General Method for Summing Divergent Series Using Mathematica and a Comparison to Other Summation Methods

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Abstract

We are interested in finding sums of some divergent series using the general method for summing divergent series discovered in our previous work[1] and symbolic mathematical computation program Mathematica. We make a comparison to other five summation methods implemented in Mathematica and show that our method is the stronger method than methods of Abel, Borel, Cesaro, Dirichlet and Euler.

Contents

1 Introduction

The aim of this paper is to show readers how to sum divergent series using the summation method discovered in our previous work[1] and symbolic mathematical computation program Mathematica and make a comparison to other five summation

[1] In our previous work we discovered a general method for summing divergent series and determination of limits of divergent sequences and functions in singular points. We also showed that the method is useful for solving divergent integrals.

methods implemented in Mathematica. As for prerequisites, the reader is expected to be familiar with real and complex analysis in one variable.
In this section, we summarize without proofs the relevant results on the general method for summing divergent series and give the sums of some divergent series from Hardy's book[2] and Ramanujan's notebook[3].
In Section 2 these sums are solved using Mathematica and general method for summing divergent series.
In Section 3 we give a table with comparison to the five most famous summation methods (Abel, Borel, Cesaro, Dirichlet and Euler) which are also used to find the sums of series from Section 2 and show that our method is the strongest (see [4] for the history of the theory of summable divergent series).
Suppose the function f has a singularity at infinity. Let's define the general limit of f(z) as z approaches infinity, denoted by $\lim^{D}_{z\to\infty} f(z)$.
We obtain the following results:
(a) If f has a pole of order m at infinity, then

$$\lim^{D}_{z\to\infty} f(z) = \int_{-1}^{0} \sum_{n=0}^{m} c_n z^n dz,$$

where $f(z) = \sum_{n=-\infty}^{m} c_n z^n$ $(|z| > R)$ is the Laurent series expansion of f about infinity.
(b) If f has a removable singularity at infinity, then

$$\lim^{D}_{z\to\infty} f(z) = c_0 = \lim_{z\to\infty} f(z),$$

where $f(z) = \sum_{n=-\infty}^{0} c_n z^n$ $(|z| > R)$ is the Laurent series expansion of f about infinity.
(c) If f has essential singularity or branch point at infinity, then

$$\lim^{D}_{z\to\infty} f(z) = c,$$

where c is constant part of any series expansion (Laurent series expansion, Puiseux series expansion, ...) of f about infinity.
Finally, if $\sum_{n=1}^{\infty} a_n$ is divergent series, then

$$\sum_{n=1}^{\infty} a_n = \lim^{D}_{z\to\infty} s(z),$$

where $s(n) = s_n = \sum_{k=1}^{n} a_k$ is nth partial sum.
Now, we give the following sums:

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + ... + (-1)^{n-1} + ... = \frac{1}{2} \tag{1}$$

$$\sum_{n=1}^{\infty} 2^{n-1} = 1 + 2 + 4 + 8 + ... + 2^{n-1} + ... = -1 \tag{2}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} 2^{n-1} = 1 - 2 + 4 - 8 + ... + (-2)^{n-1} + ... = \frac{1}{3} \tag{3}$$

$$\sum_{n=1}^{\infty} a^{n-1} = 1 + a + a^2 + a^3 + ... + a^{n-1} + ... = \frac{1}{1-a} (a > 1) \tag{4}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} n = 1 - 2 + 3 - 4 + ... + (-n)^{n-1} + ... = \frac{1}{4} \tag{5}$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + ... + n + ... = -\frac{1}{12} \tag{6}$$

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + ... + 1 + ... = -\frac{1}{2} \tag{7}$$

$$\sum_{n=1}^{\infty} (n+1) = 2 + 3 + 4 + 5 + ... + (n+1) + ... = -\frac{7}{12} \tag{8}$$

$$\sum_{n=1}^{\infty} (n-1) = 0 + 1 + 2 + 3 + ... + (n-1) + ... = \frac{5}{12} \tag{9}$$

$$\sum_{n=1}^{\infty} \ln(n) = \ln 1 + \ln 2 + \ln 3 + \ln 4 + ... + \ln(n) + ... = \frac{1}{2} \ln(2\pi) \tag{10}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \ln(n) = \ln 1 - \ln 2 + \ln 3 - \ln 4 + ... + (-1)^{n-1} \ln(n) + ... = -\frac{1}{2} \ln(\frac{1}{2}\pi) \tag{11}$$

