A Gauge Theory of Gravity in Curved Phase-Spaces *

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

December 2015, Revised January 2916

Abstract

After a cursory introduction of the basic ideas behind Born's Reciprocal Relativity theory, the geometry of the cotangent bundle of spacetime is studied via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. A novel gauge theory of gravity in the $8D$ cotangent bundle T^*M of spacetime is explicitly constructed and based on the gauge group $SO(6,2) \times_s R^8$ which acts on the tangent space to the cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$ at each point $(\mathbf{x},\mathbf{p})$. Several gravitational actions involving curvature and torsion tensors and associated with the geometry of curved phase spaces are presented. We conclude with a brief discussion of the field equations, the geometrization of matter, QFT in accelerated frames, **T**-duality, double field theory, and generalized geometry.

Keywords : Gravity, Finsler Geometry, Born Reciprocity, Phase Space.

1 Born's Reciprocal Relativity in Phase Space

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where

*Dedicated to the loving memory of Carmenza Castro Ramirez

$X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 - (dp_x)^2 - (dp_y)^2 - (dp_z)^2 \right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (1.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_l = \sqrt{\frac{\hbar\ c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar\ b}{c}}; \quad \lambda_e = \sqrt{\hbar\ b\ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G\ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (1.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (1.2) invariant are [2]

$$T' = T\ cosh\xi + \left(\frac{\xi_v\ X}{c^2} + \frac{\xi_a\ P}{b^2}\right) \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$E' \;=\; E\; cosh\xi \;+\; (-\xi_a\; X \;+\; \xi_v P)\; \frac{sinh\xi}{\xi} \tag{1.4b}$$

$$X' \;=\; X\; cosh\xi \;+\; (\xi_v\; T \;-\; \frac{\xi_a\; E}{b^2})\; \frac{sinh\xi}{\xi} \tag{1.4c}$$

$$P' \;=\; P\; cosh\xi \;+\; (\frac{\xi_v\; E}{c^2} \;+\; \xi_a\; T)\; \frac{sinh\xi}{\xi} \tag{1.4d}$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}}, \;\; \xi \;=\; \sqrt{(\frac{\xi_v}{c})^2 + (\frac{\xi_a}{b})^2} \tag{1.5}$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the spacetime proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 \;=\; (d\tau)^2 \;\left(\; 1 \;-\; \frac{F^2}{F^2_{max}}\;\right) \tag{1.6}$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\; dT \wedge E + dX \wedge dP$.

One can verify also that the transformations eqs-(1.4) are invariant under the discrete transformations

$$(T, X) \to (E, P); \quad (E, P) \to (-T, -X), \quad b \to \frac{1}{b} \tag{1.7}$$

we argued [17] that the latter transformation $b \to \frac{1}{b}$ is a manifestation of the large/small tension T-duality symmetry in string theory. In natural units of $\hbar = c = 1$, the maximal proper force $\mathbf{b}$ has the same dimensions as a string tension (energy per unit length) $(mass)^2$.

To understand the *invariant* meaning of the interval in phase space $d\sigma$, and to show the consistency of eqs-(1.4,1.5,1.6), let us describe the following scenario. A massive free particle does not experience any force, thus the momentum is conserved so that $\frac{dp_a}{d\tau} = 0$ and the flat phase space interval is $(d\sigma)^2 = (d\tau)^2$. In an accelerated frame of reference the massive particle experiences a pseudo-force which implies that $\frac{dp'_a}{d\tau'} \neq 0$. Upon choosing an infinite rapidity parameter $\xi_a = \infty$ in eqs-(1.5), the value of the pseudo-force reaches its maximal proper value $F_{max} = \mathbf{b}$. Also, $(d\tau')^2 = \infty$ when the acceleration rapidity parameter is ∞, as one can verify from eqs-(1.4) by simple inspection. Since the interval in flat phase space $(d\sigma)^2$ (1.6), in an inertial frame and accelerated frame of reference, respectively, remains invariant under the transformations (1.4) one has that $(d\sigma)^2 = (d\tau)^2 = (d\tau')^2(1-F^2/F^2_{max}) = \infty \times 0 \neq 0$. The latter product cannot be zero, because if $(d\tau)^2$ were zero, in the inertial non-accelerated frame of reference , this would mean that the massive free particle would have followed a null geodesic, which it cannot do since only massless photons can.

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

A discussion of Mach's principle within the context of Born Reciprocal Gravity in Phase Spaces was described in [17]. The Machian postulate states that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. It is also consistent with equating the maximal proper force $m_{Planck}(c^2/L_{Planck})$ to $M_{Universe}(c^2/R_{Hubble})$ and reflecting a maximal/minimal acceleration duality. By invoking Born's reciprocity between coordinates and momenta, a minimal Planck scale should correspond to a minimum momentum, and consequently to an upper scale given by the Hubble radius. Further details can be found in [17].

