.

#### Solution to the linearized Navier–Stokes equation

$$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9 ; P(x) = d\rho g_x x$$

#### Linearized Equation

$$-\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x$$

### 2. Solutions of the Euler equation

Since one has solved the Navier-Stokes equation, one has also solved the Euler equation.

$$\text{Euler equation } (\mu = 0): \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = g_x$$

$$V_x(x,y,z,t) = fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C$$

x-direction

$$P(x) = d\rho g_x x \quad (f + h + n + q + d = 1) \quad V_y \neq 0, V_z \neq 0$$

$$\frac{ng_x y}{V_y} = -\frac{n\beta_5 g_z t}{\beta_7} \pm \frac{(\sqrt{2\beta_8 g_z z})(ng_x)}{\beta_7 g_z} \Bigg\} B$$

$$\frac{qg_x z}{V_z} = \frac{(\beta_7 f g_z g_x^2 q - \beta_5 g_x g_z n q) t z \pm \sqrt{2hg_x x} \beta_7 g_x g_z q z \pm \sqrt{2\beta_8 g_z z} g_x^2 n q z}{\beta_7 \beta_6 g_z^2 x - \beta_7 q g_z g_x z} \Bigg\} C$$

## Option 6

### Solutions of 3-D Navier-Stokes Equations (Method 2)

Here, the three equations below, will be added together; and a single equation will be integrated

$$\left\{ \begin{array}{l} -\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + \rho\left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z}\right) = \rho g_x \quad (1) \\ -\mu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right) + \frac{\partial p}{\partial y} + \rho\left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z}\right) = \rho g_y \quad (2) \\ -\mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) + \frac{\partial p}{\partial z} + \rho\left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z}\right) = \rho g_z \quad (3) \end{array} \right.$$

**Step 1:** Apply the axiom, if  $a = b$  and  $c = d$ , then  $a + c = b + d$ ; and therefore, add the left sides and add the right sides of the above equations . That is, (1) + (2) + (3) =  $\rho g_x + \rho g_y + \rho g_z$

$$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} - \mu \frac{\partial^2 V_y}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_y}{\partial z^2} - \mu \frac{\partial^2 V_z}{\partial x^2} - \mu \frac{\partial^2 V_z}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \\ & + \frac{\partial p}{\partial z} + \rho \frac{\partial V_x}{\partial t} + \rho \frac{\partial V_y}{\partial t} + \rho \frac{\partial V_z}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} + \rho V_x \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_y}{\partial z} \\ & + \rho V_x \frac{\partial V_z}{\partial x} + \rho V_y \frac{\partial V_z}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = (\rho g_x + \rho g_y + \rho g_z) \end{aligned} \quad \text{(Three lines per equation)}$$

Let  $\rho g_x + \rho g_y + \rho g_z = \rho G$ , where  $G = |g_x + g_y + g_z|$  to obtain

$$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} - \mu \frac{\partial^2 V_y}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_y}{\partial z^2} - \mu \frac{\partial^2 V_z}{\partial x^2} - \mu \frac{\partial^2 V_z}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} \\ & + \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} + \rho \frac{\partial V_x}{\partial t} + \rho \frac{\partial V_y}{\partial t} + \rho \frac{\partial V_z}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} + \rho V_y \frac{\partial V_x}{\partial y} + \rho V_z \frac{\partial V_x}{\partial z} \\ & + \rho V_x \frac{\partial V_y}{\partial x} + \rho V_y \frac{\partial V_y}{\partial y} + \rho V_z \frac{\partial V_y}{\partial z} + \rho V_x \frac{\partial V_z}{\partial x} + \rho V_y \frac{\partial V_z}{\partial y} + \rho V_z \frac{\partial V_z}{\partial z} = \rho G \end{aligned}$$

**Step 2:** Solve the above 25-term equation using the ratio method. (24 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

$a, b, c, d, f, m, n, q, r, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9$

$$\begin{aligned} -\mu \frac{\partial^2 V_x}{\partial x^2} &= a\rho G; & -\mu \frac{\partial^2 V_x}{\partial y^2} &= b\rho G; & -\mu \frac{\partial^2 V_x}{\partial z^2} &= c\rho G; & -\mu \frac{\partial^2 V_y}{\partial x^2} &= d\rho G; \\ -\mu \frac{\partial^2 V_y}{\partial y^2} &= f\rho G; & -\mu \frac{\partial^2 V_y}{\partial z^2} &= h\rho G; & -\mu \frac{\partial^2 V_z}{\partial x^2} &= m\rho G; & -\mu \frac{\partial^2 V_z}{\partial y^2} &= n\rho G; \\ -\mu \frac{\partial^2 V_z}{\partial z^2} &= r\rho G; & \frac{\partial p}{\partial x} &= \beta_1\rho G; & \frac{\partial p}{\partial y} &= \beta_2\rho G; & \frac{\partial p}{\partial z} &= \beta_3\rho G; \\ \rho \frac{\partial V_x}{\partial t} &= \beta_4\rho G; & \rho \frac{\partial V_y}{\partial t} &= \beta_5\rho G; & \rho \frac{\partial V_z}{\partial t} &= \beta_6\rho G; & \rho V_x \frac{\partial V_x}{\partial x} &= \lambda_1\rho G; \\ \rho V_y \frac{\partial V_x}{\partial y} &= \lambda_2\rho G; & \rho V_z \frac{\partial V_x}{\partial z} &= \lambda_3\rho G; & \rho V_x \frac{\partial V_y}{\partial x} &= \lambda_4\rho G; & \rho V_y \frac{\partial V_y}{\partial y} &= \lambda_5\rho G; \\ \rho V_z \frac{\partial V_y}{\partial z} &= \lambda_6\rho G; & \rho V_x \frac{\partial V_z}{\partial x} &= \lambda_7\rho G; & \rho V_y \frac{\partial V_z}{\partial y} &= \lambda_8\rho G; & \rho V_z \frac{\partial V_z}{\partial z} &= \lambda_9\rho G \end{aligned}$$