$$\sum_{n=1}^{\infty} \cos(n\theta) = \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + ... + \cos(n\theta) + ... = -\frac{1}{2} (0 < \theta < 2\pi) \tag{12}$$

$$\sum_{n=1}^{\infty} \sin(n\theta) = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + ... + \sin(n\theta) + ... = \frac{1}{2} \cot\frac{\theta}{2} (0 < \theta < 2\pi) \tag{13}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cos(n\theta) = \cos\theta - \cos 2\theta + \cos 3\theta - \cos 4\theta + ... + (-1)^{n-1}\cos(n\theta) + ... = \frac{1}{2}(-\pi < \theta < \pi) \tag{14}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\sin(n\theta) = \sin\theta - \sin 2\theta + \sin 3\theta - \sin 4\theta + ... + (-1)^{n-1}\sin(n\theta) + ... = \frac{1}{2}\tan\frac{\theta}{2}(-\pi < \theta < \pi) \tag{15}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{2k} = 1^{2k} - 2^{2k} + 3^{2k} - 4^{2k} + ... + (-1)^{n-1}n^{2k} + ... = 0(k = 1, 2, 3...) \tag{16}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{2k-1} = 1^{2k-1} - 2^{2k-1} + 3^{2k-1} - 4^{2k-1} + ... + (-1)^{n-1}n^{2k-1} + ... = \frac{2^{2k}-1}{2k}B_{2k}(k = 1, 2, 3...) \tag{17}$$

$$\sum_{n=1}^{\infty}n^{k} = 1^{k} + 2^{k} + 3^{k} + 4^{k} + ... + n^{k} + ... = -\frac{B_{k+1}}{k+1}(k = 1, 2, 3, ...) \tag{18}$$

$$\sum_{n=1}^{\infty}n^{-s} = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + ... + n^{-s} + ... = \zeta(s)(Re(s) < 1) \tag{19}$$

$$\sum_{n=1}^{\infty}(1 - ((n-1) \mod 3)) = 1 + 0 + (-1) + 1 + 0 + (-1) + ... = \frac{2}{3} \tag{20}$$

$$\sum_{n=1}^{\infty}(((n+1) \mod 3) - 1) = 1 + (-1) + 0 + 1 + (-1) + 0 + ... = \frac{1}{3} \tag{21}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)! = 1 - 1! + 2! - 3! + 4! - ... + (-1)^{n-1}(n-1)! + ... = 1 - e \cdot E_2(1) \approx 0,596347 \tag{22}$$

$$\sum_{n=1}^{\infty}(n-1)! = 1 + 1! + 2! + 3! + 4! + ... + (n-1)! + ... = 1 + !(-2) \approx 0.697175 + 1.15573 \cdot i. \tag{23}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...+\frac{1}{n}+...=\gamma\approx 0.57721566 \quad (24)$$

2 General Method for Summing Divergent Series Using Mathematica

Calculate the sum of the series (1).

SumConvergence[(−1)^(n − 1) ∗ n, n]

False

Sum[(−1)^(n − 1), {n, 1, k}]

$\frac{1}{2}(1+(-1)^{1+k})$

Series[%, {k, Infinity, 10}]

$\frac{1}{2}(1+(-1)^{1+k})$

SeriesCoefficient[%, 0]

$\frac{1}{2}(1+(-1)^{1+k})$

ExpandAll[%, k]

$\frac{1}{2}+\frac{1}{2}(-1)^{1+k}$

Calculate the sum of the series (2).

SumConvergence[2^(n − 1), n]

False

Sum[2^(n − 1), {n, 1, k}]

$-1 + 2^k$

Series[%, {k, Infinity, 10}]

$-1 + 2^k$

ExpandAll[%, k]

$-1 + 2^{k}$

Calculate the sum of the series (3).

SumConvergence[(−1)^(n − 1) * 2^(n − 1), n]

False

Sum[(−1)^(n − 1) * 2^(n − 1), {n, 1, k}]

$\frac{1}{3}(1 - (-2)^k)$

Series[%, {k, Infinity, 10}]

$\frac{1}{3}(1 - (-2)^k)$

ExpandAll[%, k]

$\frac{1}{3} - \frac{(-2)^{k}}{3}$

Calculate the sum of the series (4).

SumConvergence[a^(n − 1), n]

Abs[a] < 1

Sum[a^(n − 1), {n, 1, k}]

$\frac{-1+a^k}{-1+a}$

Series[%, {k, Infinity, 10}]

$\frac{-1+a^k}{-1+a}$

ExpandAll[%, k]

$-\frac{1}{-1+a} + \frac{a^k}{-1+a}$

Calculate the sum of the series (5).

