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Mathematics with Natural Reality

— Action Flows

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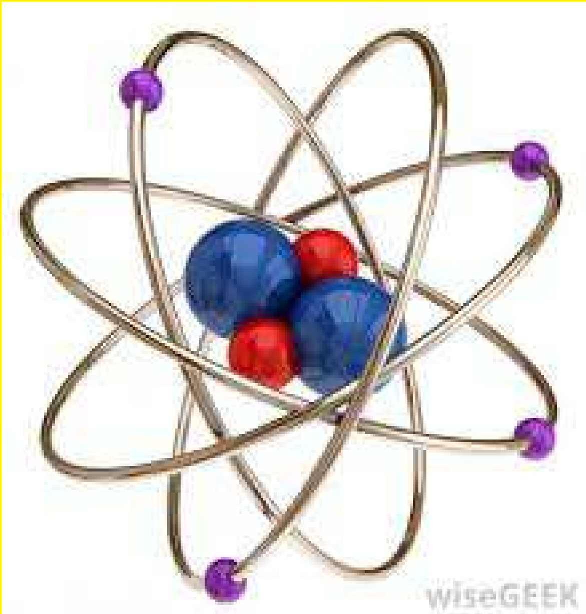

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Mathematics with Natural Reality — Action Flows

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Abstract: The universality of contradiction implies that the reality of a thing is only hold on observation with level dependent on the observer standing out or in and lead respectively to solvable equation or non-solvable equations on that thing for human beings. Notice that all contradictions are artificial, not the nature of things. Thus, holding on reality of things forces one extending contradictory systems in classical mathematics to a compatible one by combinatorial notion, particularly, *action flow* on differential equations, which is in fact an embedded oriented graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L:(v,u)\rightarrow L(v,u)$, 2 end-operators $A_{vu}^{+}:L(v,u)\rightarrow L^{A_{vu}^{+}}(v,u)$ and $A_{uv}^{+}:L(u,v)\rightarrow L^{A_{uv}^{+}}(u,v)$ on a Banach space $\mathscr{B}$ with $L(v,u)=-L(u,v)$ and $A_{vu}^{+}(-L(v,u))=-L^{A_{vu}^{+}}(v,u)$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ holding with conservation laws

$$\sum_{u\in N_G(v)} L^{A_{vu}^{+}}(v,u)=\mathbf{0}, \qquad \forall v\in V\left(\overrightarrow{G}\right).$$

The main purpose of this paper is to survey the powerful role of action flows to mathematics such as those of extended Banach $\overrightarrow{G}$-flow spaces, the representation theorem of Fréchet and Riesz on linear continuous functionals, geometry on action flows or non-solvable systems of solvable differential equations with global stability, $\cdots$ etc., and their applications to physics, ecology and other sciences. All of these makes it clear that knowing on the reality by solvable equations is local, only on coherent behaviors but by action flow on equations and generally, contradictory system is universal, which is nothing else but a mathematical combinatorics.

Key Words: Action flow, $\overrightarrow{G}$-flow, natural reality, observation, Smarandache multi-space, differential equation, topological graph, CC conjecture.

AMS(2010): 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35.

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§1. Introduction

A thing P is usually complex, even hybrid with other things but the understanding of human beings is bounded, brings about a unilateral knowledge on P identified with its known characters, gradually little by little. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be its known and $\nu_i, i \geq 1$ unknown characters at time t. Then, thing P is understood by

$$P = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right) \bigcup \left(\bigcup_{k\geq 1}\{\nu_k\}\right), \tag{1.1}$$

i.e., a *Smarandache multispace*in logic with an approximation $P^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ at time t, reveals the diversity of things such as those shown in Fig.1 for the universe,

Fig.1

and that the reality of a thing P is nothing else but the state characters (1.1) of existed, existing or will existing things whether or not they are observable or comprehensible by human beings from a macro observation at a time t.

Generally, one establishes mathematical equation

$$\mathscr{F}\left(t, x_1, x_2, x_3, \psi_t, \psi_{x_1}, \psi_{x_2}, \cdots, \psi_{x_1 x_2}, \cdots\right) = 0 \tag{1.2}$$

to determine the behavior of a thing P, for instance the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \tag{1.3}$$

on particles, where $\hbar = 6.582\times 10^{-22} MeV s$ is the Planck constant. *Can we conclude the mathematical equation (*1.2*) characterize the reality of thing P by solution ψ?* The answer is not certain, particularly, for the equation (1.3) on the superposition, i.e., in two or more possible states of being of particles, but the solution ψ of (1.3) characterizes only its one position.

Notice that things are inherently related, not isolated in the nature, observed characters are filtering sensory information on things. Whence these is a topological structure on things, i.e., an inherited topological graph G in space. On the other hand, any oriented graph $G = \left(V, \overrightarrow{E}\right)$ can be embedded into $\mathbb{R}^n$ if $n \geq 3$ because if there is an intersection p between edges $\varphi(e)$ and $\varphi(e')$ in embedding (G, φ) of G, we can always operate a surgery on curves $\varphi(e)$ and $\varphi(e')$ in a sufficient small neighborhood $N(p)$ of p such that there are no intersections again and this surgery can be operated on all intersections in (G, φ). Furthermore, if G is simple, i.e., without loops or multiple edges, we can choose n points $v_1 = (t_1, t_1^2, t_1^3)$, $v_2 = (t_2, t_2^2, t_2^3)$, $\cdots$, $v_n = (t_n, t_n^2, t_n^3)$ for different $t_i, 1 \leq i \leq n$, $n = |G|$ on curve (t, t^2, t^3). Then it is clear that the straight lines v_iv_j, v_kv_l have no intersections for any integers $1 \leq i, j, k, l \leq n$ ([26]). Thus, there is such a mapping φ in this case that all edges of (G, φ) are straight segments, i.e., rectilinear embedding in $\mathbb{R}^n$ for G if $n \geq 3$. We therefore conclude that

$$Oriented\ Graphs\ in\ \mathbb{R}^n \quad \Leftrightarrow \quad Inherent\ Structure\ of\ Natural\ Things.$$

Thus, for understanding the reality, particularly, multiple behavior of a thing P, an effective way is return P to its nature and establish a mathematical theory on embedded graphs in $\mathbb{R}^n$, $n \geq 3$, which is nothing else but *flows* in dynamical mechanics, such as the water flow in a river shown in Fig.2.

Fig.2

There are two commonly properties known to us on water flows. Thus, the rate of flow is continuous on time t, and for its any cross section C, the in-flow is always equal to the out-flow on C. Then, *how can we describe the water flow in*

*Fig.*2 *on there properties*? Certainly, we can characterize it by network flows simply. A network is nothing else but an oriented graph $G = (V, \overrightarrow{E})$ with a continuous function $f : \overrightarrow{E} \to \mathbb{R}$ holding with conditions $f(u,v) = -f(v,u)$ for $\forall (u,v) \in \overrightarrow{E}$ and $\sum\limits_{u \in N_G(v)} f(v,u) = 0$. For example, the network shown in Fig.3 is the abstracted model for water flow in Fig.2 with conservation equation $a(t) = b(t) + c(t)$, where $a(t), b(t)$ and $c(t)$ are the rates of flow on time t at the cross section of the river.

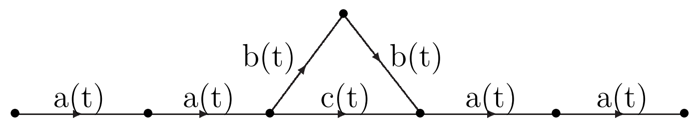


Fig.3

A further generalization of network by extending flows to elements in a Banach space with actions results in action flow following.

Definition 1.1 *An action flow* $\left(\overrightarrow{G}; L, A\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : (v,u) \to L(v,u)$, 2 *end-operators* $A_{vu}^{+} : L(v,u) \to L^{A_{vu}^{+}}(v,u)$ *and* $A_{uv}^{+} : L(u,v) \to L^{A_{uv}^{+}}(u,v)$ *on a Banach space* $\mathscr{B}$ *with* $L(v,u) = -L(u,v)$ *and* $A_{vu}^{+}(-L(v,u)) = -L^{A_{vu}^{+}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$

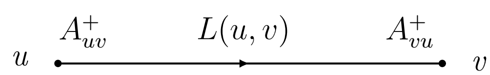


Fig.4

holding with conservation laws

$$\sum_{u \in N_G(v)} L^{A_{vu}^{+}}(v,u) = \mathbf{0} \quad for \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v *in Fig.*5 *following*

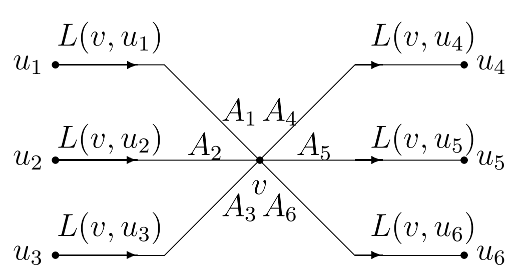


Fig.5

with a conservation law

$$-L^{A_1}(v,u_1)-L^{A_2}(v,u_2)-L^{A_4}(v,u_3)+L^{A_4}(v,u_4)+L^{A_5}(v,u_5)+L^{A_6}(v,u_6)=\mathbf{0},$$

where an embedding of G *in* $\mathscr{S}$ *is a* 2*-tuple* (G,φ) *with a* $1-1$ *continuous mapping* $\varphi: G\to\mathscr{S}$ *such that an intersection only appears at end vertices of* G *in* $\mathscr{S}$*, i.e.,* $\varphi(p)\neq\varphi(q)$ *if* $p\neq q$ *for* $\forall p,q\in G$*.*

Notice that action flows is also an expression of the *CC conjecture*, i.e., *any mathematical science can be reconstructed from or made by combinatorialization* ([7], [20]). But they are elements for hold on the nature of things.

The main purpose of this paper is to survey the powerful role of action flows in mathematics and other sciences such as those of extended Banach $\overrightarrow{G}$-flow spaces, the representation theorem of Fréchet and Riesz on linear continuous functionals, , geometry on action flows and geometry on non-solvable systems of solvable differential equations, combinatorial manifolds, global stability of action flows, $\cdots$, etc. on two cases following with applications to physics and other sciences:

Case 1. $\overrightarrow{G}$*-flows*, i.e., action flows $\left(\overrightarrow{G};L,\mathbf{1}_{\mathscr{B}}\right)$, which enable one extending Banach space to Banach $\overrightarrow{G}$-flow space and find new interpretations on physical phenomenons. Notices that an action flow with $A^+_{vu}=A^+_{uv}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ is itself a $\overrightarrow{G}$-flow if replacing $L(u,v)$ by $L^{A^+_{vu}}(v,u)$ on (v,u).

Case 2. *Differential flows*, i.e., action flows $\left(\overrightarrow{G};L,A\right)$ with ordinary differential or partial differential operators A^+_{vu} on some edges $(v,u)\in E\left(\overrightarrow{G}\right)$, which includes classical geometrical flow as the particular in cases of $\left|\overrightarrow{G}\right|=1$. Usually, if $\left|\overrightarrow{G}\right|\geq 2$, such a flow characterizes non-solvable system of physical equations.