<b>1</b> $\frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{\mu} \rho G$ $\frac{\partial V_x}{\partial x} = -\frac{a}{\mu} \rho G x + C_1$ $V_x = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x + C_2$	<b>2</b> $-\mu \frac{\partial^2 V_x}{\partial y^2} = b \rho G$ $\frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{\mu} \rho G$ $\frac{\partial V_x}{\partial y} = -\frac{b}{\mu} \rho G y + C_3$ $V_x = -\frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y + C_4$	<b>3</b> $-\mu \frac{\partial^2 V_x}{\partial z^2} = c \rho G$ $-\mu \frac{\partial^2 V_x}{\partial z^2} = c \rho G$ $\frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{\mu} \rho G$ $\frac{\partial V_x}{\partial z} = -\frac{c}{\mu} \rho G z + C_5$ $V_x = -\frac{c}{\mu} \rho G \frac{z^2}{2} + C_5 z + C_6$	
<b>4</b> $-\mu \frac{\partial^2 V_y}{\partial x^2} = d \rho G$ $-\mu \frac{\partial^2 V_y}{\partial x^2} = d \rho G$ $\frac{\partial^2 V_y}{\partial x^2} = -\frac{d}{\mu} \rho G$ $\frac{\partial V_y}{\partial x} = -\frac{d}{\mu} \rho G x + C_7$ $V_y = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x + C_8$	<b>5</b> $-\mu \frac{\partial^2 V_y}{\partial y^2} = f \rho G$ $\frac{\partial^2 V_y}{\partial y^2} = -\frac{f}{\mu} \rho G$ $\frac{\partial V_y}{\partial y} = -\frac{f}{\mu} \rho G y + C_9$ $V_y = -\frac{f}{\mu} \rho G \frac{y^2}{2} + C_9 y + C_{10}$	<b>6</b> $-\mu \frac{\partial^2 V_y}{\partial z^2} = h \rho G$ $\frac{\partial^2 V_y}{\partial z^2} = -\frac{h}{\mu} \rho G$ $\frac{\partial V_y}{\partial z} = -\frac{h}{\mu} \rho G z + C_{11}$ $V_y = -\frac{h}{\mu} \rho G \frac{z^2}{2} + C_{11} z + C_{12}$	
<b>7</b> $-\mu \frac{\partial^2 V_z}{\partial x^2} = m \rho G$ $\frac{\partial^2 V_z}{\partial x^2} = -\frac{m}{\mu} \rho G$ $\frac{\partial V_z}{\partial x} = -\frac{m}{\mu} \rho G x + C_{13}$ $V_z = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x + C_{14}$	<b>8</b> $-\mu \frac{\partial^2 V_z}{\partial y^2} = n \rho G$ $\frac{\partial^2 V_z}{\partial y^2} = -\frac{n}{\mu} \rho G$ $\frac{\partial V_z}{\partial y} = -\frac{n}{\mu} \rho G y + C_{15}$ $V_z = -\frac{n}{\mu} \rho G \frac{y^2}{2} + C_{15} y + C_{16}$	<b>9</b> $-\mu \frac{\partial^2 V_z}{\partial z^2} = r \rho G$ $\frac{\partial^2 V_z}{\partial z^2} = -\frac{r}{\mu} \rho G$ $\frac{\partial V_z}{\partial z} = -\frac{r}{\mu} \rho G z + C_{17}$ $V_z = -\frac{r}{\mu} \rho G \frac{z^2}{2} + C_{17} z + C_{18}$	
<b>10</b> $\frac{\partial p}{\partial x} = \beta_1 \rho G$ $\frac{dp}{dx} = \beta_1 \rho G$ $P(x) = \beta_1 \rho G x + C_{19}$	<b>11</b> $\frac{\partial p}{\partial y} = \beta_2 \rho G$ $\frac{dp}{dy} = \beta_2 \rho G$ $P(y) = \beta_2 \rho G y + C_{20}$	<b>12</b> $\frac{\partial p}{\partial z} = \beta_3 \rho G$ $\frac{dp}{dz} = \beta_3 \rho G$ $P(z) = \beta_3 \rho G z + C_{21}$	

