COUNTING 2-WAY MONOTONIC TERRACE FORMS OVER RECTANGULAR LANDSCAPES

RICHARD J. MATHAR

Abstract. A terrace form assigns an integer altitude to each point of a finite two-dimensional square grid such that the maximum altitude difference between a point and its four neighbors is one. It is 2-way monotonic if the sign of this altitude difference is zero or one for steps to the East or steps to the South. We provide tables for the number of 2-way monotonic terrace forms as a function of grid size and maximum altitude difference, and point at the equivalence to the number of 3-colorings of the grid.

1. A Model of Altitude Maps

A mathematical model of altitudes in a landscape is obtained if the terrain is sliced orthogonally in m length units West-to-East (W-E) and n length units North-to-South (N-S), and some (average) altitude is assigned to each of the unit squares. The topology is represented by $n \times m$ matrices of altitudes. One step into one of the four directions of the compass rose (N, E, S, W) means increasing or decreasing the column or row index of the matrix by one. By further refinement of the units of altitudes we shall assume

(1) that the altitudes can be measured in integer units, so a map is an integer matrix $A_{s,e}$ with S-E coordinate pairs $1 \le s \le n$ and $1 \le e \le m$.
(2) that the altitudes are normalized such that the altitude at the NW corner of the map is zero—at sea level—, $A_{1,1} = 0$,
(3) and that the landscape is rising monotonically with unit altitude steps if one walks either one step E or S, such that all level differences $A_{s+1,e} - A_{s,e}$ and $A_{s,e+1} - A_{s,e}$ of the matrix are either 0 or 1.

Definition 1. *The positive integer number $T_{n\times m}$ is the number of 2-way monotonic altitude maps over a terrain of $n\times m$ squares with steps of height 0 or 1 as described above.*

The unit squares that share a common altitude form terraces of the landscape. The horizontal or vertical edges of the terraces are located where the altitude changes by 1. Consider for example the 3×4 map with maximum altitude $h = 4$ represented by the 3×4 matrix

$$\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{matrix} . \quad (1)$$

Date: December 5, 2015.
2010 *Mathematics Subject Classification.* Primary 52C30, 05C15; Secondary 05B45, 52C20.

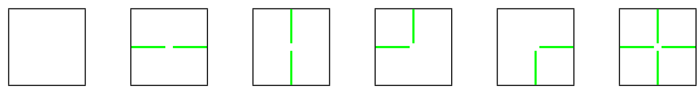

FIGURE 1. The 6 symbols on the cards that cover the $(n-1)\times(m-1)$ two-way terrace forms.

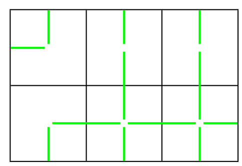

FIGURE 2. The representation of (3) by 2×3 cards taken from Figure 1.

Insert horizontal or vertical edges where the altitude differs between an entry and any of its four neighbors to the N, E, S or W:

$$\begin{array}{ccccccc} 0 & | & 1 & | & 2 & | & 3 \\ - & & & & & & \\ 1 & & 1 & | & 2 & | & 3 \\ & & - & & - & & - \\ 1 & | & 2 & | & 3 & | & 4 \end{array} . \tag{2}$$

Erase the integer entries of the matrix—which are redundant information then—and insert crosses where four edges meet and hooks where they switch direction:

$$\begin{array}{ccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & | & & | & & | & & \cdot \\ \cdot & - & \lrcorner & & | & & | & & \cdot \\ \cdot & & & & | & & | & & \cdot \\ \cdot & & \ulcorner & - & + & - & + & - & \cdot \\ \cdot & & | & & | & & | & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{3}$$

The hooks $\lrcorner$ and $\ulcorner$ may appear, but not the hooks $\urcorner$ or $\llcorner$, because the latter forms are not supported by the landscapes that rise monotonically to the East and to the South. The constraint also enforces that (i) edges do not terminate inside the rectangle but only at the four sides, and that (ii) there are no points where three edges meet.

A further transcription is obtained by covering the terrain with $(n-1)\times(m-1)$ square cards centered at the 4-way intersections which show faces with the half-edges (fins) intruding from the N or E. There is a set of 6 different cards; one card without half-edge, four different cards with two half-edges, and one card with four half-edges (Figure 1).

This transforms the representation (3) for example into Figure 2.