For example, let the $L:(v,u)\to L(v,u)\in\mathbb{R}^n\times\mathbb{R}^+$ with action operators $A^+_{vu}=a_{vu}\dfrac{\partial}{\partial t}$ and $a_{vu}\in\mathbb{R}^n$ for any edge $(v,u)\in E\left(\overrightarrow{G}\right)$ in Fig.6 following.

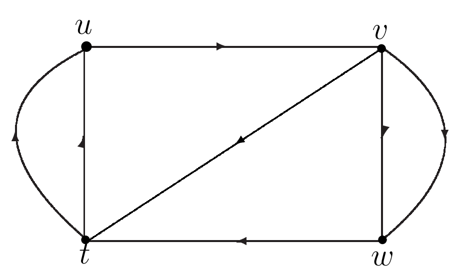


Fig.6

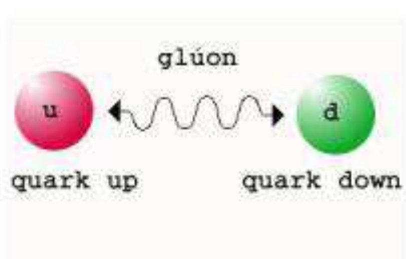
Then the conservation laws are partial differential equations

$$\begin{cases} a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t} + a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} = a_{uv}\dfrac{\partial L(u,v)}{\partial t} \\ a_{uv}\dfrac{\partial L(u,v)}{\partial t} = a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t} + a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t} + a_{vt}\dfrac{\partial L(v,t)}{\partial t} \\ a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t} + a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t} = a_{wt}\dfrac{\partial L(w,t)}{\partial t} \\ a_{wt}\dfrac{\partial L(w,t)}{\partial t} + a_{vt}\dfrac{\partial L(v,t)}{\partial t} = a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t} + a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} \end{cases}$$

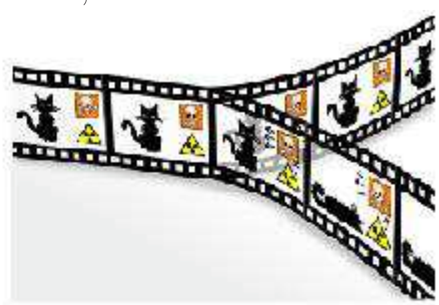
For terminologies and notations not mentioned here, we follow references [1] for mechanics, [2] for functional analysis, [11] for graphs and combinatorial geometry, [4] and [27] for differential equations, [22] for elementary particles, and [23] for Smarandache multispaces.

§2. $\overrightarrow{G}$-Flows

The divisibility of matter initiates human beings to search elementary constituting cells of matter and interpretation on the superposition of microcosmic particles such as those of quarks, leptons with those of their antiparticles, and unmatters between a matter and its antimatter([24-25]). For example, baryon and meson are predominantly formed respectively by three or two quarks in the model of Sakata, or Gell-Mann and Ne'eman, and H.Everett's multiverse ([5]) presented an interpretation for the cat in Schrödinger's paradox in 1957, such as those shown in Fig.7.



Quark Model



Multiverse on Schrödinger's Cat

Fig.7

Notice that we only hold coherent behaviors by an equation on a natural thing, not the individual because that equation is established by viewing abstractly a particle to be a geometrical point or an independent field from a macroscopic point, which leads physicists always assuming the internal structures mechanically for hold on the behaviors of matters, likewise Sakata, Gell-Mann, Ne'eman or H.Everett. However, such an assumption is a little ambiguous in mathematics, i.e., we can not even conclude which is the point or the independent field, the matter or its submatter. But $\overrightarrow{G}$-flows verify the rightness of physicists ([17]).

2.1 Algebra on Graphs

Let $\overrightarrow{G}$ be an oriented graph embedded in $\mathbb{R}^n, n \geq 3$ and let $(\mathscr{A}; \circ)$ be an algebraic system in classical mathematics, i.e., for $\forall a, b \in \mathscr{A}$, $a \circ b \in \mathscr{A}$. Denoted by $\overrightarrow{G}^L_{\mathscr{A}}$ all of those labeled graphs $\overrightarrow{G}^L$ with labeling $L: X\left(\overrightarrow{G}\right) \to \mathscr{A}$. We extend operation $\circ$ on elements in $\overrightarrow{G}^L_{\mathscr{A}}$ by a ruler following:

R : *For* $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^L_{\mathscr{A}}$, *define* $\overrightarrow{G}^{L_1} \circ \overrightarrow{G}^{L_2} = \overrightarrow{G}^{L_1 \circ L_2}$, *where* $L_1 \circ L_2 : e \to L_1(e) \circ L_2(e)$ *for* $\forall e \in E\left(\overrightarrow{G}\right)$.

For example, such an extension on graph $\overrightarrow{C}_4$ is shown in Fig.8, where, $\mathbf{a}_3 = \mathbf{a}_1 \circ \mathbf{a}_2$, $\mathbf{b}_3 = \mathbf{b}_1 \circ \mathbf{b}_2$, $\mathbf{c}_3 = \mathbf{c}_1 \circ \mathbf{c}_2$, $\mathbf{d}_3 = \mathbf{d}_1 \circ \mathbf{d}_2$.

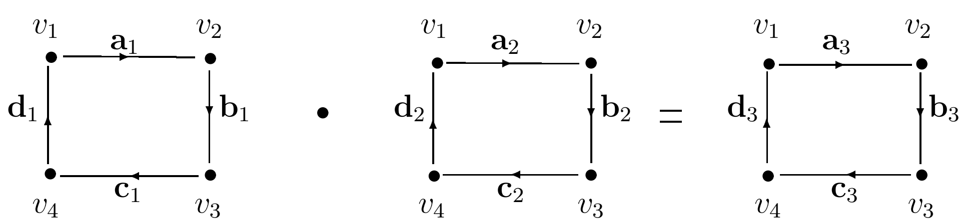


Fig.8

Notice that $\overrightarrow{G}^L_{\mathscr{A}}$ is also an algebraic system under ruler **R**, i.e., $\overrightarrow{G}^{L_1} \circ \overrightarrow{G}^{L_2} \in \overrightarrow{G}^L_{\mathscr{A}}$ by definition. Furthermore, $\overrightarrow{G}^L_{\mathscr{A}}$ is a group if $(\mathscr{A}, \circ)$ is a group because of

(1) $\left(\overrightarrow{G}^{L_1} \circ \overrightarrow{G}^{L_2}\right) \circ \overrightarrow{G}^{L_3} = \overrightarrow{G}^{L_1} \circ \left(\overrightarrow{G}^{L_2} \circ \overrightarrow{G}^{L_3}\right)$ for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_3} \in \overrightarrow{G}^L_{\mathscr{A}}$ because $(L_1(e) \circ L_2(e)) \circ L_3(e) = L_1(e) \circ (L_2(e) \circ L_3(e))$ for $e \in E\left(\overrightarrow{G}\right)$.

(2) there is an identify $\overrightarrow{G}^{L_{1_{\mathscr{A}}}}$ in $\overrightarrow{G}^L_{\mathscr{A}}$, where $L_{1_{\mathscr{A}}} : e \to 1_{\mathscr{A}}$ for $\forall e \in E\left(\overrightarrow{G}\right)$;

(3) there is an uniquely element $\overrightarrow{G}^{L^{-1}}$ holding with $\overrightarrow{G}^{L^{-1}} \circ \overrightarrow{G}^L = \overrightarrow{G}^{L_{1_{\mathscr{A}}}}$ for $\forall \overrightarrow{G}^L \in \overrightarrow{G}^L_{\mathscr{A}}$.

Thus, an algebraic system can be naturally extended on an embedded graph, and this fact holds also with those of algebraic systems of multi-operations. For example, let $\mathscr{R} = (R; +, \cdot)$ be a ring and $(\mathscr{V}; +, \cdot)$ a vector space over field $\mathcal{F}$. Then it is easily know that $\overrightarrow{G}^L_{\mathscr{R}}$, $\overrightarrow{G}^L_{\mathscr{V}}$ are respectively a ring or a vector space with zero vector $\mathbf{O} = \overrightarrow{G}^{L_\mathbf{0}}$, where $L_\mathbf{0}: \ e \to \mathbf{0}$ for $\forall e \in E\left(\overrightarrow{G}\right)$, such as those shown for $\overrightarrow{G}^L_{\mathscr{V}}$ on $\overrightarrow{C}_4$ in Fig.8 with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, $\mathbf{d}_i \in \mathscr{V}$ for $i = 1, 2, 3$, $\mathbf{x}_3$=$\mathbf{x}_1$+$\mathbf{x}_2$ for $\mathbf{x}$=$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or $\mathbf{d}$ and $\alpha \in \mathcal{F}$.

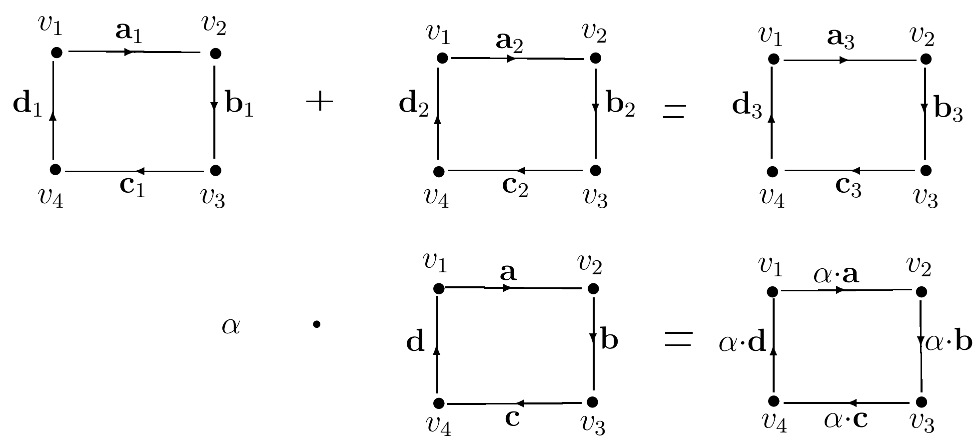


Fig.9

2.2 Action Flow Spaces

Notice that the algebra on graphs only is a formally operation system provided without the characteristics of flows, particularly, conservation, which can not be a portrayal of a natural thing because a measurable property of a physical system is usually conserved with connections. The notion wishing those of algebra on graphs with conservation naturally leads to that $\overrightarrow{G}$-flows, i.e., action flows $\left(\overrightarrow{G} : L, \mathbf{1}_{\mathscr{V}}\right)$ come into being. Thus, a $\overrightarrow{G}$-flow is a subfamily of $\overrightarrow{G}^L_{\mathscr{V}}$ limited by conservation laws. For example, if $\overrightarrow{G} = \overrightarrow{C}_4$, there must be $\mathbf{a}$=$\mathbf{b}$=$\mathbf{c}$=$\mathbf{d}$ and $\mathbf{a}_i$=$\mathbf{b}_i$=$\mathbf{c}_i$=$\mathbf{d}_i$ for $i = 1, 2, 3$ in Fig.9. Clearly, all $\overrightarrow{G}$-flows $\left(\overrightarrow{G}; L, \mathbf{1}_{\mathscr{V}}\right)$ on $\overrightarrow{G}$ for a vector space V over field $\mathscr{F}$ form a vector space by ruler $\mathbf{R}$, denoted by $\overrightarrow{G}^{\mathscr{V}}$.