<p><b>13</b></p> $\rho \frac{\partial V_x}{\partial t} = \beta_4 \rho G$ $\frac{dV_x}{dt} = \beta_4 G$ $V_x = \beta_4 G t + C_{22}$	<p><b>14</b></p> $\rho \frac{\partial V_y}{\partial t} = \beta_5 \rho G$ $\frac{dV_y}{dt} = \beta_5 G$ $V_y = \beta_5 G t + C_{23}$	<p><b>15</b></p> $\rho \frac{\partial V_z}{\partial t} = \beta_6 \rho G$ $\frac{dV_z}{dt} = \beta_6 G$ $V_z = \beta_6 G t + C_{24}$	<p><b>16</b></p> $\rho V_x \frac{\partial V_x}{\partial x} = \lambda_1 \rho G$ $V_x \frac{\partial V_x}{\partial x} = \lambda_1 G$ $V_x \frac{dV_x}{dx} = \lambda_1 G$ $V_x dV_x = \lambda_1 G dx$ $\frac{V_x^2}{2} = \lambda_1 G x$ $V_x^2 = 2\lambda_1 G x$ $V_x = \pm \sqrt{2\lambda_1 G x} + C_{25}$
<p><b>17</b></p> $\rho V_y \frac{\partial V_x}{\partial y} = \lambda_2 \rho G$ $V_y \frac{dV_x}{dy} = \lambda_2 G$ $V_y dV_x = \lambda_2 G dy$ $V_y V_x = \lambda_2 G y + \psi_y(V_y)$ $V_x = \frac{\lambda_2 G y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	<p><b>18</b></p> $\rho V_z \frac{\partial V_x}{\partial z} = \lambda_3 \rho G$ $V_z \frac{dV_x}{dz} = \lambda_3 G$ $V_z dV_x = \lambda_3 G dz$ $V_z V_x = \lambda_3 G z + \psi_z(V_z)$ $V_x = \frac{\lambda_3 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$	<p><b>19</b></p> $\rho V_x \frac{\partial V_y}{\partial x} = \lambda_4 \rho G$ $V_x \frac{dV_y}{dx} = \lambda_4 G$ $V_x dV_y = \lambda_4 G dx$ $V_x V_y = \lambda_4 G x + \psi_x(V_x)$ $V_y = \frac{\lambda_4 G x}{V_x} + \frac{\psi_x(V_x)}{V_x}$	<p><b>20</b></p> $\rho V_y \frac{\partial V_y}{\partial y} = \lambda_5 \rho G$ $V_y \frac{dV_y}{dy} = \lambda_5 G$ $V_y dV_y = \lambda_5 G dy$ $\frac{V_y^2}{2} = \lambda_5 G y$ $V_y^2 = 2\lambda_5 G y$ $V_y = \pm \sqrt{2\lambda_5 G y} + C_{26}$
<p><b>21</b></p> $\rho V_z \frac{\partial V_y}{\partial z} = \lambda_6 \rho G$ $V_z \frac{dV_y}{dz} = \lambda_6 G$ $V_z dV_y = \lambda_6 G dz$ $V_z V_y = \lambda_6 G z + \psi_z(V_z)$ $V_y = \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$	<p><b>22</b></p> $\rho V_x \frac{\partial V_z}{\partial x} = \lambda_7 \rho G$ $V_x \frac{dV_z}{dx} = \lambda_7 G$ $V_x dV_z = \lambda_7 G dx$ $V_x V_z = \lambda_7 G x + \psi_x(V_x)$ $V_z = \frac{\lambda_7 G x}{V_x} + \frac{\psi_x(V_x)}{V_x}$	<p><b>23</b></p> $\rho V_y \frac{\partial V_z}{\partial y} = \lambda_8 \rho G$ $V_y \frac{dV_z}{dy} = \lambda_8 G$ $V_y dV_z = \lambda_8 G dy$ $V_y V_z = \lambda_8 G y + \psi_y(V_y)$ $V_z = \frac{\lambda_8 G y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	<p><b>24</b></p> $\rho V_z \frac{\partial V_z}{\partial z} = \lambda_9 \rho G$ $V_z \frac{dV_z}{dz} = \lambda_9 G$ $V_z dV_z = \lambda_9 G dz$ $\frac{V_z^2}{2} = \lambda_9 G z$ $V_z^2 = 2\lambda_9 G z$ $V_z = \pm \sqrt{2\lambda_9 G z} + C_{27}$

**Step 3 :** One Collects the integrals of the sub-equations, above, for  $V_x, V_y, V_z, P(x), P(y), P(z)$

For $V_x, P(x)$	For $V_y, P(y)$	For $V_z, P(z)$
Sum of integrals from sub-equations #1, #2, #3, #13, #16, #17, #18, #10	Sum of integrals from sub-equations #4, #5, #6, #14, #19, #20, #21, #11	Sum of integrals from sub-equations #7, #8, #9, #15, #22, #23, #24, #12,
$V_x = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x + C_2$	$V_y = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x + C_8$	$V_z = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x + C_{14}$
$V_x = -\frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y + C_4$	$V_y = -\frac{f}{\mu} \rho G \frac{y^2}{2} + C_9 y + C_{10}$	$V_z = -\frac{m}{\mu} \rho G \frac{y^2}{2} + C_{15} y + C_{16}$
$V_x = -\frac{c}{\mu} \rho G \frac{z^2}{2} + C_5 z + C_6$	$V_y = -\frac{h}{\mu} \rho G \frac{z^2}{2} + C_{11} z + C_{12}$	$V_z = -\frac{r}{\mu} \rho G \frac{z^2}{2} + C_{17} z + C_{18}$
$V_x = \beta_4 G t + C_{22}$	$V_y = \beta_5 G t + C_{21}$	$V_z = \beta_6 G t + C_{24}$
$V_x = \pm \sqrt{2\lambda_1 G} x + C_{25}$	$V_y = \frac{\lambda_4 G x}{V_x} + \frac{\psi_x(V_x)}{V_x}$	$V_z = \frac{\lambda_7 G x}{V_x} + \frac{\psi_x(V_x)}{V_x}$
$V_x = \frac{\lambda_2 G y}{V_y} + \frac{\psi_y(V_y)}{V_y}$	$V_y = \pm \sqrt{2\lambda_5 G} y + C_{26}$	# $V_z = \frac{\lambda_8 G y}{V_y} + \frac{\psi_y(V_y)}{V_y}$
$V_x = \frac{\lambda_3 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$	$V_y = \frac{\lambda_6 G z}{V_z} + \frac{\psi_z(V_z)}{V_z}$	$V_z = \pm \sqrt{2\lambda_9 G} z + C_{27}$
$P(x) = \beta_1 \rho G x + C_{19}$	$P(y) = \beta_2 \rho G y + C_{20}$	$P(z) = \beta_3 \rho G z + C_{21}$