SumConvergence[(−1)^(n − 1) ∗ n, n]

False

Sum[(−1)^(n − 1) ∗ n, {n, 1, k}]

$\frac{1}{4}(1 + (-1)^{1+k} - 2(-1)^k k)$

Series[%, {k, Infinity, 10}]

$\frac{1}{4}(1 + (-1)^{1+k} + (-1)^k(-2k + O[\frac{1}{k}]^{11}))$

ExpandAll[%, k]

$\mathbf{\frac{1}{4} + \frac{1}{4}(-1)^{1+k} + (-1)^{k}(-\frac{k}{2} + O[\frac{1}{k}]^{11})}$

Calculate the sum of the series (6).

SumConvergence[n, n]

False

Sum[n, {n, 1, k}]

$\frac{1}{2}k(1+k)$

Series[%, {k, Infinity, 10}]

$\frac{k^2}{2} + \frac{k}{2} + O[\frac{1}{k}]^{11}$

Normal[%]

$\frac{k}{2} + \frac{k^2}{2}$

Integrate[%, {k, −1, 0}]

$\mathbf{-\frac{1}{12}}$

Calculate the sum of the series (7).

SumConvergence[1, n]

False

Sum[1, {n, 1, k}]

k

Series[%, {k, Infinity, 10}]

$k + O[\frac{1}{k}]^{11}$

Normal[%]

k

Integrate[%, {k, −1, 0}]

$-\frac{1}{2}$

Calculate the sum of the series (8).

SumConvergence[$n + 1$, n]

False

Sum[$n + 1$, {n, 1, k}]

$\frac{1}{2}(3k + k^2)$

Series[%, {k, Infinity, 10}]

$\frac{k^2}{2} + \frac{3k}{2} + O[\frac{1}{k}]^{11}$

Normal[%]

$\frac{3k}{2} + \frac{k^2}{2}$

Integrate[%, {k, −1, 0}]

$-\frac{\mathbf{7}}{\mathbf{12}}$

Calculate the sum of the series (9).

SumConvergence[n − 1, n]

False

Sum[n − 1, {n, 1, k}]

$\frac{1}{2}(-k + k^2)$

Series[%, {k, Infinity, 10}]

$\frac{k^2}{2} - \frac{k}{2} + O[\frac{1}{k}]^{11}$

Normal[%]

$-\frac{k}{2} + \frac{k^2}{2}$

Integrate[%, {k, −1, 0}]

$\frac{\mathbf{5}}{\mathbf{12}}$

Calculate the sum of the series (10).

SumConvergence[Log[n], n]

False

Sum[Log[n], {n, 1, k}]

Log[Pochhammer[1, k]]

Series[%, {k, Infinity, 10}]

$(-1+\text{Log}[k])k+\frac{1}{2}(-\text{Log}[\frac{1}{k}]+\text{Log}[2\pi])+\frac{1}{12k}-\frac{1}{360k^3}+\frac{1}{1260k^5}-\frac{1}{1680k^7}+\frac{1}{1188k^9}+O[\frac{1}{k}]^{21/2}$

ExpandAll[%, k]

$(-1+\text{Log}[k])k+(-\frac{1}{2}\text{Log}[\frac{1}{k}]+\frac{1}{2}\text{Log}[2\pi])+\frac{1}{12k}-\frac{1}{360k^3}+\frac{1}{1260k^5}-\frac{1}{1680k^7}+\frac{1}{1188k^9}+O[\frac{1}{k}]^{21/2}$

Calculate the sum of the series (11).

SumConvergence[(−1)^(n − 1) ∗ Log[n], n]

False

Sum[(−1)^(n − 1) ∗ Log[n], {n, 1, k}]

$\text{Log}[2]-\frac{1}{2}(-1)^k\text{Log}[2]-\frac{1}{2}\text{Log}[2\pi]+(-1)^k\text{Log}[\text{Gamma}[\frac{1+k}{2}]]+(-1)^{1+k}\text{Log}[\text{Gamma}[\frac{2+k}{2}]]$

Series[%, {k, Infinity, 10}]

$\frac{1}{2}(2\text{Log}[2]+(-1)^{1+k}\text{Log}[2]-\text{Log}[2\pi]+(-1)^k((1+\text{Log}[2]-\text{Log}[k])k+(\text{Log}[\frac{1}{k}]-\text{Log}[\pi]-\frac{1}{3k}+\frac{2}{45k^3}-\frac{16}{315k^5}+\frac{16}{105k^7}-\frac{256}{297k^9}+O[\frac{1}{k}]^{21/2})+(-1)^k((-1-\text{Log}[2]+\text{Log}[k])k+\text{Log}[2\pi]-$