The purpose of this work is to analyze the *curved* phase-space scenario in more detail and the geometry of the cotangent bundle of spacetime via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. In the case of the cotangent space of a d-dim manifold T^*M_d the metric components can be equivalently rewritten in the *block* diagonal form [10] such that the line element is given by

$$(ds)^2 \; = \; g_{ij}(x^k, p_a) \; dx^i d \; x^j \; + \; h^{ab}(x^k, p_c) \; \delta p_a \; \delta p_b,$$

$$i, j, k = 1, 2, ..., d, \quad a, b, c = 1, 2, ..., d \tag{1.8}$$

if instead of using the standard coordinate basis frames one introduces the following *nonholonomic* frames (non-coordinate basis)

$$\delta_i \; = \; \delta/\delta x^i \; = \; \partial_{x^i} \; + \; N_{ia} \; \partial^a \; = \; \partial_{x^i} \; + \; N_{ia} \; \partial_{p_a}; \;\; \partial^a \; \equiv \; \partial_{p_a} \; = \; \frac{\partial}{\partial p_a} \tag{1.9}$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \;\; \delta p_a \; = \; dp_a \; - \; N_{ja} \; dx^j \tag{1.10}$$

where the $N_{ja}(x, p)$-coefficients define a *nonlinear* connection. When $N_{ia} = 0$ and $h^{ab} = g^{ab}/b^2 = \eta^{ab}/b^2$, the interval in eq-(1.8) reduces to the Born-Green interval in eq-(1.1). In the very special case such that $N_{ja}(x,p) = \Gamma^k_{ja}(x)p_k$, the N-connection becomes linear in the momentum with $\Gamma^k_{ja}(x)$ being the underlying spacetime connection. The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some nonholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$, with nontrivial nonholonomy coefficients $W^\gamma_{\alpha\beta}$ given in terms of derivatives of N_{ia} [9], [10]. The indices α, β, γ comprise both base and fiber coordinate indices.

An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form [10], [9]

$$D_{\delta_j}(\delta_i) \ = \ H^k_{ij}\ \delta_k; \ \ D_{\delta_j}(\partial^a) \ = \ - \ H^a_{bj}\ \partial^b; \tag{1.11a}$$

$$D_{\partial^a}(\delta_i) \ = \ C^{ka}_i\ \delta_k; \ \ D_{\partial^a}(\partial^b) \ = \ - \ C^{ba}_c\ \partial^c \tag{1.11b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \ = \ \frac{1}{2}T^i_{jk}\ dx^j \wedge dx^k \ + \ C^{ia}_j\ dx^j \wedge \delta p_a \tag{1.12a}$$

$$\Omega_a \ = \ \frac{1}{2}R_{jka}\ dx^j \wedge dx^k \ + \ P^b_{aj}\ dx^j \wedge \delta p_b \ + \ \frac{1}{2}S^{bc}_a\ \delta p_b \wedge \delta p_c \tag{1.12b}$$

and the curvature 2-forms are

$$\Omega^i_j \ = \ \frac{1}{2}R^i_{jkm}\ dx^k \wedge dx^m \ + \ P^{ia}_{jk}\ dx^k \wedge \delta p_a \ + \ \frac{1}{2}S^{iab}_j\ \delta p_a \wedge \delta p_b \tag{1.13}$$

$$\Omega^a_b \ = \ \frac{1}{2}R^a_{bkm}\ dx^k \wedge dx^m \ + \ P^{ac}_{bk}\ dx^k \wedge \delta p_c \ + \ \frac{1}{2}S^{acd}_b\ \delta p_c \wedge \delta p_d \tag{1.14}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i,\ \partial^a = \partial/\partial p_a$ is given by $dx^i,\ \delta p_a = dp_a - N_{ja}dx^j$. The explicit expressions for the terms

$$T^i_{jk},\ \ C^{ia}_j,\ R_{jka},\ P^b_{aj},\ S^{bc}_a,\ R^i_{jkm},\ P^{ia}_{jk},\ S^{iab}_j,\ R^a_{bkm},\ P^{ac}_{bk},\ S^{acd}_b \tag{1.15}$$

in eqs-(1.12-1.14) are given explicitly in terms of the connection coefficients of eqs-(1.11) and the nonlinear connection and nonholonomy coefficients as shown in [10], [9]. The expressions are rather lengthy, for this reason we refer to [10], [9] for detailed calculations.

The Hamilton geometry of the phase space of particles whose motion is characterized by generalized dispersion relations was recently studied by [6]. In this framework, spacetime and momentum space are naturally *curved* and *intertwined*, allowing for a simultaneous description of both spacetime curvature and non-trivial momentum space geometry. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [7].

In the cotangent space description one has covariance under a more *restricted* set of coordinate transformations of the form [10]

$$x'^i \ = \ x'^i(x^j), \ \ p'_i \ = \ p_j \ \frac{\partial x^j}{\partial x'^i} \tag{1.16}$$

such that there is an *entanglement* of spacetime and momentum variables in the transformed momentum fiber coordinates. However, Quaplectic transformations in flat phase space have a different form $x'^i \ = \ x'^i(x^j, p_j)$ and $p'_i = p'_i(x^j, p_j)$. Thus one cannot accommodate the Quaplectic transformations in eqs-(1.4) to curved phase spaces (the cotangent bundle T^*M) in the manner described by eq-(1.16). This problem is beyond the scope of this work. A plausible solution is to *complexify* the spacetime cotangent bundle by introducing complex coordinates $z^\mu = x^\mu + ip_\mu/b$, and whose complex conjugate momenta are π_μ, along with the transformations $z'^\mu = z'^\mu(z^\nu), \pi'_\mu = \pi_\nu \frac{\partial z^\nu}{\partial z'^\mu}$. This would lead to a mixing of x^μ and p_μ encoded in the transformations of the base coordinates $z'^\mu = z'^\mu(z^\nu)$.