From above,

For  $V_x$ , Sum of integrals from sub-equations #1, #2, #3, #13, #16, #17, #18, #10

$$V_x(x, y, z, t) = -\frac{a}{\mu} \rho G \frac{x^2}{2} + C_1 x - \frac{b}{\mu} \rho G \frac{y^2}{2} + C_3 y - \frac{c}{\mu} \rho G \frac{z^2}{2} + C_5 z + \beta_4 G t \pm \sqrt{2\lambda_1 G} x + \frac{\lambda_2 G y}{V_y} + \frac{\lambda_3 G z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}$$

arbitrary functions

$$P(x) = \beta_1 \rho G x + C_{19}$$

For  $V_y$  : Sum of integrals from sub-equations #4, #5, #6, #14, #19, #20, #21, #11

$$V_y(x, y, z, t) = -\frac{d}{\mu} \rho G \frac{x^2}{2} + C_7 x - \frac{f}{\mu} \rho G \frac{y^2}{2} + C_9 y - \frac{h}{\mu} \rho G \frac{z^2}{2} + C_{11} z + \beta_5 G t \pm \sqrt{2\lambda_5 G} y + \frac{\lambda_4 G x}{V_x} + \frac{\lambda_6 G z}{V_z} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}$$

arbitrary functions

$$P(y) = \beta_2 \rho G y + C_{20}$$

For  $V_z$ : Sum of integrals from sub-equations #7, #8, #9, #15, #22, #23, #24, #12,

$$V_z = -\frac{m}{\mu} \rho G \frac{x^2}{2} + C_{13} x - \frac{n}{\mu} \rho G \frac{y^2}{2} + C_{15} y - \frac{r}{\mu} \rho G \frac{z^2}{2} + C_{17} z + \beta_6 G t \pm \sqrt{2\lambda_9 G} z + \frac{\lambda_7 G x}{V_x} + \frac{\lambda_8 G y}{V_y} + \frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}$$

arbitrary functions

$$P(z) = \beta_3 \rho G z + C_{21}$$

**Step 4:** Simplify the sums of the integrals from above..(Method 2 solutions of N-S equations

$$V_x(x,y,z,t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \beta_4Gt \pm \sqrt{2\lambda_1G}x + \frac{\lambda_2G}{V_y}y + \frac{\lambda_3G}{V_z}z$$

$$P(x) = \beta_1\rho Gx + C_{19} \quad (V_y \neq 0, V_z \neq 0) \quad + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$V_y(x,y,z,t) = -\frac{\rho G}{2\mu}(dx^2 + fy^2 + hz^2) + C_7x + C_9y + C_{11}z + C_{10}\beta_5Gt \pm \sqrt{2\lambda_5G}y + \frac{\lambda_4G}{V_x}x + \frac{\lambda_6G}{V_z}z$$

$$P(y) = \beta_2\rho Gy + C_{20} \quad (V_x \neq 0, V_z \neq 0) \quad + \underbrace{\frac{\psi_x(V_x)}{V_x} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}}$$

$$V_z(x,y,z,t) = -\frac{\rho G}{2\mu}(mx^2 + ny^2 + rz^2) + C_{13}x + C_{15}y + C_{17}z + \beta_6Gt \pm \sqrt{2\lambda_9G}z + \frac{\lambda_7G}{V_x}x + \frac{\lambda_8G}{V_y}y$$

$$P(z) = \beta_3\rho Gz + C_{21} \quad (V_x \neq 0, V_y \neq 0) \quad + \underbrace{\frac{\psi_x(V_x)}{V_x} + \frac{\psi_y(V_y)}{V_y}}_{\text{arbitrary functions}}$$

The above are solutions for  $V_x, V_y, V_z, P(x), P(y), P(z)$  .of the Navier-Stokes Equations

## Comparison of Method 1 (Option 4) and Method 2 (Option 6) of Solutions of Navier-Stokes Equations

**Method 1:**  $x$ -direction solution of Navier-Stokes equation

$$\begin{aligned}
 V_x(x,y,z,t) &= -\frac{\rho g_x}{2\mu}(ax^2+by^2+cz^2) + C_1x + C_3y + C_5z + fg_x t \pm \sqrt{2hg_x x + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z}} + \\
 P(x) &= d\rho g_x x; (a+b+c+d+h+n+q=1) \quad (V_y \neq 0, V_z \neq 0) \quad + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} + C_9 \quad (\text{A})
 \end{aligned}$$

**Method 2:**  $x$ -direction solution of Navier-Stokes equation

$$\begin{aligned}
 V_x(x,y,z,t) &= -\frac{\rho G}{2\mu}(ax^2+by^2+cz^2) + C_1x + C_3y + C_5z + \beta_4 G t \pm \sqrt{2\lambda_1 G x + \frac{\lambda_2 G y}{V_y} + \frac{\lambda_3 G z}{V_z}} + \\
 P(x) &= \beta_1 \rho G x + C_{10} \quad (V_y \neq 0, V_z \neq 0) \quad + \underbrace{\frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{arbitrary functions}} \quad (\text{B})
 \end{aligned}$$