$\frac{1}{6k} + \frac{7}{180k^3} - \frac{31}{630k^5} + \frac{127}{840k^7} - \frac{511}{594k^9} + O[\frac{1}{k}]^{21/2}))$

ExpandAll[%, *k*]

$\mathrm{Log}[\mathbf{2}] + \frac{1}{2}(-\mathbf{1})^{\mathbf{1+k}}\mathrm{Log}[\mathbf{2}] - \frac{1}{2}\mathrm{Log}[\mathbf{2}\pi] + (-\mathbf{1})^{\mathbf{k}}(\frac{1}{2}(\mathbf{1} + \mathrm{Log}[\mathbf{2}] - \mathrm{Log}[\mathbf{k}])\mathbf{k} + \frac{1}{2}(\mathrm{Log}[\frac{1}{\mathbf{k}}] - \mathrm{Log}[\pi]) - \frac{1}{6\mathbf{k}} + \frac{1}{45\mathbf{k}^3} - \frac{8}{315\mathbf{k}^5} + \frac{8}{105\mathbf{k}^7} - \frac{128}{297\mathbf{k}^9} + \mathbf{O}[\frac{1}{\mathbf{k}}]^{21/2}) + (-\mathbf{1})^{\mathbf{k}}(\frac{1}{2}(-\mathbf{1} - \mathrm{Log}[\mathbf{2}] + \mathrm{Log}[\mathbf{k}])\mathbf{k} + \frac{1}{2}\mathrm{Log}[\mathbf{2}\pi] - \frac{1}{12\mathbf{k}} + \frac{7}{360\mathbf{k}^3} - \frac{31}{1260\mathbf{k}^5} + \frac{127}{1680\mathbf{k}^7} - \frac{511}{1188\mathbf{k}^9} + \mathbf{O}[\frac{1}{\mathbf{k}}]^{21/2})$

Calculate the sum of the series (12).

SumConvergence[Cos[*n* ∗ *θ*], *n*]

False

Sum[Cos[*n* ∗ *θ*], {*n*, 1, *k*}]

$\mathrm{Cos}[\frac{1}{2}(1 + k)\theta]\mathrm{Csc}[\frac{\theta}{2}]\mathrm{Sin}[\frac{k\theta}{2}]$

Series[%, {*k*, Infinity, 10}]

$\mathrm{Cos}[\frac{\theta}{2} + \frac{k\theta}{2}]\mathrm{Csc}[\frac{\theta}{2}]\mathrm{Sin}[\frac{k\theta}{2}]$

TrigExpand[%]

$-\frac{1}{2} + \frac{1}{2}\mathrm{Cos}[\frac{\mathbf{k}\theta}{2}]^{2} + \mathrm{Cos}[\frac{\mathbf{k}\theta}{2}]\mathrm{Cot}[\frac{\theta}{2}]\mathrm{Sin}[\frac{\mathbf{k}\theta}{2}] - \frac{1}{2}\mathrm{Sin}[\frac{\mathbf{k}\theta}{2}]^{2}$

Calculate the sum of the series (13).

SumConvergence[Sin[*n* ∗ *θ*], *n*]

False

Sum[Sin[$n * \theta$], {n, 1, k}]

$\text{Csc}[\frac{\theta}{2}]\text{Sin}[\frac{k\theta}{2}]\text{Sin}[\frac{1}{2}(1+k)\theta]$

Series[%, {k, Infinity, 10}]

$\text{Csc}[\frac{\theta}{2}]\text{Sin}[\frac{k\theta}{2}]\text{Sin}[\frac{\theta}{2}+\frac{k\theta}{2}]$

TrigExpand[%]

$\frac{1}{2}\text{Cot}[\frac{\theta}{2}] - \frac{1}{2}\text{Cos}[\frac{\mathbf{k}\theta}{2}]^2\text{Cot}[\frac{\theta}{2}] + \text{Cos}[\frac{\mathbf{k}\theta}{2}]\text{Sin}[\frac{\mathbf{k}\theta}{2}] + \frac{1}{2}\text{Cot}[\frac{\theta}{2}]\text{Sin}[\frac{\mathbf{k}\theta}{2}]^2$

Calculate the sum of the series (14).