To finalize this section, we remark that in this letter we are following another approach than the one based on Hamilton geometry in investigating curved phase spaces. In the next section, a novel gauge theory of gravity in the $8D$ cotangent bundle T^*M of four-dimensional spacetime is constructed and based on the gauge group $SO(6,2) \times_s R^8$. Several gravitational actions associated with the geometry of curved phase spaces are presented. The geometry of the $8D$ tangent bundle of $4D$ spacetime and the physics of a limiting value of the proper acceleration in spacetime [4] has been studied by Brandt [3] . Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the *infinite* acceleration limit. We must emphasize that the results described in the next section are quite different than those obtained earlier by us in [13] and by [10], [9], [3], [6] among others.

2 Gauge Theories of Gravity in the Cotangent Bundle

In this section we will construct a novel gauge theory of gravity in the $8D$ cotangent bundle T^*M based on the gauge group given by the semidirect product $SO(6,2) \times_s R^8$. Let us begin with a Lie group $\mathcal{G}$; its associated Lie algebra is spanned by the generators $\mathcal{L}_A, A = 1, 2, \ldots,$ dim $\mathcal{G}$, and whose structure constants are f^C_{AB}. The Lie algebra commutator is $[\mathcal{L}_A, \mathcal{L}_B] = f^C_{AB}\mathcal{L}_C$. The components of the gauge field strength in the $8D$ cotangent bundle T^*M, and corresponding to the Lie-algebra valued gauge fields $\mathcal{A}^A_i\mathcal{L}_A, \mathcal{A}^A_a\mathcal{L}_A$, are

$$\mathcal{F}^A_{ij} \ = \ \delta_i \mathcal{A}^A_j \ - \ \delta_j \mathcal{A}^A_i \ + \ [\ \mathcal{A}_i, \ \mathcal{A}_j \ ]^A \ - \ W^L_{ij} \ \mathcal{A}^A_L \ =$$

$$(\frac{\partial}{\partial x^i} \ + \ N_{ib} \ \frac{\partial}{\partial p_b}) \ \mathcal{A}^A_j \ - \ (\frac{\partial}{\partial x^j} \ + \ N_{jb} \ \frac{\partial}{\partial p_b}) \ \mathcal{A}^A_i \ +$$

$$\mathcal{A}_i^B\ \mathcal{A}_j^C\ f_{BC}^A\ -\ W_{ij}^L\ \mathcal{A}_L^A \tag{2.1}$$

$$\mathcal{F}_{ab}^A\ =\ \frac{\partial}{\partial p^a}\mathcal{A}_b^A\ -\ \frac{\partial}{\partial p^b}\mathcal{A}_a^A\ +\ \mathcal{A}_a^B\ \mathcal{A}_b^C\ f_{BC}^A\ -\ W_{ab}^L\ \mathcal{A}_L^A \tag{2.2}$$

$$\mathcal{F}_{ia}^A\ =\ \delta_i\mathcal{A}_a^A\ -\ \partial_a\mathcal{A}_i^A\ +\ \mathcal{A}_i^B\ \mathcal{A}_a^C\ f_{BC}^A\ -\ W_{ia}^L\ \mathcal{A}_L^A \tag{2.3}$$

$$\mathcal{F}_{ai}^A\ =\ \partial_a\mathcal{A}_i^A\ -\ \delta_i\mathcal{A}_a^A\ +\ \mathcal{A}_a^B\ \mathcal{A}_i^C\ f_{BC}^A\ -\ W_{ai}^L\ \mathcal{A}_L^A \tag{2.4}$$

where the nontrivial nonholonomy coefficients are $W_{MN}^L = -W_{NM}^L$. The indices M, N, L comprise both base x^i and fiber coordinate p_a indices. The nonholonomic frame fields in eq-(1.9) satisfy the relations $\delta_M\delta_N - \delta_N\delta_M = W_{MN}^L\delta_L$. There is anti-symmetry in the indices $\mathcal{F}_{ia}^A = -\mathcal{F}_{ai}^A$. From eq-(1.10) one has that the Lie-algebra-valued two-form field strengths are given by $\mathcal{F}_{ia}^A dx^i \wedge \delta p^a$, where $dx^i \wedge \delta p^a\ \ =\ -\ \delta p^a \wedge dx^i$; and by $\mathcal{F}_{ij}^A dx^i \wedge dx^j$, $\mathcal{F}_{ab}^A \delta p^a \wedge \delta p^b$, respectively. In nonholonomic frames one must include the nonholonomy coefficients W_{MN}^L in the definition of the gauge field strengths resulting from the exterior differential of a one-form $\mathbf{dA}$. In a coordinate-free form it is given by $\mathbf{dA}(\mathbf{X}, \mathbf{Y}) = \mathbf{X}(\mathbf{A}(\mathbf{Y})) - \mathbf{Y}(\mathbf{A}(\mathbf{X})) - \mathbf{A}([\mathbf{X}, \mathbf{Y}])$ [14]. In a holonomic frame, the basis vectors commute $[\mathbf{X}, \mathbf{Y}] = 0$ and one recovers the standard definition of the exterior derivative of a one-form.