Generally, a *conservative action family* is a pair $\{\{\mathbf{v}\}, \{\mathbf{A}(\mathbf{v})\}\}$ with vectors $\{\mathbf{v}\} \subset \mathscr{V}$ and operators A on $\mathscr{V}$ such that $\sum\limits_{\mathbf{v} \in V} \mathbf{v}^{A(\mathbf{v})} = \mathbf{0}$. Clearly, action flow consists of conservation action families. The result following establishes its inverse.

Theorem 2.1([17]) *An action flow* $\left(\overrightarrow{G}; L, A\right)$ *exists on* $\overrightarrow{G}$ *if and only if there are conservation action families* $L(v)$ *in a Banach space* $\mathscr{V}$ *associated an index set* V

with

$$L(v) = \{L^{A_{vu}^+}(v,u) \in \mathscr{V} \ for\ some\ u \in V\}$$

such that $A_{vu}^+(-L(v,u)) = -L^{A_{vu}^+}(v,u)$ *and*

$$L(v) \bigcap (-L(u)) = L(v,u) \quad or \quad \emptyset.$$

2.3 Banach $\overrightarrow{G}$-Flow Space

Let $(\mathscr{V}; +, \cdot)$ be a Banach or Hilbert space with inner product $\langle\cdot,\cdot\rangle$. We can furthermore introduce the *norm* and *inner product* on $\overrightarrow{G}^{\mathscr{V}}$ by

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(u,v)\in E\left(\overrightarrow{G}\right)} \|L(u,v)\|$$

and

$$\left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \right\rangle = \sum_{(u,v)\in E\left(\overrightarrow{G}\right)} \langle L_1(u,v), L_2(u,v)\rangle$$

for $\forall \overrightarrow{G}^L, \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$, where $\|L(u,v)\|$ is the norm of $L(u,v)$ in $\mathscr{V}$. Then, it can be easily verified that ([17]):

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi \left\|\overrightarrow{G}^L\right\|$ for any scalar ξ;

(3) $\left\|\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right\| \leq \left\|\overrightarrow{G}^{L_1}\right\| + \left\|\overrightarrow{G}^{L_2}\right\|$;

(4) $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle \geq 0$ and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$;

(5) $\left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \right\rangle = \overline{\left\langle \overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_1} \right\rangle}$ for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$;

(6) For $\overrightarrow{G}^L, \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}}$ and $\lambda, \mu \in \mathscr{F}$,

$$\left\langle \lambda\overrightarrow{G}^{L_1} + \mu\overrightarrow{G}^{L_2}, \overrightarrow{G}^L \right\rangle = \lambda \left\langle \overrightarrow{G}^{L_1}, \overrightarrow{G}^L \right\rangle + \mu \left\langle \overrightarrow{G}^{L_2}, \overrightarrow{G}^L \right\rangle.$$

Thus, $\overrightarrow{G}^{\mathscr{V}}$ is also a normed space by (1)-(3) or inner space by (4)-(6). By showing that any Cauchy sequence in $\overrightarrow{G}^{\mathscr{V}}$ is converged also holding with conservation laws in [17], we know the result following.

Theorem 2.2 *For any oriented graph $\overrightarrow{G}$ embedded in topological space $\mathscr{S}$, $\overrightarrow{G}^{\mathscr{V}}$ is a Banach space, and furthermore, if $\mathscr{V}$ is a Hilbert space, so is $\overrightarrow{G}^{\mathscr{V}}$.*

A $\overrightarrow{G}^L$-flow is *orthogonal* to $\overrightarrow{G}^{L'}$ if $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^{L'} \right\rangle = \mathbf{O}$. We know the orthogonal decomposition of Hilbert space $\overrightarrow{G}^{\mathscr{V}}$ following.

Theorem 2.3([17]) *Let $\mathscr{V}$ be a Hilbert space with an orthogonal decomposition $\mathscr{V} = \mathbf{V} \oplus \mathbf{V}^{\perp}$ for a closed subspace $\mathbf{V} \subset \mathscr{V}$. Then there is also an orthogonal decomposition*

$$\overrightarrow{G}^{\mathscr{V}} = \widetilde{\mathbf{V}} \oplus \widetilde{\mathbf{V}}^{\perp},$$

where, $\widetilde{\mathbf{V}} = \left\{ \overrightarrow{G}^{L_1} \in \overrightarrow{G}^{\mathscr{V}} \middle| L_1 : X\left(\overrightarrow{G}\right) \to \mathbf{V} \right\}$ and $\widetilde{\mathbf{V}}^{\perp} = \left\{ \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{V}} \middle| L_2 : X\left(\overrightarrow{G}\right) \to \mathbf{V}^{\perp} \right\}$, i.e., for $\forall \overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$, there is a uniquely decomposition $\overrightarrow{G}^L = \overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}$ with $L_1 : X\left(\overrightarrow{G}\right) \to \mathbf{V}$ and $L_2 : X\left(\overrightarrow{G}\right) \to \mathbf{V}^{\perp}$.

2.4 Actions on $\overrightarrow{G}$-Flow Spaces

Let $\mathscr{V}$ be a Hilbert space consisting of measurable functions $f(x_1, x_2, \cdots, x_n)$ on the functional space $L^2[\Delta]$ with inner product

$$\langle f(\mathbf{x}), g(\mathbf{x}) \rangle = \int_{\Delta} \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} \quad \text{for} \quad f(\mathbf{x}), g(\mathbf{x}) \in L^2[\Delta]$$

and

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \int_{\Delta}, \quad \overline{\int}_{\Delta}$$

are respectively differential operators and integral operators linearly defined by $D\overrightarrow{G}^L = \overrightarrow{G}^{DL(u^v)}$ and

$$\begin{aligned} \int_{\Delta} \overrightarrow{G}^L &= \int_{\Delta} K(\mathbf{x}, \mathbf{y}) \overrightarrow{G}^{L[\mathbf{y}]} d\mathbf{y} = \overrightarrow{G}^{\int_{\Delta} K(\mathbf{x},\mathbf{y}) L(u,v)[\mathbf{y}] d\mathbf{y}}, \\ \overline{\int}_{\Delta} \overrightarrow{G}^L &= \int_{\Delta} \overline{K(\mathbf{x}, \mathbf{y})} \overrightarrow{G}^{L[\mathbf{y}]} d\mathbf{y} = \overrightarrow{G}^{\int_{\Delta} \overline{K(\mathbf{x},\mathbf{y})} L(u,v)[\mathbf{y}] d\mathbf{y}} \end{aligned}$$

for $\forall (u, v) \in E\left(\overrightarrow{G}\right)$, where a_i, $\dfrac{\partial a_i}{\partial x_j} \in \mathbb{C}^0(\Delta)$ for integers $1 \leq i, j \leq n$ and $K(\mathbf{x}, \mathbf{y}) : \Delta \times \Delta \to \mathbb{C} \in L^2(\Delta \times \Delta, \mathbb{C})$ with

$$\int_{\Delta \times \Delta} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} < \infty.$$

For example, let let $f(t)=t$, $g(t)=e^t$, $K(t,\tau)=t^2+\tau^2$ for $\Delta=[0,1]$ and let $\overrightarrow{G}^L$ be the $\overrightarrow{G}$-flow shown on the left in Fig.10,

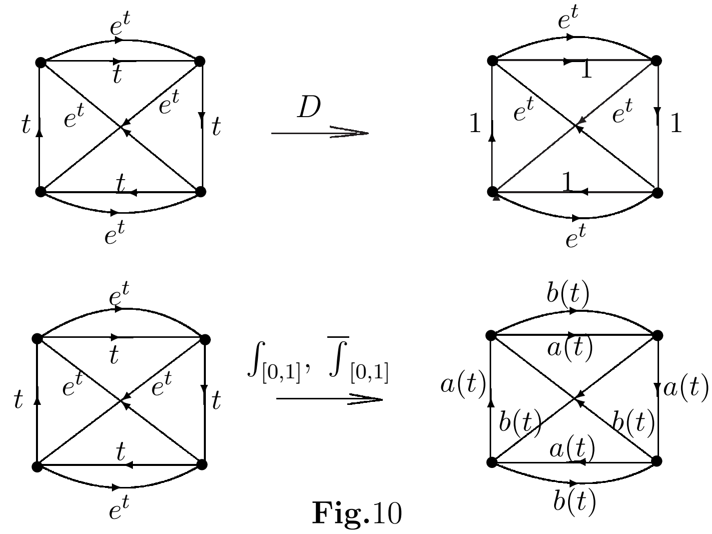


Fig.10

where $a(t)=\dfrac{t^2}{2}+\dfrac{1}{4}$ and $b(t)=(e-1)t^2+e-2$. We know the result following.

Theorem 2.4([17]) *$D: \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ and $\displaystyle\int_\Delta: \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$.*

Thus, operators D, $\displaystyle\int_\Delta$ and $\displaystyle\overline{\int}_\Delta$ are linear operators action on $\overrightarrow{G}^{\mathscr{V}}$.

Generally, let $\mathscr{V}$ be Banach space $\mathscr{V}$ over a field $\mathscr{F}$. An operator $\mathbf{T}: \overrightarrow{G}^{\mathscr{V}} \to \overrightarrow{G}^{\mathscr{V}}$ is linear if

$$\mathbf{T}\left(\lambda\overrightarrow{G}^{L_1}+\mu\overrightarrow{G}^{L_2}\right)=\lambda\mathbf{T}\left(\overrightarrow{G}^{L_1}\right)+\mu\mathbf{T}\left(\overrightarrow{G}^{L_2}\right)$$

for $\forall\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\overrightarrow{G}^{\mathscr{V}}$ and $\lambda,\mu\in\mathscr{F}$, and is continuous at a $\overrightarrow{G}$-flow $\overrightarrow{G}^{L_0}$ if there always exist such a number $\delta(\varepsilon)$ for $\forall\epsilon>0$ that

$$\left\|\mathbf{T}\left(\overrightarrow{G}^{L}\right)-\mathbf{T}\left(\overrightarrow{G}^{L_0}\right)\right\|<\varepsilon \quad if \quad \left\|\overrightarrow{G}^{L}-\overrightarrow{G}^{L_0}\right\|<\delta(\varepsilon).$$

The following result extends the Fréchet and Riesz representation theorem on linear continuous functionals to linear functionals $\mathbf{T}: \overrightarrow{G}^{\mathscr{V}} \to \mathbb{C}$ on $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{V}}$, where $\mathbb{C}$ is the complex field.