SumConvergence[(−1)^(n − 1) * Cos[$n * \theta$], n]

$\text{SumConvergence}[(-1)^{-1+n}\text{Cos}[n\theta], n]$

Sum[(−1)^(n − 1) * Cos[$n * \theta$], {n, 1, k}]

$\frac{1}{2}(1+(-1)^{1+k}\text{Cos}[\frac{1}{2}(1+2k)\theta]\text{Sec}[\frac{\theta}{2}])$

Series[%, {k, Infinity, 10}]

$\frac{1}{2}(1+(-1)^{1+k}\text{Cos}[\frac{\theta}{2}+k\theta]\text{Sec}[\frac{\theta}{2}])$

TrigExpand[%]

$\frac{1}{2} - \frac{1}{2}(-1)^{\mathbf{k}}\text{Cos}[\mathbf{k}\theta] + \frac{1}{2}(-1)^{\mathbf{k}}\text{Sin}[\mathbf{k}\theta]\text{Tan}[\frac{\theta}{2}]$

Calculate the sum of the series (15).

SumConvergence[(−1)^(n − 1) ∗ Sin[n ∗ θ], n]

$\text{SumConvergence}[(-1)^{-1+n}\text{Sin}[n\theta], n]$

Sum[(−1)^(n − 1) ∗ Sin[n ∗ θ], {n, 1, k}]

$\text{Csc}[\theta]\text{Sin}[\frac{\theta}{2}](\text{Sin}[\frac{\theta}{2}] + (-1)^{1+k}\text{Sin}[\frac{1}{2}(1 + 2k)\theta])$

Series[%, {k, Infinity, 10}]

$\text{Csc}[\theta]\text{Sin}[\frac{\theta}{2}](\text{Sin}[\frac{\theta}{2}] + (-1)^{1+k}\text{Sin}[\frac{\theta}{2} + k\theta])$

TrigExpand[%]

$-\frac{\mathbf{1}}{\mathbf{4}}\text{Cot}[\frac{\theta}{\mathbf{2}}]+\frac{\mathbf{1}}{\mathbf{4}}(-\mathbf{1})^{\mathbf{k}}\text{Cos}[\mathbf{k}\theta]\text{Cot}[\frac{\theta}{\mathbf{2}}]+\frac{\mathbf{1}}{\mathbf{4}}\text{Csc}[\frac{\theta}{\mathbf{2}}]\text{Sec}[\frac{\theta}{\mathbf{2}}]-\frac{\mathbf{1}}{\mathbf{4}}(-\mathbf{1})^{\mathbf{k}}\text{Cos}[\mathbf{k}\theta]\text{Csc}[\frac{\theta}{\mathbf{2}}]\text{Sec}[\frac{\theta}{\mathbf{2}}]-\frac{\mathbf{1}}{\mathbf{2}}(-\mathbf{1})^{\mathbf{k}}\text{Sin}[\mathbf{k}\theta]+\frac{\mathbf{1}}{\mathbf{4}}\text{Tan}[\frac{\theta}{\mathbf{2}}]-\frac{\mathbf{1}}{\mathbf{4}}(-\mathbf{1})^{\mathbf{k}}\text{Cos}[\mathbf{k}\theta]\text{Tan}[\frac{\theta}{\mathbf{2}}]$

Calculate the sum of the series (16).

SumConvergence[(−1)^(n − 1) ∗ n^(2 ∗ k), n]

$-2\text{Re}[k] > 1$

Sum[(−1)^(n − 1) ∗ n^(2 ∗ k), {n, 1, m}]

$\text{Zeta}[-2k]-2^{1+2k}\text{Zeta}[-2k]+(-1)^{1+m}2^{2k}\text{Zeta}[-2k, \frac{1+m}{2}, \text{IncludeSingularTerm} \to \text{False}]+(-1)^{m}2^{2k}\text{Zeta}[-2k, \frac{2+m}{2}, \text{IncludeSingularTerm} \to \text{False}]$

Series[%, {m, Infinity, 10}]

$\text{Zeta}[-2k] - 2^{1+2k}\text{Zeta}[-2k] + (-1)^{1+m}2^{2k}\text{Zeta}[-2k, \frac{1}{2} + \frac{m}{2}, \text{IncludeSingularTerm} \to$

False] + $(-1)^m 2^{2k}$Zeta[$-2k, 1+\frac{m}{2}$, IncludeSingularTerm → False]

ExpandAll[%]

Zeta[$-\mathbf{2k}$] $- \mathbf{2^{1+2k}}$Zeta[$-\mathbf{2k}$]$+(-\mathbf{1})^{\mathbf{1+m}}\mathbf{2^{2k}}$Zeta[$-\mathbf{2k}, \frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{m}}{\mathbf{2}}$, IncludeSingularTerm →

False] + $(-\mathbf{1})^{\mathbf{m}}\mathbf{2^{2k}}$Zeta[$-\mathbf{2k}, \mathbf{1}+\frac{\mathbf{m}}{\mathbf{2}}$, IncludeSingularTerm → False]

Calculate the sum of the series (17).