From eq-(1.9) one can evaluate the nonholonomy coefficients W_{MN}^L which turn out to be given by

$$W_{ij}^a\ =\ \delta_i N_j^a\ -\ \delta_j N_i^a\ =\ (\partial_i\ +\ N_i^b\partial_b)\ N_j^a\ -\ (\partial_j\ +\ N_j^b\partial_b)\ N_i^a$$

$$W_{ia}^b = -W_{ai}^b\ =\ -\,\partial_a N_i^b \tag{2.5a}$$

and the rest are vanishing

$$W_{ij}^k\ =\ 0,\ \ W_{ia}^j\ =\ -\ W_{ai}^b\ =\ 0,\ \ W_{ab}^i\ =\ 0,\ \ W_{ab}^c\ =\ 0 \tag{2.5b}$$

We shall choose the gauge group to be the semidirect product $SO(6,2) \times_s R^8$ which is the extension of the $4D$ Poincare group $SO(3,1) \times_s R^4$ given by the semidirect product of the Lorentz group with the translations. The flat metric in the tangent space to the cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$, at the point $(\mathbf{x}, \mathbf{p})$, is $\eta_{AB} = diag\ (-, +, +, +, -, +, +, +)$. There are two timelike directions corresponding to the temporal coordinate x^0 and the energy p^0.

The $SO(6,2)$ Lie algebra generators $\mathcal{L}_{AB}$ obey the commutation relations

$$[\mathcal{L}_{AB},\ \mathcal{L}_{CD}]\ =\ (\eta_{BC}\mathcal{L}_{AD} - \eta_{AC}\mathcal{L}_{BD} - \eta_{BD}\mathcal{L}_{AC} + \eta_{AD}\mathcal{L}_{BC}). \tag{2.6a}$$

The other commutators associated with the translation generators $\mathcal{P}_A$ are

$$[\mathcal{L}_{AB},\ \mathcal{P}_C]\ =\ (\ \eta_{BC}\ \mathcal{P}_A\ -\ \eta_{AC}\ \mathcal{P}_B\ );\quad [\mathcal{P}_A,\ \mathcal{P}_B]\ =\ 0 \tag{2.6b}$$

In a holonomic frame (a coordinate basis) the metric G_{MN} in the $8D$ cotangent bundle T^*M is given by

$$G_{MN} \ = \ G_{MN}(x,p) \ = $$

$$\begin{pmatrix} g_{ij}(x,p) \ + \ h_{ab}(x,p) \ N_i^a(x,p) \ N_j^b(x,p) & - \ N_i^a(x,p) \ h_{ab}(x,p) \\ - \ N_j^b(x,p) \ h_{ab}(x,p) & h_{ab}(x,p) \end{pmatrix} \quad (2.7)$$

In a non-holonomic frame (a non-coordinate basis) the metric G_{MN} is (g_{ij}, h_{ab}) *block diagonal* as depicted in eq-(1.8). The entries of G_{MN} have different units, one could introduce suitable factors of **b** in order to have the same units for all the entries of G_{MN} if one wishes. For simplicity we shall set $\mathbf{b} = 1$. One could also have complex (Hermitian) metrics of the form $G_{MN} = G_{(MN)} + iG_{[MN]}$ with an antisymmetric piece $G_{[MN]}$. We refer to [11] for a study of gauge theories of Born Reciprocal Gravity based on the Quaplectic group [2] given by the semidirect product of the (pseudo) unitary group with the Weyl-Heisenberg group.

The frame E_M^A fields are introduced such that

$$G_{MN} \ = \ E_M^A \ E_N^B \ \eta_{AB} \quad (2.8)$$

where $A, B = 1, 2, \ldots, 8$ are the indices of the tangent space to the $8D$ cotangent bundle $T_{(\mathbf{x},\mathbf{p})}T^*M$, at each point $(\mathbf{x}, \mathbf{p})$. $M, N = 1, 2, \ldots, 8$ are the indices of the cotangent bundle T^*M of the $4D$ spacetime manifold M.

The Lie-algebra valued gauge field is

$$\mathbf{A}_M \ = \ \Omega_M^{AB} \ \mathcal{L}_{AB} \ + \ E_M^A \ \mathcal{P}_A \quad (2.9)$$

where Ω_M^{AB} (analog of the spin connection) is the field that gauges the $SO(6,2)$ symmetry. E_M^A gauges the (Abelian) local translations in $T_{(\mathbf{x},\mathbf{p})}T^*M$. Defining the derivative operators as

$$\hat{\partial}_M \ \equiv \ (\delta_i, \ \partial_a) \ = \ (\frac{\partial}{\partial x^i} \ + \ N_{ib}\frac{\partial}{\partial p_b}, \ \frac{\partial}{\partial p_a}) \quad (2.10)$$

where we take the index M to comprise both the base spacetime and fiber indices. The Lie-algebra valued field strength is given by

$$\mathbf{F}_{MN} \ = \ \hat{\partial}_M \mathbf{A}_N \ - \ \hat{\partial}_N \mathbf{A}_M \ + \ [\mathbf{A}_M, \ \mathbf{A}_N] \ - \ W_{MN}^L \ \mathbf{A}_L \quad (2.11)$$