Theorem 2.5([17]) *Let $\mathbf{T}: \overrightarrow{G}^{\mathscr{V}} \to \mathbb{C}$ be a linear continuous functional, where $\mathscr{V}$ is a Hilbert space. Then there is a unique $\overrightarrow{G}^{\widehat{L}}\in\overrightarrow{G}^{\mathscr{V}}$ such that $\mathbf{T}\left(\overrightarrow{G}^{L}\right)=\left\langle\overrightarrow{G}^{L},\overrightarrow{G}^{\widehat{L}}\right\rangle$ for $\forall\overrightarrow{G}^{L}\in\overrightarrow{G}^{\mathscr{V}}$.*

2.5 $\overrightarrow{G}$-Flows on Equations

Let $\overrightarrow{G}$ be an oriented graph embedded in space $\mathbb{R}^n, n \geq 3$ and let

$$f(x_1, x_2, \cdots, x_n) = 0$$

be a solvable equation in a field $\mathscr{F}$. We are naturally consider its $\mathscr{F}$-extension equation

$$f(X_1, X_2, \cdots, X_n) = \mathbf{O}$$

in $\overrightarrow{G}^{\mathscr{F}}$ by viewing an element $b \in \mathscr{F}$ as $b = \overrightarrow{G}^{L}$ if $L(u^v) = b$ for $(u, v) \in X\left(\overrightarrow{G}\right)$ and $0 \neq a \in \mathscr{F}$. For example, the extension of equation $ax = b$ is $\overrightarrow{G}^{L_a} X = \overrightarrow{G}^{L_b}$ in $\overrightarrow{G}^{\mathscr{F}}$ with a $\overrightarrow{G}$-flow solution $x = \overrightarrow{G}^{a^{-1}L}$, such as those shown in Fig.11 for $\overrightarrow{G} = \overrightarrow{C}_4$, $a = 3$ and $b = 5$. Thus we can entrust a combinatorial structure $\overrightarrow{G}$ on its solution.

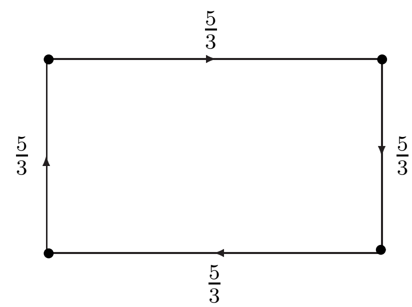


Fig.11

Generally, for a solvable system of linear equations, let $[L_{ij}]_{m\times n}$ be a matrix with entries $L_{ij}:\ u^v \to \mathscr{V}$. Denoted by $[L_{ij}]_{m\times n}(u, v)$ the matrix $[L_{ij}(u, v)]_{m\times n}$. A result on $\overrightarrow{G}$-flow solutions of linear systems was known in [17] following.

Theorem 2.6 *A linear system* (LES_m^n) *of equations*

$$\left\{\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = \overrightarrow{G}^{L_1} \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = \overrightarrow{G}^{L_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = \overrightarrow{G}^{L_m} \end{array}\right. \qquad (LES_m^n)$$

with $a_{ij} \in \mathbb{C}$ *and* $\overrightarrow{G}^{L_i} \in \overrightarrow{G}^{\mathscr{V}}$ *for integers* $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *is solvable for* $X_i \in \overrightarrow{G}^{\mathscr{V}}$, $1 \leq i \leq m$ *if and only if*

$$\text{rank}\,[a_{ij}]_{m\times n} = \text{rank}\,[a_{ij}]^{+}_{m\times(n+1)}(u, v)$$

for $\forall (u,v) \in \overrightarrow{G}$, *where*

$$[a_{ij}]^{+}_{m\times(n+1)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & L_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & L_2 \\ \cdots & \cdots & \cdot\cdot & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & L_m \end{bmatrix}.$$

For $\overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{V}}$, let

$$\frac{\partial \overrightarrow{G}^{L}}{\partial t} = \overrightarrow{G}^{\frac{\partial L}{\partial t}} \quad \text{and} \quad \frac{\partial \overrightarrow{G}^{L}}{\partial x_i} = \overrightarrow{G}^{\frac{\partial L}{\partial x_i}},\ 1 \le i \le n.$$

We consider the Cauchy problem on heat equation in $\overrightarrow{G}^{\mathscr{V}}$, i.e.,

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial values $X|_{t=t_0}$ and constant $c \neq 0$.

Theorem 2.7([17]) *For* $\forall \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathscr{V}}$ *and a non-zero constant* c *in* $\mathbb{R}$, *the Cauchy problems on differential equations*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial value $X|_{t=t_0} = \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathscr{V}}$ *is solvable in* $\overrightarrow{G}^{\mathscr{V}}$ *if* $L'(u,v)$ *is continuous and bounded in* $\mathbb{R}^n$ *for* $\forall (u,v) \in X\left(\overrightarrow{G}\right)$.

For an integral kernel $K(\mathbf{x},\mathbf{y})$, $\mathscr{N}, \mathscr{N}^* \subset L^2[\Delta]$ are defined respectively by

$$\begin{aligned} \mathscr{N} &= \left\{ \phi(\mathbf{x}) \in L^2[\Delta] | \int_\Delta K(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y}) d\mathbf{y} = \phi(\mathbf{x}) \right\}, \\ \mathscr{N}^* &= \left\{ \varphi(\mathbf{x}) \in L^2[\Delta] | \int_\Delta \overline{K(\mathbf{x},\mathbf{y})} \varphi(\mathbf{y}) d\mathbf{y} = \varphi(\mathbf{x}) \right\}. \end{aligned}$$

Then

Theorem 2.8([17]) *For* $\forall G^L \in \overrightarrow{G}^{\mathscr{V}}$, *if* $\dim \mathscr{N} = 0$ *the integral equation*

$$\overrightarrow{G}^{X} - \int_\Delta \overrightarrow{G}^{X} = G^L$$

is solvable in $\overrightarrow{G}^{\mathscr{V}}$ *with* $\mathscr{V} = L^2[\Delta]$ *if and only if*

$$\left\langle \overrightarrow{G}^{L}, \overrightarrow{G}^{L'} \right\rangle = 0, \quad \forall \overrightarrow{G}^{L'} \in \mathscr{N}^*.$$

In fact, if $\overrightarrow{G}$ is circuit decomposable, we can generally extend solutions of an equation to $\overrightarrow{G}$-flows following.

Theorem 2.9([17]) *If the topological graph* $\overrightarrow{G}$ *is strong-connected with circuit decomposition* $\overrightarrow{G} = \bigcup_{i=1}^{l} \overrightarrow{C}_i$ *such that* $L(e) = L_i(\mathbf{x})$ *for* $\forall e \in E\left(\overrightarrow{C}_i\right)$, $1 \leq i \leq l$ *and the Cauchy problem*

$$\begin{cases} \mathscr{F}_i\left(\mathbf{x}, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots\right) = 0 \\ u|_{\mathbf{x}_0} = L_i(\mathbf{x}) \end{cases}$$

is solvable in a Hilbert space $\mathscr{V}$ *on domain* $\Delta \subset \mathbb{R}^n$ *for integers* $1 \leq i \leq l$, *then the Cauchy problem*

$$\begin{cases} \mathscr{F}_i\left(\mathbf{x}, X, X_{x_1}, \cdots, X_{x_n}, X_{x_1x_2}, \cdots\right) = 0 \\ X|_{\mathbf{x}_0} = \overrightarrow{G}^{L} \end{cases}$$

such that $L(e) = L_i(\mathbf{x})$ *for* $\forall e \in E\left(\overrightarrow{C}_i\right)$ *is solvable for* $X \in \overrightarrow{G}^{\mathscr{V}}$.

§3. Geometry on Action Flows

In physics, a thing P, particularly, a particle such as those of water molecule H_2O and its hydrogen or oxygen atom shown in Fig.12

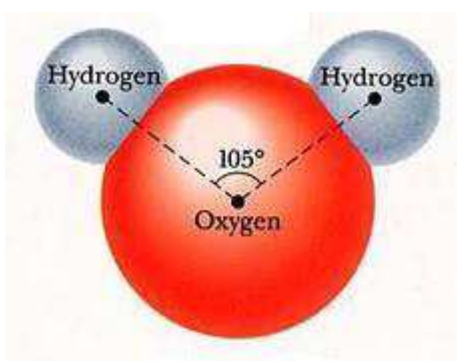


Fig.12

is characterized by differential equation established on observed characters of $\mu_1, \mu_2, \cdots, \mu_n$ for its state function $\psi(t, x)$ by the principle of stationary action $\delta S = 0$ in

$\mathbb{R}^4$ with

$$S = \int_{t_1}^{t_2} dtL\left(q(t), \dot{q}(t)\right) \quad \text{or} \quad S = \int_{\tau_2}^{\tau_1} d^4x\mathcal{L}(\phi, \partial_\mu\psi), \tag{3.1}$$

i.e., the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0, \tag{3.2}$$

where $q(t), \dot{q}(t)$, ψ are the generalized coordinates, the velocities, the state function, and $L\left(q(t), \dot{q}(t)\right)$, $\mathcal{L}$ are the *Lagrange function* or *density* on P, respectively by viewing P as an independent system or a field. For examples, let

$$\mathcal{L}_S = \frac{i\hbar}{2}\left(\frac{\partial\psi}{\partial t}\overline{\psi} - \frac{\partial\overline{\psi}}{\partial t}\psi\right) - \frac{1}{2}\left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + V|\psi|^2\right).$$

Then we get the Schrödinger equation by (1.3) and similarly, the Dirac equation

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi(t, x) = 0 \tag{3.3}$$

for a free fermion $\psi(t, x)$, the Klein-Gordon equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi(x, t) + \left(\frac{mc}{\hbar}\right)^2\psi(x, t) = 0 \tag{3.4}$$

for a free boson $\psi(t, x)$ on particle with masses m hold in relativistic forms, where $\hbar = 6.582 \times 10^{-22} MeVs$ is the Planck constant.

Notice that the equation (1.3) is dependent on observed characters $\mu_1, \mu_2, \cdots, \mu_n$ and different position maybe results in different observations. For example, if an observer receives information stands out of H_2O by viewing it as a geometrical point then he only receives coherent information on atoms H and O with H_2O ([18]), but if he enters the interior of the molecule, he will view a different sceneries for atom H and atom O with a non-solvable system of 3 dynamical equations following ([19]).

$$\begin{cases} -i\hbar\dfrac{\partial\psi_O}{\partial t} &= \dfrac{\hbar^2}{2m_O}\nabla^2\psi_O - V(x)\psi_O \\ -i\hbar\dfrac{\partial\psi_{H_1}}{\partial t} &= \dfrac{\hbar^2}{2m_{H_1}}\nabla^2\psi_{H_1} - V(x)\psi_{H_1} \\ -i\hbar\dfrac{\partial\psi_{H_2}}{\partial t} &= \dfrac{\hbar^2}{2m_{H_2}}\nabla^2\psi_{H_2} - V(x)\psi_{H_2} \end{cases}$$

Thus, an in-observation on a physical thing P results in a non-solvable system of solvable equations, which is also in accordance with individual difference in epistemology. However, the atoms H and O are compatible in the water molecule H_2O

without contradiction. Thus, accompanying with the establishment of compatible systems, we are also needed those of contradictory systems, particularly, non-solvable equations for holding on the reality of things ([15]).

3.1 Geometry on Equations

Physicist characterizes a natural thing usually by solutions of differential equations. However, if they are non-solvable such as those of equations for atoms H and O on in-observation, how to determine their behavior in the water molecule H_2O? Holding on the reality of things motivates one to leave behind the solvability of equation, extend old notion to a new one by machinery. The knowledge of human beings concludes the social existence determine the consciousness. However, if we can not characterize a thing until today, we can never conclude that it is nothingness, particularly on those of non-solvable system consisting of solvable equations. For example, consider the two systems of linear equations following:

$$(LES_4^N) \quad \begin{cases} x+y &= 1 \\ x+y &= -1 \\ x-y &= -1 \\ x-y &= 1 \end{cases} \qquad\qquad (LES_4^S) \quad \begin{cases} x=y \\ x+y=2 \\ x=1 \\ y=1 \end{cases}$$

Clearly, (LES_4^N) is non-solvable because $x+y=-1$ is contradictious to $x+y=1$, and so $x-y=-1$ to $x-y=1$. But (LES_4^S) is solvable with $x=1$ and $y=1$.