SumConvergence[(−1)^(*n* − 1) ∗ *n*^(2 ∗ *k* − 1), *n*]

Re[k] < 0

Sum[(−1)^(*n* − 1) ∗ *n*^(2 ∗ *k* − 1), {*n*, 1, *m*}]

$\frac{1}{2}$(2Zeta[$1-2k$]-2^{1+2k}Zeta[$1-2k$]$+(-1)^{1+m}2^{2k}$Zeta[$1-2k, \frac{1+m}{2}$, IncludeSingularTerm →

False] + $(-1)^m 2^{2k}$Zeta[$1-2k, \frac{2+m}{2}$, IncludeSingularTerm → False])

Series[%, {*m*, Infinity, 10}]

$\frac{1}{2}$(2Zeta[$1-2k$]-2^{1+2k}Zeta[$1-2k$]$+(-1)^{1+m}2^{2k}$Zeta[$1-2k, \frac{1}{2}+\frac{m}{2}$, IncludeSingularTerm →

False] + $(-1)^m 2^{2k}$Zeta[$1-2k, 1+\frac{m}{2}$, IncludeSingularTerm → False])

ExpandAll[%]

Zeta[$\mathbf{1}-\mathbf{2k}$] $- \mathbf{2^{2k}}$Zeta[$\mathbf{1}-\mathbf{2k}$]$+(-\mathbf{1})^{\mathbf{1+m}}\mathbf{2^{-1+2k}}$Zeta[$\mathbf{1}-\mathbf{2k}, \frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{m}}{\mathbf{2}}$, IncludeSingularTerm →

False] + $(-\mathbf{1})^{\mathbf{m}}\mathbf{2^{-1+2k}}$Zeta[$\mathbf{1}-\mathbf{2k}, \mathbf{1}+\frac{\mathbf{m}}{\mathbf{2}}$, IncludeSingularTerm → False]

Calculate the sum of the series (18).

SumConvergence[n^k, n]

$1+\mathrm{Re}[k]<0$

Sum[n^k, {n, 1, m}]

HarmonicNumber$[m,-k]$

Series[%, {m, Infinity, 10}]

$$m^k\left(\frac{m}{1+k}+\frac{1}{2}+\frac{k}{12m}+\frac{-2k+3k^2-k^3}{720m^3}+\frac{(-4+k)(-3+k)(-2+k)(-1+k)k}{30240m^5}-\frac{(-6+k)(-5+k)(-4+k)(-3+k)(-2+k)(-1+k)k}{1209600m^7}+\frac{(-8+k)(-7+k)(-6+k)(-5+k)(-4+k)(-3+k)(-2+k)(-1+k)k}{47900160m^9}+O\left[\frac{1}{m}\right]^{11}\right)+\mathrm{Zeta}[-k]$$

ExpandAll[%]

$$m^k\left(\frac{m}{1+k}+\frac{1}{2}+\frac{k}{12m}+\frac{-\frac{k}{360}+\frac{k^2}{240}-\frac{k^3}{720}}{m^3}+\frac{\frac{k}{1260}-\frac{5k^2}{3024}+\frac{k^3}{864}-\frac{k^4}{3024}+\frac{k^5}{30240}}{m^5}+\frac{-\frac{k}{1680}+\frac{7k^2}{4800}-\frac{29k^3}{21600}+\frac{7k^4}{11520}-\frac{k^5}{6912}+\frac{k^6}{57600}-\frac{k^7}{1209600}}{m^7}+\frac{\frac{k}{1188}-\frac{761k^2}{332640}+\frac{29531k^3}{11975040}-\frac{89k^4}{63360}+\frac{1069k^5}{2280960}-\frac{k^6}{10560}+\frac{13k^7}{1140480}-\frac{k^8}{1330560}+\frac{k^9}{47900160}}{m^9}+O\left[\frac{1}{m}\right]^{11}\right)+\mathrm{Zeta}[-\mathbf{k}]$$

Calculate the sum of the series (19).