It is pleasantly surprising that the above solutions (A) and (B) are almost identical (except for the constants), even though they were obtained by different approaches as in Option 4 and Option 6. Such an agreement confirms the validity of the solution method for the system of magnetohydrodynamic equations (see viXra:1405.0251.. For the system of magnetohydrodynamic equations, there is only a single "driver" equation. For the system of N-S equations, there are three driver equations, since each equation contains the gravity term. Therefore, one was able to solve each of the three simultaneous equations separately (as in Method 1); but in addition, one obtained an identical solution (except for the constants) in solving the simultaneous N-S system by adding the three equations in the system and integrating a single driver equation. In Method 1, the gravity term was  $\rho g$ . In Method 2, the gravity term was  $\rho G$ , where  $G$  is the magnitude of the vector sum of the gravity terms. Note that in Method 1, the sum of the ratio terms (8 ratio terms for each equation) equals unity, but in Method 2, the sum of the ratio terms (24 ratio terms) for the single driver equation solved equals unity. Note that in Method 2, only a single "driver" equation was solved, but in Method 1, three "driver" equations were solved. In Method 2, one could say that the system of N-S equations was "more simultaneously" solved than in Method 1.

To summarize, solving the Navier-Stokes equations by the first method helped one to solve the magnetohydrodynamic equations (not presented in this paper.. See viXra:1405.0251) and solving the magnetohydrodynamic equations encouraged one to solve the Navier-Stokes equations by the second method.

(" Navier-Stokes equations "scratched the back" of magnetohydrodynamic equations; and in return, magnetohydrodynamic equations "scratched the back" of Navier-Stokes equations")

### About integrating only a single equation

If one asked for help in solving the N-S equations, and one was told to add the three equations together and then solve them, one would think that one was being given a nonsensical advice; but now, after studying the above Option 6 method, one would appreciate such a suggestion.



## Option 7

### Solutions of 3-D Linearized Navier-Stokes Equations

#### Method 2

Here, the three equations below, will be added together; and a single equation will be integrated.

$$\left\{ \begin{array}{l} -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x \quad (1) \\ -\mu \frac{\partial^2 V_y}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} = \rho g_y \quad (2) \\ -\mu \frac{\partial^2 V_z}{\partial x^2} - \mu \frac{\partial^2 V_z}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} = \rho g_z \quad (3) \end{array} \right.$$

**Step 1:** Apply the axiom, if  $a = b$  and  $c = d$ , then  $a + c = b + d$ ; and therefore, add the left sides and add the right sides of the above equations. That is, (1) + (2) + (3) =  $\rho g_x + \rho g_y + \rho g_z$

$$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} - \mu \frac{\partial^2 V_y}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} \\ & -\mu \frac{\partial^2 V_z}{\partial x^2} - \mu \frac{\partial^2 V_z}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} = \rho g_x + \rho g_y + \rho g_z \quad (\text{Two lines per equation}) \end{aligned}$$

Let  $\rho g_x + \rho g_y + \rho g_z = \rho G$ , where  $G = |g_x + g_y + g_z|$  to obtain

$$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial x^2} - \mu \frac{\partial^2 V_x}{\partial y^2} - \mu \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} - \mu \frac{\partial^2 V_y}{\partial x^2} - \mu \frac{\partial^2 V_y}{\partial y^2} - \mu \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial p}{\partial y} + 4\rho \frac{\partial V_y}{\partial t} \\ & -\mu \frac{\partial^2 V_z}{\partial x^2} - \mu \frac{\partial^2 V_z}{\partial y^2} - \mu \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial p}{\partial z} + 4\rho \frac{\partial V_z}{\partial t} = \rho G \quad (\text{Two lines per equation}) \end{aligned}$$

**Step 2:** Solve the above 15-term equation using the ratio method. (14 ratio terms)

The ratio terms to be used are respectively the following: (Sum of the ratio terms = 1)

$a, b, c, d, f, h, j, m, n, q, r, s, u, v, w.$  (Sum of the ratio terms = 1)

$$\begin{aligned} -\mu \frac{\partial^2 V_x}{\partial x^2} &= a\rho G; & -\mu \frac{\partial^2 V_x}{\partial y^2} &= b\rho G; & -\mu \frac{\partial^2 V_x}{\partial z^2} &= c\rho G; & \frac{\partial p}{\partial x} &= d\rho G \\ 4\rho \frac{\partial V_x}{\partial t} &= f\rho G; & -\mu \frac{\partial^2 V_y}{\partial x^2} &= h\rho G; & -\mu \frac{\partial^2 V_y}{\partial y^2} &= j\rho G; & -\mu \frac{\partial^2 V_y}{\partial z^2} &= m\rho G; \\ \frac{\partial p}{\partial y} &= n\rho G; & 4\rho \frac{\partial V_y}{\partial t} &= q\rho G; & -\mu \frac{\partial^2 V_z}{\partial x^2} &= r\rho G; & -\mu \frac{\partial^2 V_z}{\partial y^2} &= s\rho G; \\ -\mu \frac{\partial^2 V_z}{\partial z^2} &= u\rho G; & \frac{\partial p}{\partial z} &= v\rho G; & 4\rho \frac{\partial V_z}{\partial t} &= w\rho G \end{aligned}$$