What is the geometrical essence of a system of linear equations? In fact, each linear equation $ax+by=c$ with $ab \neq 0$ is in fact a point set $L_{ax+by=c} = \{(x,y)|ax+by=c\}$ in $\mathbb{R}^2$, such as those shown in Fig.13 for the linear systems (LES_4^N) and (LES_4^S).

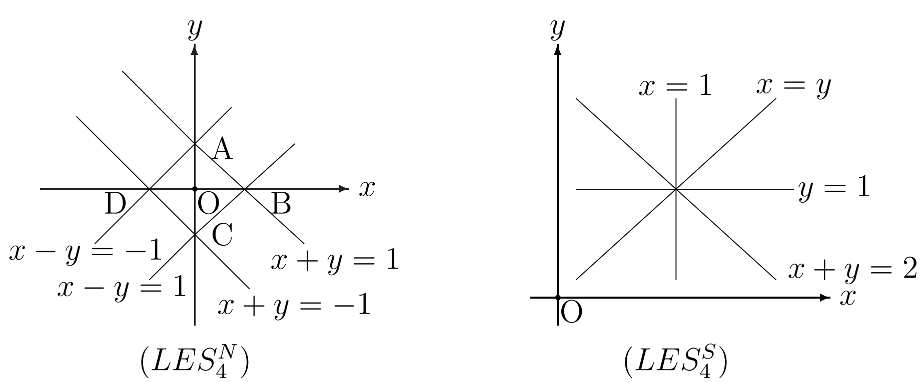


Fig.13

Clearly,

$$L_{x+y=1}\bigcap L_{x+y=-1}\bigcap L_{x-y=1}\bigcap L_{x-y=-1}=\emptyset$$

but

$$L_{x=y}\bigcap L_{x+y=2}\bigcap L_{x=1}\bigcap L_{y=1}=(1,1)$$

in the Euclidean plane $\mathbb{R}^2$.

Generally, a solution manifold of an equation $f(x_1,x_2,\cdots,x_n,y)=0, n\geq 1$ is defined to be an n-manifold

$$S_f=(x_1,x_2,\cdots,x_n,y(x_1,x_2,\cdots,x_n))\subset\mathbb{R}^{n+1}$$

if it is solvable, otherwise $\emptyset$ in topology. Clearly, a system

$$(ES_m)\quad\begin{cases} f_1(x_1,x_2,\cdots,x_n)=0\\ f_2(x_1,x_2,\cdots,x_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ f_m(x_1,x_2,\cdots,x_n)=0\end{cases}$$

of algebraic equations with initial values $f_i(0),\ 1\leq i\leq m$ in Euclidean space $\mathbb{R}^{n+1}$ is solvable or not dependent on $\bigcap\limits_{i=1}^{m}S_{f_i}\neq\emptyset$ or $=\emptyset$ in geometry.

Particularly, let $(PDES_m)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\\ F_2(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1,x_2,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n},u_{x_1x_2},\cdots,u_{x_1x_n},\cdots)=0\end{cases}$$

on a function $u(x_1,\cdots,x_n,t)$. Its *symbol* is determined by

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,p_1,\cdots,p_n,p_1p_2,\cdots,p_1p_n,\cdots)=0\\ F_2(x_1,x_2,\cdots,x_n,u,p_1,\cdots,p_n,p_1p_2,\cdots,p_1p_n,\cdots)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1,x_2,\cdots,x_n,u,p_1,\cdots,p_n,p_1p_2,\cdots,p_1p_n,\cdots)=0,\end{cases}$$

i.e., substitute $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ into $(PDES_m)$ for the term $u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}$, where $\alpha_i\geq 0$ for integers $1\leq i\leq n$.

Definition 3.1 *A non-solvable $(PDES_m)$ is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

For example, the system of partial differential equations following

$$\begin{cases} u_x + 2u_y + 3u_z = 2 + y^2 + z^2 \\ yzu_x + xzu_y + xyu_z = x^2 - y^2 - z^2 \\ (yz+1)u_x + (xz+2)u_y + (xy+3)u_z = x^2 + 1 \end{cases}$$

is algebraically contradictory because its symbol

$$\begin{cases} p_1 + 2p_2 + 3p_3 = 2 + y^2 + z^2 \\ yzp_1 + xzp_2 + xyp_3 = x^2 - y^2 - z^2 \\ (yz+1)p_1 + (xz+2)p_2 + (xy+3)p_3 = x^2 + 1 \end{cases}$$

is non-solvable. A necessary and sufficient condition on the solvability of Cauchy problem on $(PDES_m)$ was found in [16] following.

Theorem 3.2 *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases} \qquad (PDES_m)$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \le k \le m$$

is algebraically contradictory, in this case, there must be an integer $k_0, \ 1 \le k_0 \le m$ *such that*

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer $j_0, \ 1 \le j_0 \le n-1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

Particularly, we immediately get a conclusions on quasilinear partial differential equations following.

Corollary 3.3 *A Cauchy problem $(PDES_m^C)$ of quasilinear partial differential equations with initial values $u|_{x_n=x_n^0} = u_0$ is non-solvable if and only if the system $(PDES_m)$ of partial differential equations is algebraically contradictory.*

Geometrically, the behavior of (ES_m) is completely characterized by a union $\bigcup\limits_{i=1}^{m} S_{f_i}$, i.e., a Smarandache multispace with an inherited graph $G^L[ES_m]$ following:

$$V\left(G^L[ES_m]\right) = \{S_{f_i},\ 1 \leq i \leq m\},$$

$$E\left(G^L[ES_m]\right) = \{\ (S_{f_i}, S_{f_j}) \mid S_{f_i} \bigcap S_{f_j} \neq \emptyset, 1 \leq i, j \leq m\}$$

with a vertex and edge labeling

$$L:\ S_{f_i} \to S_{f_i} \text{ and } L:\ (S_{f_i}, S_{f_j}) \to S_{f_i} \bigcap S_{f_j} \text{ if}$$

for integers $1 \leq i \leq m$ and $\left(S_{f_i}, S_{f_j}\right) \in E\left(G^L[ES_m]\right)$.

For example, it is clear that $L_{x+y=1} \bigcap L_{x+y=-1} = \emptyset = L_{x-y=1} \bigcap L_{x-y=-1} = \emptyset$, $L_{x+y=1} \bigcap L_{x-y=-1} = \{A\}$, $L_{x+y=1} \bigcap L_{x-y=1} = \{B\}$, $L_{x+y=-1} \bigcap L_{x-y=1} = \{C\}$, $L_{x+y=-1} \bigcap L_{x-y=-1} = \{D\}$ for the system (LES_4^N) with an inherited graph C_4^L shown in Fig.14.

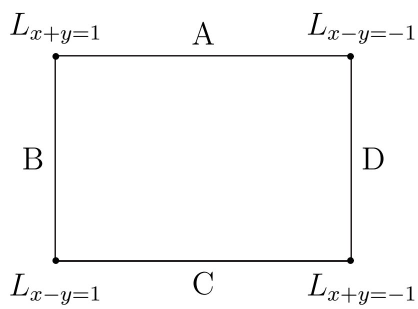


Fig.14

Generally, we can determine the graph $G\left[\widetilde{S}\right]$. In fact, let $\mathscr{C}(f_i)$ be a maximal contradictory class including equation $f_i = 0$ in (ES_m) for an integer $1 \leq i \leq m$ and let classes $\mathscr{C}^1, \mathscr{C}^2, \cdots, \mathscr{C}^l$ be a partition of equations in (ES_m). Then we are easily know that $G\left[\widetilde{S}\right] \simeq K\left(\mathscr{C}^1, \mathscr{C}^2, \cdots, \mathscr{C}^l\right)$. Particularly, a result on Cauchy problem of partial differential equations following. .

Theorem 3.4([16]) *A Cauchy problem on system ($PDES_m$) of partial differential equations of first order with initial values* $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ *for the kth equation in ($PDES_m$),* $1 \leq k \leq m$ *such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G-solvable, i.e., $G[PDES_m^C]$ *is uniquely determined.*

3.2 Geometry on Action Flows

Let $\left(\overrightarrow{G}; L, A\right)$ be an action flow on Banach space $\mathscr{B}$. By the closed graph theorem in functional analysis, i.e., *if X and Y are Banach spaces with a linear operator $\varphi : X \to Y$, then φ is continuous if and only if its graph*

$$\Gamma[X, Y] = \{(\overline{x}, \overline{y}) \in X \times Y | T\overline{x} = \overline{y}\}$$

is closed in $X \times Y$, if $L(v, u): \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{C}^r$ differentiable for $\forall (v, u) \in E\left(\overrightarrow{G}\right)$, then

$$\Gamma[v, u] = \{((x_1, \cdots, x_n), L(v, u)) | (x_1, \cdots, x_n) \in \mathbb{R}^n\}$$

is a $\mathbb{C}^{r_{vu}}$ differentiable n-dimensional manifold, where $r_{vu} \geq 0$ is an integer. Whence, the geometry of action flow $\left(\overrightarrow{G}; L, A\right)$ is nothing else but a combination of $\mathbb{C}^{r_{vu}}$ differentiable manifolds for $r_{vu} \geq 0, (v, u) \in E\left(\overrightarrow{G}\right)$, such as those combinatorial manifolds (a) and (b) shown in Fig.15 for $r = 0$.

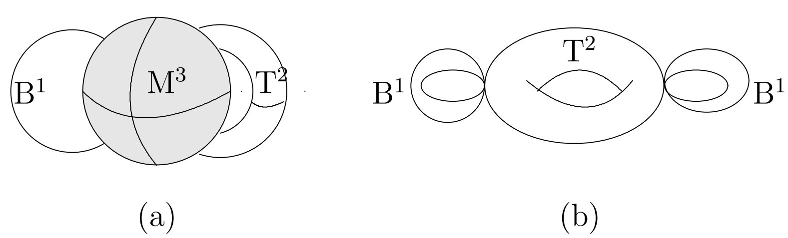


Fig.15

Definition 3.5 *For a given integer sequence* $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, *a combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p \in \widetilde{M}$, *there is a local chart* (U_p, φ_p) *of* p, *i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), \cdots, n_{s(p)}(p))$ *with*

$$\left\{n_1(p), \cdots, n_{s(p)}(p)\right\} \subseteq \{n_1, \cdots, n_m\}, \quad \bigcup_{p \in \widetilde{M}} \left\{n_1(p), \cdots, n_{s(p)}(p)\right\} = \{n_1, \cdots, n_m\},$$

denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \left\{(U_p, \varphi_p)\left| p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\right.\right\}$$

its an atlas. Particularly, a combinatorial manifold $\widetilde{M}$ *is finite if it is just combined by finite manifolds without one manifold contained in the union of others.*

Similarly, an inherent structure $G^L\left[\widetilde{M}\right]$ on combinatorial manifolds $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is defined by

$$\begin{aligned} V(G^L[\widetilde{M}]) &= \{M_1, M_2, \cdots, M_m\}, \\ E(G^L[\widetilde{M}) &= \{\ (M_i, M_j) \mid M_i \bigcap M_j \neq \emptyset, 1 \leq i, j \leq n\} \end{aligned}$$

with a labeling mapping L determined by

$$L:\ M_i \rightarrow M_i, \quad L:\ (M_i, M_j) \rightarrow M_i \bigcap M_j$$

for integers $1 \leq i, j \leq m$. The result following enables one to construct $\mathbb{C}^r$ differentiable combinatorial manifolds.