SumConvergence[n^(−s), n]

$\mathrm{Re}[s]>1$

Sum[n^(−s), {n, 1, k}]

HarmonicNumber$[k,s]$

Series[%, {k, Infinity, 10}]

$$k^{-s}\left(-\frac{k}{-1+s}+\frac{1}{2}-\frac{s}{12k}+\frac{s(1+s)(2+s)}{720k^3}-\frac{s(1+s)(2+s)(3+s)(4+s)}{30240k^5}+\frac{s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1209600k^7}-\right.$$

$$\frac{s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47900160k^9} + O[\tfrac{1}{k}]^{11}) + \mathrm{Zeta}[s]$$

ExpandAll[%]

$$k^{-s}\left(-\frac{k}{-1+s}+\frac{1}{2}-\frac{s}{12k}+\frac{\frac{s}{360}+\frac{s^2}{240}+\frac{s^3}{720}}{k^3}+\frac{-\frac{s}{1260}-\frac{5s^2}{3024}-\frac{s^3}{864}-\frac{s^4}{3024}-\frac{s^5}{30240}}{k^5}+\frac{\frac{s}{1680}+\frac{7s^2}{4800}+\frac{29s^3}{21600}+\frac{7s^4}{11520}+\frac{s^5}{6912}+\frac{s^6}{57600}+\frac{s^7}{1209600}}{k^7}+\frac{-\frac{s}{1188}-\frac{761s^2}{332640}-\frac{29531s^3}{11975040}-\frac{89s^4}{63360}-\frac{1069s^5}{2280960}-\frac{s^6}{10560}-\frac{13s^7}{1140480}-\frac{s^8}{1330560}-\frac{s^9}{47900160}}{k^9} + O[\tfrac{1}{k}]^{11}\right) + \mathrm{Zeta}[\mathbf{s}]$$

Calculate the sum of the series (20).

SumConvergence[Mod[2 + 2*n*, 3, −1], *n*]

False

Sum[Mod[2 + 2*n*, 3, −1], {*n*, 1, *k*}]

$1 + \mathrm{Floor}[\frac{1}{3}(-1+k)] - \mathrm{Floor}[\frac{k}{3}]$

Series[%, {*k*, Infinity, 10}]

$1 + \mathrm{Floor}[-\frac{1}{3} + \frac{k}{3}] - \mathrm{Floor}[\frac{k}{3}]$

ExpandAll[%]

$\mathbf{1} + \mathrm{Floor}[-\frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{k}}{\mathbf{3}}] - \mathrm{Floor}[\frac{\mathbf{k}}{\mathbf{3}}]$[2]

Calculate the sum of the series (21).

SumConvergence[Mod[*n*, 3, −1], *n*]

False

[2]We know if x is not an integer we have $\lfloor x \rfloor = x - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{2\pi k x}{k}$.

Sum[Mod[n, 3, −1], {n, 1, k}]

$-\text{Floor}[\frac{1}{3}(-2+k)]+\text{Floor}[\frac{1}{3}(-1+k)]$

Series[%, {k, Infinity, 10}]

$-\text{Floor}[-\frac{2}{3}+\frac{k}{3}]+\text{Floor}[-\frac{1}{3}+\frac{k}{3}]$

ExpandAll[%]

$-\text{Floor}[-\frac{\mathbf{2}}{\mathbf{3}}+\frac{\mathbf{k}}{\mathbf{3}}]+\text{Floor}[-\frac{\mathbf{1}}{\mathbf{3}}+\frac{\mathbf{k}}{\mathbf{3}}]$

Calculate the sum of the series (22).

SumConvergence[(−1)^(n − 1) ∗ (n − 1)!, n]

False

RSolve[{s[n] == s[n − 1] + (−1)^(n − 1) ∗ (n − 1)!, s[1] == 1}, s[n], n]

$\{\{s[n]\to 1-e\text{ExpIntegralE}[2,1]+(-1)^{1+n}e\text{ExpIntegralE}[1+n,1]n!\}\}$

Series[1 − eExpIntegralE[2, 1] + (−1)$^{1+n}$$e$ExpIntegralE[1 + n, 1]n!, .
{k, Infinity, 10}]

$1-e\text{ExpIntegralE}[2,1]+(-1)^{1+n}e\text{ExpIntegralE}[1+n,1]n!$

ExpandAll[%]

$1-e\text{ExpIntegralE}[2,1]+(-1)^{1+n}e\text{ExpIntegralE}[1+n,1]n!$

$N[1 - e\text{ExpIntegralE}[2,1]]$

0.596347

Calculate sum of the series (23).