$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial x^2} = a\rho G \\ & \frac{\partial^2 V_x}{\partial x^2} = -\frac{a}{\mu} \rho G \\ \mathbf{1.} \quad & \frac{\partial V_x}{\partial x} = -\frac{a}{\mu} \rho G x + C_1 \\ & V_x = -\frac{\rho G a}{2\mu} x^2 + C_1 x + C_2 \end{aligned}$	$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial y^2} = b\rho G \\ & \frac{\partial^2 V_x}{\partial y^2} = -\frac{b}{\mu} \rho G \\ \mathbf{2.} \quad & \frac{\partial V_x}{\partial y} = -\frac{b}{\mu} \rho G y + C_3 \\ & V_x = -\frac{\rho G b}{2\mu} y^2 + C_3 y + C_4 \end{aligned}$	$\begin{aligned} & -\mu \frac{\partial^2 V_x}{\partial z^2} = c\rho G \\ & \frac{\partial^2 V_x}{\partial z^2} = -\frac{c}{\mu} \rho G \\ \mathbf{3.} \quad & \frac{\partial V_x}{\partial z} = -\frac{c}{\mu} \rho G z + C_5 \\ & V_x = -\frac{\rho G c}{2\mu} z^2 + C_5 z + C_6 \end{aligned}$
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<p><b>4</b></p> $\frac{\partial p}{\partial x} = d\rho G$ $P(x) = d\rho Gx + C_7$	<p><b>5</b></p> $4\rho \frac{\partial V_x}{\partial t} = f\rho G$ $\frac{\partial V_x}{\partial t} = \frac{fG}{4}$ $V_x = \frac{fG}{4}t + C_8$	<p><b>6</b></p> $-\mu \frac{\partial^2 V_y}{\partial x^2} = h\rho G$ $\frac{\partial^2 V_y}{\partial x^2} = -\frac{h}{\mu} \rho G$ $\frac{\partial V_y}{\partial x} = -\frac{h}{\mu} \rho Gx + C_9$ $V_y = -\frac{\rho Gh}{2\mu} x^2 + C_9 x + C_{10}$
<p><b>7</b></p> $-\mu \frac{\partial^2 V_y}{\partial y^2} = j\rho G$ $\frac{\partial^2 V_y}{\partial y^2} = -\frac{j}{\mu} \rho G$ $\frac{\partial V_y}{\partial y} = -\frac{j}{\mu} \rho Gy + C_{11}$ $V_y = -\frac{\rho Gj}{2\mu} y^2 + C_{11}y + C_{12}$	<p><b>8</b></p> $-\mu \frac{\partial^2 V_y}{\partial z^2} = m\rho G$ $\frac{\partial^2 V_y}{\partial z^2} = -\frac{m}{\mu} \rho G$ $\frac{\partial V_y}{\partial z} = -\frac{m}{\mu} \rho Gz + C_{13}$ $V_y = -\frac{\rho Gm}{2\mu} z^2 + C_{13}z + C_{14}$	<p><b>9</b></p> $\frac{\partial p}{\partial y} = n\rho G$ $P(y) = n\rho Gy + C_{15}$
<p><b>10</b></p> $4\rho \frac{\partial V_y}{\partial t} = q\rho G$ $\frac{\partial V_y}{\partial t} = \frac{qG}{4}$ $V_y = \frac{qG}{4}t + C_{16}$	<p><b>11</b></p> $-\mu \frac{\partial^2 V_z}{\partial x^2} = r\rho G$ $\frac{\partial^2 V_z}{\partial x^2} = -\frac{r}{\mu} \rho G$ $\frac{\partial V_z}{\partial x} = -\frac{r}{\mu} \rho Gx + C_{17}$ $V_z = -\frac{\rho Gr}{2\mu} x^2 + C_{17}x + C_{18}$	<p><b>12</b></p> $-\mu \frac{\partial^2 V_z}{\partial y^2} = s\rho G$ $\frac{\partial^2 V_z}{\partial y^2} = -\frac{s}{\mu} \rho G$ $\frac{\partial V_z}{\partial y} = -\frac{s}{\mu} \rho Gy + C_{19}$ $V_z = -\frac{\rho Gs}{2\mu} y^2 + C_{19}y + C_{20}$
<p><b>13</b></p> $-\mu \frac{\partial^2 V_z}{\partial z^2} = u\rho G$ $\frac{\partial^2 V_z}{\partial z^2} = -\frac{u}{\mu} \rho G$ $\frac{\partial V_z}{\partial z} = -\frac{u}{\mu} \rho Gz + C_{21}$ $V_z = -\frac{\rho Gu}{2\mu} z^2 + C_{21}z + C_{22}$	<p><b>14</b></p> $\frac{\partial p}{\partial z} = v\rho G$ $P(z) = v\rho Gz + C_{23}$	<p><b>15</b></p> $4\rho \frac{\partial V_z}{\partial t} = w\rho G$ $\frac{\partial V_z}{\partial t} = \frac{wG}{4}$ $V_z = \frac{wG}{4}t + C_{24}$

**Step 3:** One collect the solutions from Step 2 for  $(V_x, V_y, V_z, P(x), P(y), P(z))$

For  $V_x$ , Sum of integrals from sub-equations #1, #2, #3, #5, and for  $P(x)$ , from #4

$$V_x(x, y, z, t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = d\rho Gx + C_7$$

For  $V_y$  Sum of integrals from sub-equations #6, #7, #8, #10, and for  $P(y)$ , from #9.

$$V_y(x, y, z, t) = -\frac{\rho G}{2\mu}(hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = n\rho Gy + C_{15}$$

For  $V_z$ : Sum of integrals from sub-equations #11, #12, #13, and for  $P(z)$ , from #14

$$V_z(x, y, z, t) = -\frac{\rho G}{2\mu}(rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}$$

### Comparison of the above methods for the solutions of Linearized Navier-Stokes Equations

Note below that the solutions by the two different methods are the same except for the constants involved. Now, one has two different methods for solving the system of Navier-Stokes equations. Such an agreement and consistency confirm the validity of the method used in solving the magnetohydrodynamic equations.