The curvature two-form associated with the spin connection $\Omega_M^{AB} \ = \ - \ \Omega_M^{BA}$ is

$$\mathcal{R}_{MN}^{AB} \ \equiv \ \mathcal{F}_{MN}^{AB} \ = \ \hat{\partial}_M \Omega_N^{AB} \ - \ \hat{\partial}_N \Omega_M^{AB} \ + \ \Omega_{[M}^{AC} \ \Omega_{N]}^{CB} \ - \ W_{MN}^L \ \Omega_L^{AB} \quad (2.12)$$

and whose explicit components are

$$\mathcal{R}^{AB}_{ij} \equiv \mathcal{F}^{AB}_{ij} = (\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b})\,\Omega^{AB}_j - (\frac{\partial}{\partial x^j} + N_{jb}\frac{\partial}{\partial p_b})\,\Omega^{AB}_i + \Omega^{AC}_{[i}\,\Omega^{CB}_{j]} - W^L_{ij}\,\Omega^{AB}_L \tag{2.13}$$

$$\mathcal{R}^{AB}_{ab} \equiv \mathcal{F}^{AB}_{ab} = \frac{\partial}{\partial p^a}\,\Omega^{AB}_b - \frac{\partial}{\partial p^b}\,\Omega^{AB}_a + \Omega^{AC}_{[a}\,\Omega^{CB}_{b]} - W^L_{ab}\,\Omega^{AB}_L \tag{2.14}$$

$$\mathcal{R}^{AB}_{ia} \equiv \mathcal{F}^{AB}_{ia} = (\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b})\,\Omega^{AB}_a - \frac{\partial}{\partial p^a}\,\Omega^{AB}_i + \Omega^{AC}_{[i}\,\Omega^{CB}_{a]} - W^L_{ia}\,\Omega^{AB}_L \tag{2.15}$$

and $\mathcal{F}^{AB}_{ai} = -\mathcal{F}^{AB}_{ia}$. A summation over the repeated indices is implied and $[MN]$ denotes the anti-symmetrization of indices with weight one.

The explicit components of the torsion two-form defined as

$$\mathcal{T}^A_{MN} \equiv \mathcal{F}^A_{MN} = \hat{\partial}_M E^A_N - \hat{\partial}_N E^A_M + \Omega^{AC}_{[M}\,E^C_{N]} - W^L_{MN}\,E^A_L \tag{2.16}$$

are

$$\mathcal{T}^A_{ij} \equiv \mathcal{F}^A_{ij} = (\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b})\,E^A_j - (\frac{\partial}{\partial x^j} + N_{jb}\frac{\partial}{\partial p_b})\,E^A_i + \Omega^{AC}_{[i}\,E^C_{j]} - W^L_{ij}\,E^A_L \tag{2.17}$$

$$\mathcal{T}^A_{ab} \equiv \mathcal{F}^A_{ab} = \frac{\partial}{\partial p^a}\,E^A_b - \frac{\partial}{\partial p^b}\,E^A_a + \Omega^{AC}_{[a}\,E^C_{b]} - W^L_{ab}\,E^A_L \tag{2.18}$$

$$\mathcal{T}^A_{ia} \equiv \mathcal{F}^A_{ia} = (\frac{\partial}{\partial x^i} + N_{ib}\frac{\partial}{\partial p_b})\,E^A_a - \frac{\partial}{\partial p^a}\,E^A_i + \Omega^{AC}_{[i}\,E^C_{a]} - W^L_{ia}\,E^A_L \tag{2.19}$$

and $\mathcal{F}^A_{ai} = -\mathcal{F}^A_{ia}$.

The frame fields allow us to construct the curvature tensor on the cotangent bundle T^*M as follows

$$\mathcal{R}^Q_{MNP} \equiv \mathcal{R}^{AB}_{MN}\;E^Q_A\,E_{BP} = \mathcal{F}^{AB}_{MN}\;E^Q_A\,E_{BP} \tag{2.20}$$

where the explicit components $\mathcal{F}^{AB}_{MN}$ are obtained in eqs- (2.13-2.15) . E^M_A is the inverse frame field such that $E^M_A\,E^B_M = \delta^B_A$ and $E_{AM}\,E^M_B = \eta_{AB}$. The contraction of indices yields the Ricci-like tensors.

$$\mathcal{R}_{MP} = \delta^N_Q\,\mathcal{R}^Q_{MNP} \tag{2.21a}$$

A further contraction yields the generalized Ricci scalar

$$\mathcal{R} \;=\; G^{MP}\;\mathcal{R}_{MP} \tag{2.21b}$$

The Torsion tensors are

$$\mathcal{T}_{MNQ} \;=\; \mathcal{F}^{A}_{MN}\;E_{AQ},\;\;\mathcal{T}^{Q}_{MN} \;=\; \mathcal{F}^{A}_{MN}\;E^{Q}_{A},\;\;\mathcal{T}_{M} \;=\; \delta^{N}_{Q}\;\mathcal{T}^{Q}_{MN} \tag{2.22}$$

A Lagrangian, linear in the curvature scalar and quadratic in torsion, can be chosen to be

$$\mathcal{L} \;=\; c_1\;\mathcal{R} \;+\; c_2\;\mathcal{T}_{MNQ}\;\mathcal{T}^{MNQ} \;+\; c_3\;\mathcal{T}_M\;\mathcal{T}^M. \tag{2.23}$$

where c_1, c_2, c_3 are numerical coefficients. The action is

$$S \;=\; \frac{1}{2\kappa^2}\int_{\Omega_8} d^8Y\;\sqrt{|\;det\;G_{MN}\;|}\;\mathcal{L} \tag{2.24}$$

where κ^2 is the analog of the gravitational coupling constant and the $8D$ measure of integration involves

$$d^8Y \;\equiv\; dx^1\wedge dx^2\wedge dx^3\wedge dx^4\wedge \delta p_1\wedge \delta p_2\wedge \delta p_3\wedge \delta p_4 \tag{2.25}$$

with

$$\delta p_a \;=\; dp_a \;-\; N_{ai}\;dx^i \tag{2.26}$$

Other measures besides $\sqrt{|detG_{MN}|}$ in eq-(2.24) can be used in tangent/cotangent bundles. For example, see the discussion on the Busemann-Hausdorff and Holmes-Thompson measure in [12]. For simplicity we shall retain the ordinary measure in eq-(2.24).