$\text{SumConvergence}[(n-1)!, n]$

False

$\text{RSolve}[\{s[n] == s[n-1] + (n-1)!, s[1] == 1\}, s[n], n]$

$\{\{s[n] \to 1 + \text{Subfactorial}[-2] + (-1)^n n!\text{Subfactorial}[-1-n]\}\}$

$\text{Series}[1 + \text{Subfactorial}[-2] + (-1)^n n!\text{Subfactorial}[-1-n], \{k, \text{Infinity}, 10\}]$

$1 + \text{Subfactorial}[-2] + (-1)^n n!\text{Subfactorial}[-1-n]$

$\text{ExpandAll}[\%]$

$1 + \text{Subfactorial}[-2] + (-1)^n n!\text{Subfactorial}[-1-n]$

$N[1 + \text{Subfactorial}[-2]]$

$0.697175 + 1.15573i$

Calculate the sum of the series (24).

$\text{SumConvergence}[1/n, n]$

False

Sum[1/n, {n, 1, k}]

HarmonicNumber[k]

Series[%, {k, Infinity, 10}]

$(\text{EulerGamma} - \text{Log}[\frac{1}{k}]) + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} + \frac{1}{240k^8} - \frac{1}{132k^{10}} + O[\frac{1}{k}]^{11}$

ExpandAll[%]

$(\text{EulerGamma} - \text{Log}[\frac{1}{k}]) + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} + \frac{1}{240k^8} - \frac{1}{132k^{10}} + O[\frac{1}{k}]^{11}$

3 Comparison to Other Summation Methods

We can obtain the same answers for some of above sums using the *Regularization* option for *Sum* as follows. For example, calculate the sum (1).

Sum[(−1)^(n − 1), {n, 1, Infinity}, Regularization → "Abel"]

$\frac{1}{2}$

Sum[(−1)^(n − 1), {n, 1, Infinity}, Regularization → "Borel"]

$\frac{1}{2}$

Sum[(−1)^(n − 1), {n, 1, Infinity}, Regularization → "Cesaro"]

$\frac{1}{2}$

Sum[(−1)^(n − 1), {n, 1, Infinity}, Regularization → "Dirichlet"]

$\frac{1}{2}$

Sum[(−1)^(n − 1), {n, 1, Infinity}, Regularization → "Euler"]

$\frac{1}{2}$

The results of our method and other five are summarized in following table:

Divergent series	Sumation method					
	Abel	*Borel*	*Cesaro*	*Dirichlet*	*Euler*	*Bubonja*
$\sum_{n=1}^{\infty}(-1)^{n-1}$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty} 2^{n-1}$	✗	✓	✗	✗	✗	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}2^{n-1}$	✗	✓	✗	✗	✓	✓
$\sum_{n=1}^{\infty} a^{n-1}$	✗	✓	✗	✗	✗	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}n$	✓	✓	✗	✓	✓	✓
$\sum_{n=1}^{\infty} n$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty} 1$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty}(n+1)$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty}(n-1)$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty} \ln(n)$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}\ln(n)$	✗	✗	✗	✓	✗	✓
$\sum_{n=1}^{\infty} \cos(n\theta)$	✗	✓	✗	✓	✗	✓
$\sum_{n=1}^{\infty} \sin(n\theta)$	✗	✓	✗	✓	✗	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}\cos(n\theta)$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}\sin(n\theta)$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}n^{2k}$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}n^{2k+1}$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty} n^{k}$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty} n^{-s}$	✓	✓	✓	✓	✓	✓
$\sum_{n=1}^{\infty}(1-((n-1) \mod 3))$	✓	✓	✓	✓	✗	✓
$\sum_{n=1}^{\infty}(((n+1) \mod 3)-1)$	✓	✓	✓	✓	✗	✓
$\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!$	✗	✓	✗	✗	✗	✓
$\sum_{n=1}^{\infty}(n-1)!$	✗	✓	✗	✗	✗	✓
$\sum_{n=1}^{\infty} \frac{1}{n}$	✗	✗	✗	✗	✗	✓

It is easily seen that our method is strongest method around for summing divergent series (see for instance description of the method in Section 1 and results in above table).

References

[1] Sinisa Bubonja. *General Method for Summing Divergent Series. Determination of Limits of Divergent Sequences and Functions in Singular Points.* Preprint, viXra:1502.0074.

[2] G. H. Hardy, *Divergent series*, Oxford at the Clarendon Press (1949)

[3] Bruce C. Brendt, *Ramanujan's Notebooks*, Springer-Verlag New York Inc. (1985)

[4] John Tucciarone, *The Development of the Theory of Summable Divergent Series from 1880 to 1925*, Archive for History of Exact Sciences, Vol. 10, No. 1/2, (28.VI.1973), 1-40 1

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