Theorem 3.6([8]) *Let* $\widetilde{M}$ *be a finitely combinatorial manifold. If* $\forall M \in V\left(G^L\left[\widetilde{M}\right]\right)$ *is* C^r*-differential for integer* $r \geq 0$ *and* $\forall (M_1, M_2) \in E\left(G\left[\widetilde{M}\right]\right)$ *there exist atlas*

$$\mathcal{A}_1 = \{(V_x; \varphi_x) \,|\forall x \in M_1\} \quad \mathcal{A}_2 = \{(W_y; \psi_y) \,|\forall y \in M_2\}$$

such that $\varphi_x|_{V_x \bigcap W_y} = \psi_y|_{V_x \bigcap W_y}$ *for* $\forall x \in M_1, y \in M_2$*, then there is a differential structures*

$$\widetilde{\mathcal{A}} = \left\{(U_p; [\varpi_p]) \,|\forall p \in \widetilde{M}\right\}$$

such that $\left(\widetilde{M}; \widetilde{\mathcal{A}}\right)$ *is a combinatorial* C^r*-differential manifold.*

For the basis of tangent and cotangent vectors on combinatorial manifold $\widetilde{M}$, we know results following in [8].

Theorem 3.7 *For any point* $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ *with a local chart* $(U_p; [\varphi_p])$*, the dimension of* $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ *is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix $\left[\frac{\partial}{\partial \overline{x}}\right]_{s(p)\times n_{s(p)}} =$

$$\left[\begin{array}{ccccccccc} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{array}\right]$$

where $x^{il} = x^{jl}$ *for* $1 \leq i,j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, *namely there is a smoothly functional matrix* $[v_{ij}]_{s(p)\times n_{s(p)}}$ *such that for any tangent vector* $\overline{v}$ *at a point* p *of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = \left\langle [v_{ij}]_{s(p)\times n_{s(p)}}, [\frac{\partial}{\partial \overline{x}}]_{s(p)\times n_{s(p)}} \right\rangle,$$

where $\langle [a_{ij}]_{k\times l}, [b_{ts}]_{k\times l}\rangle = \sum_{i=1}^{k}\sum_{j=1}^{l} a_{ij}b_{ij}$, *the inner product on matrixes.*

Theorem 3.8 *For* $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ *with a local chart* $(U_p; [\varphi_p])$, *the dimension of* $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ *is*

$$\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p))$$

with a basis matrix $[d\overline{x}]_{s(p)\times n_{s(p)}} =$

$$\left[\begin{array}{ccccccccc} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}} \end{array}\right]$$

where $x^{il} = x^{jl}$ *for* $1 \leq i,j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, *namely for any co-tangent vector* d *at a point* p *of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$, *there is a smoothly functional matrix* $[u_{ij}]_{s(p)\times s(p)}$ *such that,*

$$d = \left\langle [u_{ij}]_{s(p)\times n_{s(p)}}, [d\overline{x}]_{s(p)\times n_{s(p)}} \right\rangle.$$

Then, we can establish tensor theory with connections on smoothly combinatorial manifolds ([8]) and [11]. For example, we can get the curvature $\widetilde{R}$ formula following.

Theorem 3.9([8]) *Let $\widetilde{M}$ be a finite combinatorial manifold, $\widetilde{R}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow C^{\infty}(\widetilde{M})$ a curvature on $\widetilde{M}$. Then for $\forall p\in\widetilde{M}$ with a local chart $(U_p;[\varphi_p])$,*

$$\widetilde{R}=\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}dx^{\sigma\varsigma}\otimes dx^{\eta\theta}\otimes dx^{\mu\nu}\otimes dx^{\kappa\lambda},$$

where

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}}+\frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}})\\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)}g_{(\xi o)(\vartheta\iota)}-\Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}}g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

and $g_{(\mu\nu)(\kappa\lambda)}=g(\frac{\partial}{\partial x^{\mu\nu}},\frac{\partial}{\partial x^{\kappa\lambda}})$.

All these results on differentiable combinatorial manifolds enable one to characterize the combination of classical fields, such as the Einstein's gravitational fields and other fields on combinatorial spacetimes and hold their behaviors (see [10] for details).

3.3 Classification

Definition 3.10 *Let $\left(\overrightarrow{G}_1;L_1,A_1\right)$ and $\left(\overrightarrow{G}_2;L_2,A_2\right)$ be 2 action flows on Banach space $\mathscr{B}$ with $\overrightarrow{G}_1\simeq\overrightarrow{G}_2$. Then they are said to be combinatorially homeomorphic if there is a homeomorphism h on $\mathscr{B}$ and a $1-1$ mapping $\varphi:V\left(\overrightarrow{G}_1\right)\rightarrow V\left(\overrightarrow{G}_2\right)$ such that $h(L_1(v,u))=L_2(\varphi(v,u))$ and $A_{vu}=A_{\varphi(vu)}$ for $\forall(v,u)\in V\left(\overrightarrow{G}_1\right)$, denoted by $\left(\overrightarrow{G}_1;L_1,A_1\right)\overset{h}{\sim}\left(\overrightarrow{G}_2;L_2,A_2\right)$. Particularly, if $\mathscr{B}=\mathbb{R}^n$ for an integer $n\geq 3$, h is an isometry, they are said to be combinatorially isometric, denoted by $\left(\overrightarrow{G}_1;L_1,A_1\right)\overset{h}{\simeq}\left(\overrightarrow{G}_2;L_2,A_2\right)$, and identical if $h=\mathbf{1}_{\mathbb{R}^n}$, denoted by $\left(\overrightarrow{G}_1;L_1,A_1\right)=\left(\overrightarrow{G}_2;L_2,A_2\right)$.*

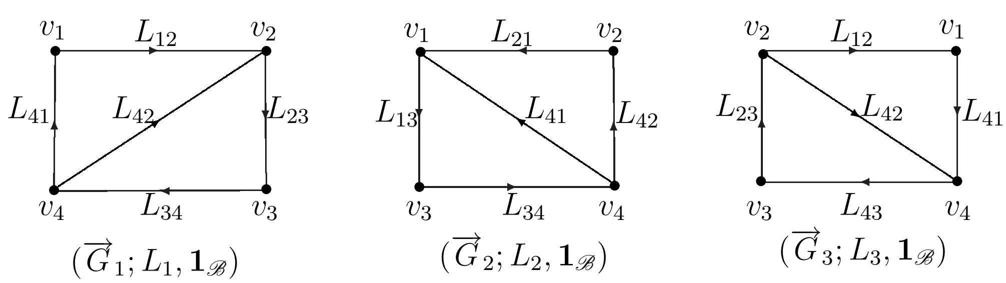


Fig.16

Notice that the mapping φ in Definition 3.10 maybe not a graph isomorphism.

For example, the action flows $\left(\overrightarrow{G}_1; L_1, \mathbf{1}_{\mathbb{R}^n}\right) = \left(\overrightarrow{G}_2; L_2, \mathbf{1}_{\mathbb{R}^n}\right)$ because there is a 1-1 mapping $\varphi = (v_1v_2)(v_3)(v_4) : V\left(\overrightarrow{G}_1\right) \to \left(\overrightarrow{G}_2\right)$ holding with $L(u,v) = L(\varphi(u,v))$ for $\forall(u,v) \in E\left(\overrightarrow{G}\right)$, which is not a graph isomorphism between $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ but $\left(\overrightarrow{G}_1; L_1, \mathbf{1}_{\mathbb{R}^n}\right) \neq \left(\overrightarrow{G}_3; L_3, \mathbf{1}_{\mathbb{R}^n}\right)$ for $\overrightarrow{G}_1 \not\simeq \overrightarrow{G}_3$ in Fig.16. Thus if we denote by $\mathrm{Aut}\left(\overrightarrow{G}; L, A\right)$ all such 1-1 mappings $\varphi : V\left(\overrightarrow{G}\right) \to V\left(\overrightarrow{G}\right)$ holding with $L(u,v) = L(\varphi(u,v))$ and $A_{uv} = A_{\varphi(uv)}$ for $\forall(u,v) \in E\left(\overrightarrow{G}\right)$, then it is clearly a group itself holding with the following result.

Theorem 3.11 *If* $V\left(\overrightarrow{G}\right) = \{v_1, v_2, \cdots, v_p\}$, *then* $\mathrm{Aut}\left(\overrightarrow{G}; L, A\right) = \mathrm{Aut}\overrightarrow{G} \bigotimes (S_p)_{\overrightarrow{G}}$, *particularly,* $\mathrm{Aut}\left(\overrightarrow{G}; L, A\right) \succ \mathrm{Aut}\overrightarrow{G}$, *where* $(S_p)_{\overrightarrow{G}}$ *is the stabilizer of symmetric group* S_p *on* $\Delta = \{1, 2, \cdots, p\}$.

For an isometry h on $\mathbb{R}^n$, let $\left(\overrightarrow{G}; L, A\right)^h = \left(\overrightarrow{G}; hLh^{-1}, A\right)$ be an action flow, i.e., replacing $x_1, x_2, \cdots, x_n$ by $h(x_1), h(x_2), \cdots, h(x_n)$. The result following is clearly known by definition.

Theorem 3.12 $\left(\overrightarrow{G}_1; L_1, A_1\right) \overset{h}{\simeq} \left(\overrightarrow{G}_2; L_2, A_2\right)$ *if and only if* $\left(\overrightarrow{G}_1; L_1, A_1\right)^h = \left(\overrightarrow{G}_2; L_2, A_2\right)$.

Certainly, we can also classify action flows geometrically. For example, two finitely combinatorial manifolds $\widetilde{M}_1$, $\widetilde{M}_2$ are said to be *homotopically equivalent* if there exist continuous mappings $f : \widetilde{M}_l \to \widetilde{M}_2$ and $g : \widetilde{M}_2 \to \widetilde{M}_1$ such that $gf \simeq$identity: $\widetilde{M}_2 \to \widetilde{M}_2$ and $fg \simeq$identity: $\widetilde{M}_1 \to \widetilde{M}_1$. Then we know

Theorem 3.13([7]) *Let* $\widetilde{M}_1$ *and* $\widetilde{M}_2$ *be finitely combinatorial manifolds with an equivalence* $\varpi : G^L[\widetilde{M}_1 \to G^L[\widetilde{M}_2]$. *If for* $\forall M_1, M_2 \in V(G^L[\widetilde{M}_1])$, M_i *is homotopic to* $\varpi(M_i)$ *with homotopic mappings* $f_{M_i} : M_i \to \varpi(M_i)$, $g_{M_i} : \varpi(M_i) \to M_i$ *such that* $f_{M_i}|_{M_i \bigcap M_j} = f_{M_j}|_{M_i \bigcap M_j}$, $g_{M_i}|_{M_i \bigcap M_j} = g_{M_j}|_{M_i \bigcap M_j}$ *providing* $(M_i, M_j) \in E(G^L[\widetilde{M}_1])$ *for* $1 \leq i, j \leq m$, *then* $\widetilde{M}_1$ *is homotopic to* $\widetilde{M}_2$.