The curvature (2.13-2.15) depends on the geometric quantities g_{ij}, h_{ab}, N_{ia} that describe the metric (2.7) and Ω^{AB}_M. The number of degrees of freedom $d(2d+1)$ associated with g_{ij}, h_{ab}, N_{ia} is the same as the number of degrees of freedom of a metric G_{MN} in $2d$ dimensions. Furthermore, if the torsion (2.16) is set to zero one can solve Ω^{AB}_M in terms of E^A_M. To sum up, in the absence of torsion, the action (2.24) represents effectively a Poincare-like gauge theory of gravity in 8 dimensions, written in a *nonholonomic* coordinate basis, and where the gauge group is $SO(6,2)\times_s R^8$.

Bars [15] has proposed a gauge symmetry in phase space. One of the consequences of this gauge symmetry is a new formulation of physics in spacetime. Instead of one time there must be *two* times, while phenomena described by one-time physics in $3+1$ dimensions appear as various shadows of the same phenomena that occur in $4+2$ dimensions with one extra space and one extra time dimensions (more generally, $d+2$). Problems of ghosts and causality are resolved automatically by the $Sp(2,R)$ gauge symmetry in phase space.

The ordinary $4D$ Einstein-Hilbert action can be written in terms of the vielbeins e^a_i and spin connection ω^{ab}_i as

$$S \;=\; \frac{1}{16\pi G} \int \; e_i^a \wedge e_j^b \wedge R_{kl}^{cd}(\omega_i^{ab}, e_i^a) \; \epsilon_{abcd} \; \epsilon^{ijkl} \tag{2.27}$$

The natural extension of (2.27) to the $8D$ cotangent bundle T^*M is

$$\frac{1}{2\kappa^2} \int \; E_{M_1}^{A_1} \wedge E_{M_2}^{A_2} \wedge E_{M_3}^{A_3} \wedge E_{M_4}^{A_4} \wedge E_{M_5}^{A_5} \wedge E_{M_6}^{A_6} \wedge \mathcal{R}_{M_7 M_8}^{A_7 A_8} \; \epsilon_{A_1 A_2 \ldots A_8} \; \epsilon^{M_1 M_2 \ldots M_8} \tag{2.28}$$

One could also introduce Lanczos-Lovelock-like Lagrangians in D-dimensions, written in terms of the generalized Kronecker deltas,

$$\delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \;=\; \frac{1}{n!} \; \delta^{\mu_1 \nu_1}_{[\alpha_1 \beta_1} \; \delta^{\mu_2 \nu_2}_{\alpha_2 \beta_2} \; \ldots \; \delta^{\mu_n \nu_n}_{\alpha_n \beta_n]} \tag{2.29}$$

as

$$\mathcal{L} \;=\; \sum_{n=0}^{|D/2|} a_n \; \mathcal{R}^{(n)}, \quad \mathcal{R}^{(n)} \;=\; \frac{1}{2^n} \; \delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \; \prod \; \mathcal{R}^{\alpha_r \beta_r}_{\mu_r \nu_r} \tag{2.30}$$

where $|D/2|$ is the integer part of $D/2$; a_n are coupling constants of dimensions $(length)^{2n-D}$. In the $8D$ cotangent bundle case T^*M the range of indices is $\alpha, \beta \;=\; 1, 2, \ldots, 8;\; \mu, \nu, \ldots, 8$. The first four indices correspond to the four-dim spacetime, and the last four indices to the momentum space. Despite the product of curvatures, the advantage of Lanczos-Lovelock Lagrangians is that they lead to field equations containing only derivatives of the metric up to *second* order, and in arbitrary number of dimensions.

The field equations associated with the above actions **S** are obtained via an Euler variation with respect to the independent fields appearing in the description of the metric of the cotangent bundle G_{MN} displayed in eq-(2.7)

$$\frac{\delta \mathbf{S}}{\delta g_{ij}} = 0, \quad \frac{\delta \mathbf{S}}{\delta h_{ab}} = 0, \quad \frac{\delta \mathbf{S}}{\delta N_i^a} = 0 \tag{2.31}$$

it is beyond the scope of this letter to find solutions to these very complicated set of differential equations. One could also follow a different approach to gravity in curved phase spaces described in section **1**. By recurring to eqs-(1.11-1.15), and writing the metric in *block* diagonal form which allows to factorize the determinant of the metric as $(det g_{ij})(det h_{ab})$, one could study the analog of the Einstein vacuum field equations

$$\mathcal{R}_{ij} \;-\; \frac{1}{2} (\; \mathcal{R} + \mathcal{S} \;) \; g_{ij} \;= 0 \;; \quad S_{ab} \;-\; \frac{1}{2} (\; \mathcal{R} + \mathcal{S} \;) \; h_{ab} \;= 0 \tag{2.32}$$

and supplemented by the equations

$$\frac{\delta \mathcal{R}}{\delta N_i^a} \;+\; \frac{\delta \mathcal{S}}{\delta N_i^a} \;=\; 0 \tag{2.33}$$

where the spacetime and internal space scalar curvatures are, respectively,

$$\mathcal{R} \;=\; \delta_i^j \; R^i_{kjl} \; g^{kl}; \;\; \mathcal{S} \;=\; \delta_b^d \; S_d^{abc} \; h_{ac} \tag{2.34}$$