#### Solutions by Method 1

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x x$$

$$V_y(x, y, z, t) = -\frac{\rho g_y}{2\mu}(hx^2 + jy^2 + mz^2) + C_1x + C_3y + C_5z + \frac{qg_y}{4}t + C; \quad P(y) = n\rho g_y y$$

$$V_z(x, y, z, t) = -\frac{\rho g_z}{2\mu}(rx^2 + sy^2 + uz^2) + C_1x + C_3y + C_5z + \frac{wg_z}{4}t + C; \quad P(z) = v\rho g_z z$$

#### Solutions by Method 2

$$V_x(x, y, z, t) = -\frac{\rho G}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fG}{4}t + C_8; \quad P(x) = d\rho Gx + C_7$$

$$V_y(x, y, z, t) = -\frac{\rho G}{2\mu}(hx^2 + jy^2 + mz^2) + C_9x + C_{11}y + C_{13}z + \frac{qG}{4}t + C_{16}; \quad P(y) = n\rho Gy + C_{15}$$

$$V_z(x, y, z, t) = -\frac{\rho G}{2\mu}(rx^2 + sy^2 + uz^2) + C_{17}x + C_{19}y + C_{21}z + \frac{wG}{4}t + C_{24}; \quad P(z) = v\rho Gz + C_{23}$$

## Overall Conclusion

The Navier-Stokes (N-S) equations in 3-D and 4-D have been solved analytically for the first time by two different methods. In Method 1, the three equations were separately integrated.

In Method 2, the three equations were first added together and a single equation was integrated. The solutions from these two methods were the same, except for the constants involved. The N-S solution is unique. The experience gained in solving the linearized equation helped the author to propose a new law, the law of definite ratio for incompressible fluid flow. This law states that in incompressible fluid flow, the other terms of the fluid flow equation divide the gravity term in a definite ratio, and each term utilizes gravity to function. The sum of the terms of the ratio is always unity. The application of this law helped speed-up the solutions of the non-linearized N-S equations, since there was no more experimentation as to the subject of the equation. It was also shown that without gravity forces on earth, there would be no incompressible fluid flow on earth as is known.

The solutions and relations revealed the role of each term of the Navier-Stokes equations in fluid flow. Most importantly, the gravity term is the indispensable term in fluid flow, and it is involved in the parabolic as well as the forward motion. The pressure gradient term is also involved in the parabolic motion. The viscosity terms are involved in parabolic, periodic and decreasingly exponential motion. As the viscosity increases, periodicity increases. The variable acceleration term is also involved in the periodic and decreasingly exponential motion. The convective acceleration terms produce square root function behavior and behavior of fractional terms containing square root functions with variables in the denominator. In terms of the velocity profile, the first three terms characterize parabolas. If one assumes that in laminar flow, the axis of symmetry of the parabola for horizontal velocity flow profile is in the direction of fluid flow, then in turbulent flow, the axis of symmetry of the parabola would have been rotated 90 degrees from that for laminar flow. The characteristic curve for the  $x$ -nonlinear term is such a parabola whose axis of symmetry has been rotated 90 degrees from that of laminar flow. The  $y$ -nonlinear term is similar parabolically to the  $x$ -nonlinear term. The characteristic curve for the  $z$ -nonlinear term is a combination of two similar parabolas and a hyperbola. If the above  $x$ -direction flow is repeated simultaneously in the  $y$ - and  $z$ - directions, the flow is chaotic and consequently turbulent.

The following statements can be made:

- (a) The N-S equations have unique solutions; (b) The N-S equations have parabolic solutions; 3. The N-S equations have square root function solutions. 4. The N-S equations do not have periodic solutions but have periodic relations. 5. The N-S equations do not have decreasingly exponential solutions but have decreasingly exponential relations.

In applications, the ratio terms  $a, b, c, d, f, h, n, q$  and others may perhaps be determined using information such as initial and boundary conditions or may have to be determined experimentally. The author came to the experimental determination conclusion after referring to preliminaries...The question is how did the grandmother determine the terms of the ratio for her grandchildren? Note that so far as the general solutions of the N-S equations are concerned, one needs not find the specific values of the ratio terms.

Finally, for any fluid flow design, one should always maximize the role of gravity for cost-effectiveness, durability, and dependability. Perhaps, Newton's law for fluid flow should read "Sum of everything else equals  $\rho g$ "; and this would imply the proposed new law that the other terms divide the gravity term in a definite ratio, and each term utilizes gravity to function.

**P.S.** The liquid pressure,  $P$  at the bottom of a liquid of depth  $h$  units is given by  $P = \rho gh$ .

From the N-S solution in this paper,  $P(x) = d\rho gx$  from integrating  $\frac{dp}{dx} = d\rho g$  where  $d$  is ratio term. Each of the other terms in the N-S equation must also be set equal to the product of a ratio term and  $\rho g$ . This result implies that the approach used in solving the N-S equations is valid..

## Option 8

### Spin-off: CMI Millennium Prize Problem Requirements

#### Proof 1

##### For the linearized Navier-Stokes equations

##### Proof of the existence of solutions of the Navier-Stokes equations

Since from page 11, it has been shown that the smooth equations given by

$$V_x(x, y, z, t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \frac{fg_x}{4}t + C_9; \quad P(x) = d\rho g_x x$$
 are solutions

of the linearized equation,  $-\mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \frac{\partial p}{\partial x} + 4\rho \frac{\partial V_x}{\partial t} = \rho g_x$ , it has been shown that smooth solutions to the above differential equation exist. and the proof is complete.