§4. Stable Action Flows

The importance of stability for a model on natural things P results in determining the prediction and controlling of its behaviors. The same also happens to those of action flows for the perturbation of things such as those shown in Fig.17 on operating of the universe.

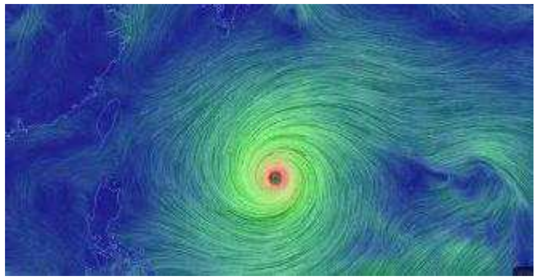


Fig.17

As we shown in Theorem 3.4, the Cauchy problem on partial differential equations of first order is uniquely G-solvable. Thus it is significant to consider the stability of action flows. Let $\left(\overrightarrow{G}; L(t), A\right)$ be an action flow on Banach space $\mathscr{B}$ with initial values $\left(\overrightarrow{G}; L(t_0), A\right)$ and let $\omega : \left(\overrightarrow{G}; L, A\right) \to \mathbb{R}$ be an index function. It is said to be *ω-stable* if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that

$$\left\|\omega\left(\overrightarrow{G}; L_1(t) - L_2(t), A\right)\right\| < \varepsilon,$$

or furthermore, *asymptotically ω-stable* if

$$\lim_{t\to\infty}\left\|\omega\left(\overrightarrow{G}; L_1(t) - L_2(t), A\right)\right\| = 0$$

if initial values holding with

$$\|L_1(t_0)(v,u) - L_2(t_0)(v,u)\| < \delta(\varepsilon)$$

for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$, for instance the *norm-stable* or *sum-stable* by letting

$$\omega\left(\overrightarrow{G}; L, A\right) = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \left\|L^{A^{+}_{vu}}(v,u)\right\|.$$

Particularly, ;et

$$\omega\left(\overrightarrow{G}; L, \mathbf{1}_{\mathscr{B}}\right) = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \|L(v,u)\|$$

or

$$\left(\overrightarrow{G}; L, A\right) = \|\sum_{(v,u)\in E\left(\overrightarrow{G}\right)} L(v,u)\|, \quad A \neq \mathbf{1}_{\mathscr{B}}.$$

The following result on the stability of $\overrightarrow{G}$-flow solution was obtained in [17], which is a commonly norm-stability on $\overrightarrow{G}$-flows.

Theorem 4.1 *Let $\mathscr{V}$ be the Hilbert space $L^2[\Delta]$. Then, the $\overrightarrow{G}$-flow solution X of equation*

$$\begin{cases} \mathscr{F}(\mathbf{x}, X, X_{x_1}, \cdots, X_{x_n}, X_{x_1x_2}, \cdots) = 0 \\ X|_{\mathbf{x}_0} = \overrightarrow{G}^L \end{cases}$$

in $\overrightarrow{G}^{\mathscr{V}}$ is norm-stable if and only if the solution $u(\mathbf{x})$ of equation

$$\begin{cases} \mathscr{F}(\mathbf{x}, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots) = 0 \\ u|_{\mathbf{x}_0} = \varphi(\mathbf{x}) \end{cases}$$

on (v,u) is stable for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$.

In fact, we only need to consider the stability of $\left(\overrightarrow{G}; \mathbf{O}, A\right)$ after letting flows $\mathbf{0} = L(t)(v,u) - L(t)(v,u)$ on $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ without loss of generality.

Similarly, if there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbb{R}, n \geq 1$ on $\overrightarrow{G}$ with $\mathscr{O} \subset \mathbb{R}^n$ open such that $L(\omega(t)) \geq 0$ with equality hold only if $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and if $t \geq t_0$, $\dot{L}(\omega(t)) \leq 0$, then it can be likewise Theorem 3.8 of [12] to know the next result, where $\dot{L}(\omega) = \dfrac{dL(\omega)}{dt}$.

Theorem 4.2 *If there is a Liapunov sum-function $L(\omega(t)) : \mathscr{O} \to \mathbf{R}$ on $\overrightarrow{G}$, then $\left(\overrightarrow{G}; \mathbf{O}, A\right)$ is ω-stable, and furthermore, if $\dot{L}(\omega(t)) < 0$ for $\left(\overrightarrow{G}; L(t), A\right) \neq \mathbf{O}$, then $\left(\overrightarrow{G}; \mathbf{O}, A\right)$ is asymptotically ω-stable.*

For example, let $\left(\overrightarrow{G}; L, A\right)$ be the action flow with operators $A_{z_{i+1}z_i} = -\dfrac{d}{dt}$ for $z = v, u, \cdots, w$ and $A^+_{v_iv_{i+1}} = \lambda_{1i}$, $A^+_{u_iu_{i+1}} = \lambda_{2i}$, $\cdots$, $A^+_{w_iw_{i+1}} = \lambda_{ni}$ for integer $i \equiv (\mathrm{mod} n)$, such as those shown in Fig.18.

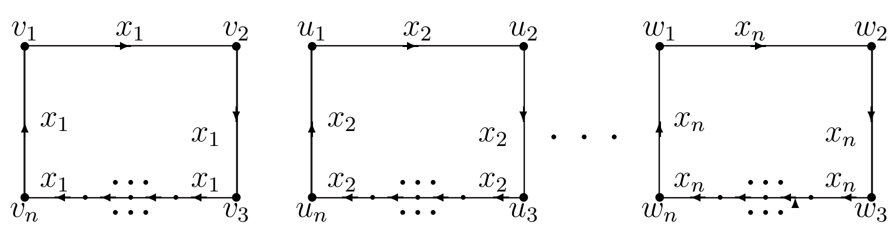


Fig.18

Then its conservation equations are respectively

$$\begin{cases} \dot{x}_1 = \lambda_{11}x_1 \\ \dot{x}_1 = \lambda_{12}x_1 \\ \cdots\cdots\cdots \\ \dot{x}_1 = \lambda_{1n}x_1 \end{cases}, \quad \begin{cases} \dot{x}_2 = \lambda_{21}x_2 \\ \dot{x}_2 = \lambda_{22}x_2 \\ \cdots\cdots\cdots \\ \dot{x}_2 = \lambda_{2n}x_2 \end{cases}, \quad \cdots, \quad \begin{cases} \dot{x}_n = \lambda_{n1}x_n \\ \dot{x}_n = \lambda_{n2}x_n \\ \cdots\cdots\cdots \\ \dot{x}_n = \lambda_{nn}x_n \end{cases},$$

where all λ_{ij}, $1 \leq i, j \leq n$ are real and $\lambda_{ij_1} \neq \lambda_{ij_2}$ if $j_1 \neq j_2$ for integers $1 \leq i \leq n$. Let $L = x_1^2 + x_2^2 + \cdots + x_n^2$. Then $\dot{L} = \lambda_{i_1 1}x_1^2 + \lambda_{i_2 2}x_2^2 + \cdots + \lambda_{i_n n}x_n^2$ for integers $1 \leq i \leq n$, where $1 \leq i_j \leq n$ for integers $1 \leq j \leq n$. Whence, it is a Liapunov ω-function for action flow $\left(\overrightarrow{G}; L, A\right)$ if $\lambda_{ij} < 0$ for integers $1 \leq i, j \leq n$.

§5. Applications

As a powerful theory, action flow extends classical mathematics on embedded graph, which can be used as a model nearly for moving things in the nature, particularly, applying to physics and mathematical ecology.

5.1 Physics

For diversity of things, two typical examples are respectively the superposition behavior of microcosmic particle and the quarks model of Sakata, or Gell-Mann and Ne'eman by assuming internal structures of hadrons and gluons, which can not be commonly understanding.

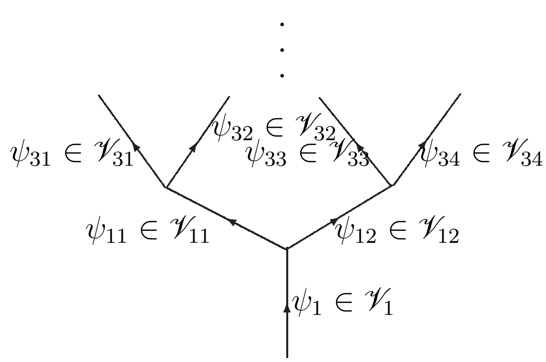
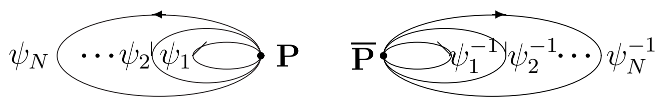


Fig.19

Certainly, H.Everett's multiverse interpretation in Fig.6 presented the superposition of particles but with a little machinery, i.e., viewed different worlds in different quantum mechanics and explained the superposition of a particle to be 2 branch tree such as those shown in Fig.19, where the multiverse is $\bigcup\limits_{i\geq 1}\mathscr{V}_i$ with $\mathscr{V}_{kl}=\mathscr{V}$ for integers $k\geq 1,\ 1\leq l\leq 2^k$ but in different positions.

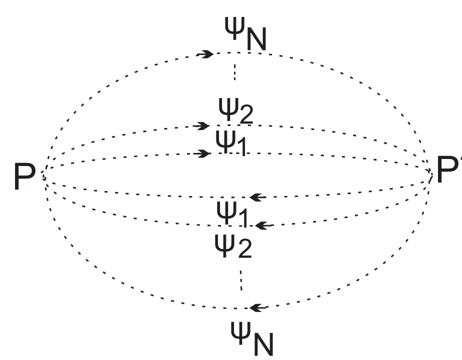
Similarly, the quark model assumes internal structures K_2, K_3 respectively on hadrons and gluons mechanically for hold the behaviors of particles. However, such an assumption is a little ambiguous in logic, i.e., we can not even conclude which is the point, the hadron and gluon or its subparticle, the quark. However, the action flows imply the rightness of H.Everett's multiverse interpretation, also the assumption of physicists on the internal structures for hold the behaviors of particles because there are infinite many such graphs $\overrightarrow{G}$ satisfying conditions of Theorem 2.9.



Particle **Antiparticle**

Fig.20

For example, let $\overrightarrow{G}=\overrightarrow{B}_n$ or $\overrightarrow{D^{\perp}}_{0,2N,0}$, i.e., a bouquet or a dipole. Then we can respectively establish a $\overrightarrow{G}$-flow model for fermions, leptons, quark P with an antiparticle $\overline{P}$, and the mediate interaction particles quanta presented in Banach space $\overrightarrow{B}_N^{\mathscr{V}}$ or $\overrightarrow{D^{\perp}}_{0,2N,0}^{\mathscr{V}}$, such as those shown in Figs.20 and 21,



$\overrightarrow{D^{\perp}}_{0,2N,0}^{L_\psi}$

Fig.21

where, the vertex P, P' denotes particles, and arcs or loops with state functions $\psi_1,\psi_2,\cdots,\psi_N$ are its states with inverse functions $\psi_1^{-1},\psi_2^{-1},\cdots,\psi_N^{-1}$. Notice that $\overrightarrow{B}_N^{L_\psi}$ and $\overrightarrow{D^{\perp}}_{0,2N,0}^{L_\psi}$ both are a union of N circuits. We know the following result.