These type of equations were studied by Vacaru [9] and some solutions were found in some special cases. We leave the study of the field equations described by eqs-(2.31) for future work. Another issue that warrants further investigation is the issue of ghosts since the cotangent bundle metric signature has two timelike directions. Physical theories with two times encounter ghosts in general except if there are additional symmetries like $Sp(2,R)$ that allow the removal of ghosts [15]. Since the $2d$-dim cotangent bundle is a symplectic space we may recur to the symplectic group $Sp(2d,R)$ to see if a similar mechanism allows the ghosts removal.

3 Conclusions : Towards the Geometrization of Matter and T-Duality

The results of this work leads us to believe that a *geometrization* of matter is of paramount importance in the quantization program of gravity based on the geometry of cotangent spaces (phase spaces). For instance, in $4D$ Riemannian spacetimes, one finds that Einstein's field equations, in units of $8\pi G = c = 1$,

$$R_{\mu\nu} \ - \ \frac{1}{2} \ g_{\mu\nu} \ R \ = \ T_{\mu\nu} \tag{3.1}$$

exhibit a geometry/matter reciprocity symmetry, because after replacing

$$R_{\mu\nu} \leftrightarrow T_{\mu\nu}, \quad R = g^{\mu\nu} R_{\mu\nu} \leftrightarrow T = g^{\mu\nu} T_{\mu\nu} \tag{3.2}$$

in eq-(3.1) it yields

$$T_{\mu\nu} \ - \ \frac{1}{2} \ g_{\mu\nu} \ T \ = \ R_{\mu\nu} \tag{3.3}$$

one can notice that the last eq-(3.3) is indeed *equivalent* to eq-(3.1) after simply taking the trace of eq-(3.1) in $D = 4$ and leading to $T = -R$. In this respect four dimensions is singled out.

In other dimensions than $D = 4$ one can look back at eqs-(2.32)

$$\mathcal{R}_{ij} \ - \ \frac{1}{2} \ (\ \mathcal{R} + \mathcal{S} \ ) \ g_{ij} \ = 0 \ \Rightarrow$$

$$\mathcal{R}_{ij} \ - \ \frac{1}{2} \ \mathcal{R} \ = \ \frac{1}{2} \ g_{ij} \ \mathcal{S} \ = \ T_{ij} \tag{3.4}$$

such that when all the quantities in eq-(3.4) solely depend on the coordinates x^i (and not on the momenta p_a) one finds that the scalar *curvature* $\mathcal{S}$ in *momentum* space (times $g_{ij}/2$) plays the role of an effective stress energy tensor T_{ij} in the horizontal spacetime M. Hence, matter sources (mass in particular) can be effectively *geometrized* (mimicked) by the *momentum* space *curvature*.

In classical mechanics, inertial mass is that property of matter which opposes acceleration. The Quaplectic group transformations in flat phase spaces, implementing Born's Reciprocal Relativity principle [1], implies the physical equivalence of accelerated frames of reference [2]. Likewise, Special Relativity is based on the physical equivalence of inertial frames in flat Minkowski spacetime via Lorentz transformations. One of the most salient features of the Quaplectic group transformations is the *mixing* of spacetime coordinates with the energy-momentum coordinates as described in section **1**.

This picture of the equivalence of accelerated frames in flat phase space differs considerably from the one in ordinary Quantum Field Theory (QFT). The physics behind accelerated frames in Minkowski space is essential in the Fulling-Davies-Unruh effect, where an accelerating observer will observe black-body radiation where an inertial observer would observe none. From the viewpoint of the accelerating observer, the vacuum of the inertial observer will look like a state containing many particles in thermal equilibrium (a warm gas of photons). The Unruh temperature [19] is the effective temperature experienced by a uniformly accelerating detector in a vacuum field. It is given by $T = \frac{\hbar a}{2\pi c k_B}$, where a is the local acceleration, and k_B is the Boltzmann constant. The Unruh temperature has the same form as the Hawking temperature after replacing a for the surface gravity at the black hole horizon.

Recently, Dasgupta [20] re-investigated the Bogoliubov transformations which relate the Minkowski inertial vacuum to the vacuum of an accelerated observer. He implemented the transformation using a non-unitary operator used in formulations of irreversible systems by Prigogine [21]. An attempt was discussed to generalize Quantum Field Theory (QFT) for accelerated frames using this new connection to Prigogine transformations. It is warranted to build a generalized QFT in accelerated frames which is compatible with the Quaplectic group transformations in Born's Reciprocal Relativity [1]. This may shed some light into the resolution of the black hole information paradox by recurring to *novel* physical principles and which are beyond the many current proposals based on standard QFT in curved Riemannian spacetimes.