From, above, if  $y = 0, z = 0$ ,  $V_x(x, t) = -\frac{\rho g_x}{2\mu}ax^2 + C_1x + \frac{fg_x}{4}t + C_9$ ;  $P(x) = d\rho g_x x + C_{10}$

Therefore,  $V_x(x, 0) = V_x^0(x) = -\frac{\rho g_x}{2\mu}ax^2 + C_{10}x + C_9$

**Finding**  $P(x, t)$

1.  $V_x(x, t) = -\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9$ ;  $P(x) = d\rho g_x x$     2.  $\frac{\partial p}{\partial x} = d\rho g_x$

**Required:** To find  $P(x, t)$  (that is, find a formula for  $P$  in terms of  $x$  and  $t$ )

$$\frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx} V_x \quad \left(\frac{dx}{dt} = V_x\right)$$

$$\frac{dp}{dt} = d\rho g_x \left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9\right) \quad \left(\frac{dp}{dx} = d\rho g_x\right)$$

$$P(x, t) = \int d\rho g_x \left(-\frac{\rho g_x}{2\mu}(ax^2) + C_1x + \frac{fg_x}{4}t + C_9\right) dt$$

$$P(x, t) = d\rho g_x \left(-\frac{a\rho g_x}{2\mu}x^2t + C_1xt + \frac{fg_x}{8}t^2 + C_9t\right) + C_{10}$$

For the corresponding coverage for the original Navier-Stokes equation, see the next page

## Proof 2

### For the Non-linearized Navier-Stokes equations (Original Equations)

#### Proof of the existence of solutions of the Navier-Stokes equations

From page 22, if  $y = 0, z = 0$  in

<b>Solution to Linear part</b>
$V_x(x,y,z,t) = -\frac{\rho g_x}{2\mu}(ax^2 + by^2 + cz^2) + C_1x + C_3y + C_5z + \underbrace{fg_x t \pm \sqrt{2hg_x x} + \frac{ng_x y}{V_y} + \frac{qg_x z}{V_z} + \frac{\psi_y(V_y)}{V_y} + \frac{\psi_z(V_z)}{V_z}}_{\text{continued! solution of Euler equation}}$
$P(x) = d\rho g_x x$

one obtains

$$V_x(x,t) = -\frac{\rho g_x}{2\mu} ax^2 + C_1x + fg_x t \pm \sqrt{2hg_x x} + C_9; \quad P(x) = d\rho g_x x;$$

$$V_x(x,0) = V_x^0(x) = -\frac{\rho g_x}{2\mu} ax^2 + C_1x \pm \sqrt{2hg_x x} + C_9; \quad P(x) = d\rho g_x x;$$

Since previously, from p.21, it has been shown that the smooth equations given by

$$V_x(x,t) = -\frac{\rho g_x}{2\mu} ax^2 + C_1x + fg_x t \pm \sqrt{2hg_x x} + C_9; \quad P(x) = d\rho g_x x; \text{ are solutions of}$$

$-\mu \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial p}{\partial x} + \rho \frac{\partial V_x}{\partial t} + \rho V_x \frac{\partial V_x}{\partial x} = \rho g_x$  (deleting the  $y$ - and  $z$ - terms of (A)), p.20, one has shown that smooth solutions to the above differential equation exist, and the proof is complete.

**Finding**  $P(x,t)$ :

$$1. \quad V_x(x,t) = -\frac{\rho g_x}{2\mu} ax^2 + C_1x + fg_x t \pm \sqrt{2hg_x x} + C_9; \quad P(x) = d\rho g_x x; \quad 2. \quad \frac{\partial p}{\partial x} = d\rho g_x;$$

$$\frac{dp}{dt} = \frac{dp}{dx} \frac{dx}{dt}$$

$$\frac{dp}{dt} = \frac{dp}{dx} V_x \quad \left( \frac{dx}{dt} = V_x \right)$$

$$\frac{dp}{dt} = d\rho g_x \left( -\frac{\rho g_x}{2\mu} (ax^2) + C_1x \pm \sqrt{2hg_x x} + fg_x t + C_9 \right) \quad \left( \frac{dp}{dx} = d\rho g_x \right)$$

$$P(x,t) = \int d\rho g_x \left( -\frac{\rho g_x}{2\mu} (ax^2) + C_1x \pm \sqrt{2hg_x x} + fg_x t + C_9 \right) dt$$

$$P(x,t) = d\rho g_x \left( -\frac{a\rho g_x}{2\mu} x^2 t + C_1 x t \pm t \sqrt{2hg_x x} + \frac{fg_x t^2}{2} + C_9 t \right) + C_{10}$$

#### References:

For paper edition of the above paper, see Chapter 11 & Appendix 7 of the book entitled "Power of Ratios" by A. A. Frempong, published by Yellowtextbooks.com. Without using ratios or proportion, the author would never be able to split-up the Navier-Stokes equations into sub-equations which were readily integrable. The impediment to solving the Navier-Stokes equations for over 150 years (whether linearized or non-linearized) has been due to finding a way to split-up the equations. Since ratios were the key to splitting the Navier-Stokes equations, and solving them, the solutions have also been published in the "Power of Ratios" book which covers definition of ratio and applications of ratio in mathematics, science, engineering, economics and business fields.