Theorem 5.1([18]) *For any integer $N \geq 1$, there are indeed $\overrightarrow{D^{\perp}}^{L_{\psi}}_{0,2N,0}$-flow solution on Klein-Gordon equation* (3.5)*, and $\overrightarrow{B}^{L_{\psi}}_{N}$-flow solution on Dirac equation* (3.6)*.*

For a particle $\widetilde{P}$ consisted of l elementary particles $P_1, P_1, \cdots, P_l$ underlying a graph $\overrightarrow{G}\left[\widetilde{P}\right]$, its $\overrightarrow{G}$-flow is obtained by replace vertices v by $\overrightarrow{B}^{L_{\psi_v}}_{N_v}$ and arcs e by $\overrightarrow{D^{\perp}}^{L_{\psi_e}}_{0,2N_e,0}$ in $\overrightarrow{G}\left[\widetilde{P}\right]$, denoted by $\overrightarrow{G}^{L_{\psi}}\left[\overrightarrow{B}_v, \overrightarrow{D}_e\right]$. Then we know that

Theorem 5.2([18]) *If $\widetilde{P}$ is a particle consisted of elementary particles $P_1, P_1, \cdots, P_l$, then $\overrightarrow{G}^{L_{\psi}}\left[\overrightarrow{B}_v, \overrightarrow{D}_e\right]$ is a $\overrightarrow{G}$-flow solution on the Schrödinger equation* (1.1) *whenever its size index λ_G is finite or infinite, where*

$$\lambda_G = \sum_{v \in V\left(\overrightarrow{G}\right)} N_v + \sum_{e \in V\left(\overrightarrow{G}\right)} N_e.$$

5.2 Mathematical Ecology

Action flows can applied to be a model of ecological systems. For example, let u and v denote respectively the density of two species that compete for a common food supply. Then the equations of growth of the two populations may be characterized by ([6])

$$\begin{cases} \dot{u} = M(u,v)u \\ \dot{v} = N(u,v)v \end{cases} \tag{5.1}$$

particularly, the Lotaka-Volterra competition model is given by

$$\begin{cases} \dot{u} = a_1 u\left(1 - u/K_1 - \alpha_{12} v/K_1\right) \\ \dot{v} = a_2 v\left(1 - v/K_2 - \alpha_{21} u/K_2\right) \end{cases} \tag{5.2}$$

in ordinary differentials ([21]), or

$$\begin{cases} u_t = d_1 \Delta u + a_1 u(1 - K_1 u - \alpha_{12} v/K_1) \\ v_t = d_2 \Delta v + a_2 v(1 - K_2 u - \alpha_{21} v/K_2) \end{cases} \tag{5.3}$$

in partial differentials on a boundary domain $\Omega \subset \mathbb{R}^n$ for an integer $n \geq 1$ with initial conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on unit normal out vector ν, $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ ([28]), where $u(x,t), v(x,t)$ are respectively the density of 2 competitive species at $(x,t) \in \Omega \times (0,\infty)$, M, N and positive parameters a_1, a_2 are the growth rates, K_1, K_2 are the carrying capacities, α_{ij} denotes the interaction between the two species, i.e., the effect of species i on species j for $i, j = 1$ or 2, and d_1, d_2 are the diffusion rate of species 1 and 2, respectively. This system is nothing else but an action flow on loop B_1 on a boundary domain $\Omega \subset \mathbb{R}^n$ for an integer $n \geq 1$

Fig.22

with initial conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on unit normal out vector ν and $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ for $(x, t) \in \Omega \times (0, \infty)$ such as those shown in Fig.22, where $A(u, v) = (uM(u, v), vN(u, v))$. For example, $M(u, v) = a_1 (1 - u/K_1 - \alpha_{12}v/K_1)$, $N(u, v) = a_2 (1 - v/K_2 - \alpha_{21}u/K_2)$ in equations (5.2) or $M(u, v) = d_1\Delta u/u + a_1(1 - K_1u - \alpha_{12}v/K_1)$, $N(u, v) = d_2\Delta v/v + a_2(1 - K_2u - \alpha_{21}v/K_2)$ in equations (5.3) for $\forall(x, t) \in \Omega \times (0, \infty)$.

Similarly, assume there are four kind groups in persons at time t, i.e., susceptible $S(t)$, infected but in the incubation period $E(t)$, infected with infectious $I(t)$ and recovered $R(t)$ and new recognition Λ with removal rates κ, α, contact rate β and natural mortality rate μ, such as the action flow shown in Fig.23.

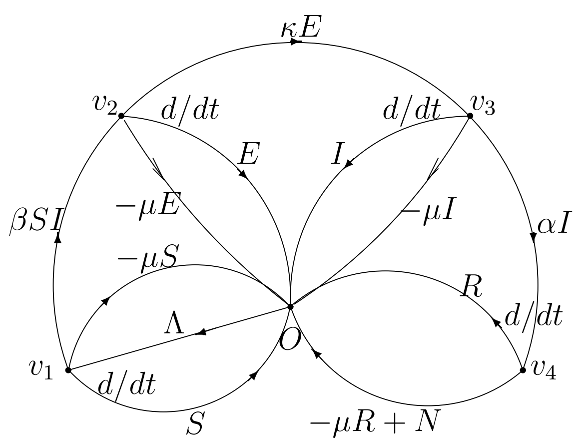


Fig.23

Then, we are easily to get the SEIR model on infectious by conservative laws respectively at vertices v_1, v_2, v_3 and v_4 following:

$$\begin{cases} \dot{S} = \Lambda - \mu S - \beta SI \\ \dot{E} = \beta SI - (\mu+\kappa)E \\ \dot{I} = \kappa E - (\mu+\alpha)I \\ \dot{R} = \alpha I - \mu R \end{cases} \tag{5.4}$$

where, $N = S+R+E+I-\mu(S+R+E+I)$ and all end-operators are **1** if it is not labeled in Fig.23. Notice that the systems (5.1)-(5.4) of differential equations are solvable. Whence, the behavior of action flows in Figs.22 and 23 can be characterized respectively by solution of system (5.1)-(5.4).

Generally, an ecological system is such an action flow $\left(\overrightarrow{G}; L, A\right)$ on an oriented graph with a loop on its each vertex, where flows on loops and other edges denote respectively the density of species or interactions of one species action on another. If the conservation laws of an action flow are not solvable, then holding on the reality of competitive species by solution of equations will be not suitable again.

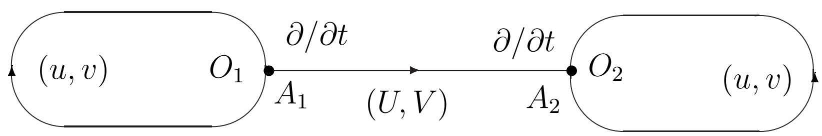


Fig.24

For example, the action flow shown in Fig.24 is such an ecological system with conservation laws

$$\begin{cases} u_t = A_1(u,t) + U(u,t) \\ v_t = A_1(u,t) + V(u,t) \end{cases}, \qquad \begin{cases} u_t = A_2(u,t) - U(u,t) \\ v_t = A_2(u,t) - V(u,t) \end{cases}$$

under initial conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$, $u(x,0) = u_0(x), v(x,0) = v_0(x)$ for $(x,t) \in \Omega \times (0,\infty)$, where

$$\begin{aligned} A_1(u,v) &= (d_1\Delta u + a_1 u(1 - K_1 u - \alpha_{12} v/K_1), d_2\Delta v + a_2 v(1 - K_2 u - \alpha_{21} v/K_2)) \\ A_2(u,v) &= (d_3\Delta u + a_3 u(1 - K_3 u - \alpha_{34} v/K_3), d_4\Delta v + a_4 v(1 - K_4 u - \alpha_{43} v/K_4)) \end{aligned}$$

for $\forall (x,t) \in \Omega \times (0,\infty)$ and U, V are known functions. They are non-solvable in general but we can characterize its behaviors, for instance, the global stability by application of Theorem 4.2.

In fact, all ecological systems are interaction fields in physics. Let $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_m$ be m interaction fields with respective Hamiltonians $H^{[1]}, H^{[2]}, \cdots, H^{[m]}$, where

$$H^{[k]}: \ (q_1, \cdots, q_n, p_2, \cdots, p_n, t) \to H^{[k]}(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$$

for integers $1 \leq k \leq m$, i.e.,

$$\left.\begin{array}{l} \dfrac{\partial H^{[k]}}{\partial p_i} = \dfrac{dq_i}{dt} \\ \dfrac{\partial H^{[k]}}{\partial q_i} = -\dfrac{dp_i}{dt}, \quad 1 \leq i \leq n \end{array}\right\} \quad 1 \leq k \leq m.$$

Such a system is equivalent to the Cauchy problem on the system of partial differential equations

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = H_k(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[k]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\} \quad 1 \leq k \leq m. \qquad (PDES_m)$$

and is in fact an action flows on m dipoles $\overrightarrow{D}_{0,2,0}$. For example, a system of interaction field is shown in Fig.25 in $m = 4$ with $A = A' = \dfrac{\partial}{\partial t}$ and $A_1 = A_1 = 1$.

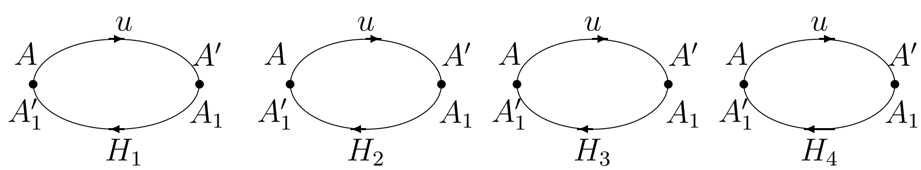


Fig.25

By choosing Liapunov sum-function $L(\omega(t))(X) = \sum\limits_{i=1}^{m} H_i(X)$ on $\overrightarrow{G}$ in Theorem 3.15, the following result was obtained in [15] on the stability of system $(PDES_m)$.

Theorem 5.3 *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in $(PDES_m)$ for each integer $1 \leq i \leq m$. If $\sum\limits_{i=1}^{m} H_i(X) > 0$ and $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} \leq 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then the system $(PDES_m)$ is sum-stability, i.e., $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} < 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Sigma}{\to} G[0]$.*

§6. Conclusion

The main function of mathematics is provide quantitative analysis tools or ways for holding on the reality of things by observing from a macro or micro view. In fact, the out or macro observation is basic but the in-observation is cardinal, and an in-observation characterizes the individual behavior of things but with non-solvable equations in mathematics. However, the trend of mathematical developing in 20th century shows that a mathematical system is more concise, its conclusion is more extended, but farther to the true face of the natural things. *Is a mathematical true inevitable lead to the natural reality of a thing*? Certainly not because more characters of thing P have been abandoned in its mathematical model. Then, *is there a mathematical envelope theory on classical mathematics reflecting the nature of things*? Answer this question motivates the *mathematical combinatorics*, i.e., extending mathematical systems on topological graphs $\overrightarrow{G}$ because the reality of things is nothing else but a multiverse on a topological structure under action, i.e., *action flows*, which is an appropriated way for understanding the nature because things are in connection, also with contradiction.

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