Finally we add that earlier on in eq-(1.7) we argued how Born Reciprocal Relativity could provide a physical mechanism to understand T-duality in string theory [17]. Nowadays it is pursued via Double Field Theory (DFT) [18]. The idea behind DFT is to introduce a doubled space with coordinates $X^M = (x^i, \tilde{x}^i), M = 1, \ldots, 2D$, on which $O(D, D)$ acts naturally in the fundamental representation. One has doubled the number of all spacetime coordinates. This idea is actually well motivated by string theory on toroidal backgrounds, where these coordinates are dual both to momentum and winding modes. An extension of DFT to exceptional groups, now commonly referred to as exceptional field theory, allows us to settle open problems in Kaluza-Klein truncations of supergravity that, although of conventional nature, were impossible to solve with standard techniques [18]. We have not addressed in this work how to accommodate DFT to Born Reciprocal Relativity and the geometry of (co) tangent bundles. It is becoming more clear that generalized geometries (like Finsler geometry) warrant further investigation.

Acknowledgements

We thank M. Bowers for very kind assistance.

References

[1] M. Born, Proc. Royal Society **A 165** (1938) 291.

M. Born, Rev. Mod. Physics **21** (1949) 463.

[2] S. Low, Jour. Phys **A** Math. Gen **35** (2002) 5711.

S. Low, Il Nuovo Cimento **B 108** (1993) 841;

S. Low, Found. Phys. **36** (2007) 1036.

S. Low, J. Math. Phys **38** (1997) 2197.

[3] H. Brandt, Contemporary Mathematics **196** (1996) 273.

H. Brandt, Found Phys. Letts **4** (1991) 523.

H. Brandt, Chaos, Solitons and Fractals **10** (2-3) (1999) 267.

[4] E. Caianiello, Lett. Nuovo Cimento **32** (1981) 65.

[5] C. Castro, " Some consequences of Born's Reciprocal Relativity in Phase Spaces" Foundations of Physics **35**, no.6 (2005) 971.

[6] L. Barcaroli, L. Brunkhorst, G. Gubitosi, N. Loret and C. Pfeifer, "Hamilton geometry: Phase space geometry from modified dispersion relations" arXiv : 1507.00992.

[7] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "Relative locality: A deepening of the relativity principle," Gen. Rel .Grav. **43** (2011) 2547, arXiv:1106.0313 [hep-th].

[8] Jan J. Slawianowski, *Geometry of Phase Spaces*, (John Wiley and Sons, 1990).

[9] S. Vacaru and Heinz Dehnen, "Locally Anisotropic Nonlinear Connections in Einstein and Gauge Gravity" [arXiv:gr-qc/0009039].

S. Vacaru and Y. Goncharenko, "Yang-Mills and Gauge Gravity on Generalized Lagrange and Finsler Spaces", Int. J. Theor. Phys. **34** (1995) 1955.

S. Vacaru, " Gauge Gravity and Conservation Laws in Higher Order Anisotropic Spaces" hep-th/9810229.

S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, *"Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity"* (Geometry Balkan Press, 693 pages).

S. Vacaru, "Finsler-Lagrange Geometries and Standard Theories in Physics: New Methods in Einstein and String Gravity" [arXiv : hep-th/0707.1524].

[10] R. Miron, D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* (Kluwer Academic Publishers, Dordrecht, Boston, 2001).

R. Miron, *Lagrangian and Hamiltonian geometries. Applications to Analytical Mechanics*, arXiv: 1203.4101 [math.DG].

[11] C. Castro, Phys. Letts **B 668** (2008) 442.

[12] H. Busemann, Ann. of Math., 48, 234 (1947). Comment. Math. Helvet. 24, 156 (1950).

R. Holmes and A. Thompson Pac. Jour. Math., 85, 1 (1979).

J. Silva, R. Maluf and C. A. S. Almeida, "A nonlinear dynamics for the scalar field in Randers spacetime" arXiv: 1511.00769.

[13] C. Castro, "Gravity in Curved Phase-Spaces, Finsler Geometry and Two Times Physics" Int. J. Mod.Phys. **A27** (2012) 1250069

C. Castro, "Born's Reciprocal Gravity in Curved Phase-Spaces and the Cosmological Constant " Found. Phys. **42**, issue 8 (2012) 1031-1055

[14] M. Nakahara, *Geometry, Toplogy and Physics* (IOP Publishing, Bristol and Philadelphia 1998).

[15] I. Bars, Int. J. Mod. Phys. **A 25** (2010). 5235.

I. Bars, Phys. Rev. **D 54** (1996) 5203,

[16] D. Lovelock, "The Einstein tensor and its generalizations". Journal of Mathematical Physics **12** no. (3) (1971) 498.

[17] C. Castro, "On Dual Phase Space Relativity, the Machian Principle and Modified Newtonian Dynamics" Prog. in Phys. **1** (April 2005) 20.

[18] C. Hull and B. Zwiebach, "Double Field Theory" JHEP 1204 (2012) 020.

M. Duff, J. Lu, R. Percacci, C. Pope, H. Samtleben and E. Sezgin, " Membrane Duality Revisited" arXiv:1509.02915.

L. Motls, http://motls.blogspot.com/2015/07/introduction-to-double-field-theory-12.html

[19] W. Unruh , "Notes on black-hole evaporation". Physical Review **D 14**, no. 4 (1976) 870.

http://en.wikipedia.org/wiki/Unruh_effect

[20] A. Dasgupta, "Quantum Field Theory in Accelerated Frames" arXiv: 1508.01940.

[21] I. Prigogine, *From Being to Becoming : Time and Complexity in the Physical Sciences* (WH. Freeman and Co. 1980).