Remark 1. *There are further interpretations of these objects:*

(1) *The edges may represent light beams that enter an optical switch table at up to $m-1$ equidistant ports from the North and up to $n-1$ equidistant ports from the East. The optical table offers $(n-1)(m-1)$ places optionally equipped with flat mirrors with surfaces pointing NW and SE. The mirrors deflect beams running S to W, and deflect beams running W to S. At places marked with crosses beams are reflected by both surfaces of a mirror. The beams leave either at the S or at the W rim of the table. $T_{n\times m}$ counts a number of switch tables of that type.*

(2) *Representing the presence or absence of a light beam at the $n+m-2$ ports of entrance at the optical table with a 1 or a 0, and the presence or absence at the $n+m-2$ ports of exit again with a 1 or 0, $T_{n\times m}$ counts a family of gate-array logic which preserves the number of bits set on entry.*

(3) *The terraces form polyominos that cover the $n \times m$ rectangle. $T_{n\times m}$ counts covers by polyominos of any size which do not need the two absent hooks mentioned above to construct two forms "internal" bending. The example (3) covers the rectangle with a total surface of $3 \times 4 = 12$ units by four 1×1 mono-tiles (one at the NW corner, three at the South rim), one 4-omino with a double-L shape, and two dominos.*

2. Terrace Refinement According to Hill Height

With these constraints the maximum altitude in the map is in the SE corner, which is the value of $A_{n,m}$, the hill height h of the geography.

Definition 2. *The number $T_{n\times m}(h)$, $n, m \ge 1$, is the number of 2-way monotonic altitude maps with steps of height 0 or 1 as described above with maximum altitude $h = A_{n,m}$.*

Example 1. *If the maximum altitude at the SE corner is zero, the landscape is entirely flat, and there is only a single configuration (without edges) that supports this:*

$$T_{n\times m}(0) = 1. \tag{4}$$

The maps can be counted by summing over all possible heights at the corners opposite to the minimum altitude:

$$T_{n\times m} = \sum_{h=0}^{n+m-2} T_{n\times m}(h). \tag{5}$$

Example 2. *A simple case counts the staircase shapes along uni-directional stairs, $n = 1$. There are $m-1$ steps which may individually be up-steps that change the altitude by $+1$ or flat stapes that do not change the altitude. The binomial numbers count the number of ways of distributing the up-steps over the $m-1$ steps:*

$$T_{1\times m}(h) = \binom{m-1}{h}; \quad T_{1\times m} = \sum_{h=0}^{m-1} \binom{m-1}{h} = 2^{m-1}. \tag{6}$$

Example 3. *The maximum hill height is $h = m+n-2$ because walking from the NW to the SE corner implies $m-1$ steps to the East and $n-1$ steps to the South. To gain that maximum height, all steps must be up-steps, which is one single configuration:*

$$T_{n\times m}(m+n-2) = 1. \tag{7}$$

3. Terrace Symmetries

3.1. **Swapping Rows and Columns.** A 2-way monotonic altitude map stays 2-way monotonic if the associated matrices A are mirrored along the diagonal, because this swaps the meaning of steps to the S and steps to the E. Therefore $T_{n\times m}$ and $T_{n\times m}(h)$ are symmetric with respect to exchange of width and height of the rectangle:

$$T_{n\times m} = T_{m\times n}, \tag{8}$$

$$T_{n\times m}(h) = T_{m\times n}(h). \tag{9}$$

3.2. **Conjugation.** Each 2-way monotonic altitude map has a unique *conjugate* 2-way monotonic map which is constructed by replacing each up-step to the East or to the South (entering another terrace) by a leveled/flat step (staying on the terrace) and replacing each leveled step to the East or South by an up-step.

Conjugation replaces each of the cards of Figure 1 by a conjugate card; the conjugate card has half-edges where the card has not, and vice versa. (There are three pairs of conjugate cards in Figure 1.)

If a walk form the NW to the SE corner in the original map is represented by the binary number of the step heights, the walk through the conjugate map is represented by the binary complement, therefore

$$T_{n\times m}(h) = T_{n\times m}(n+m-2-h). \tag{10}$$

3.3. **Inversion.** A rotation of the terrace form by 180 degrees is again a terrace form; the N rim changes place with the S rim and the E rim changes place with the W rim. A rotation by 180 degrees acts like a permutation on the set of cards of Figure 1 and flips their row and column order in representations like (3).

In the matrix representation this is equivalent to a transformation which (i) converts all up-steps to down-steps and keeps the flat steps, so the former hill becomes a valley and all matrix values switch sign, (ii) swaps the matrix elements by the rotation, so steps to the E or to the S are up-steps or flat steps again, and (iii) adds h to each matrix element: $A_{s,e} \to h - A_{m-s+1,n-e+1}$.

4. Terrace Recurrence: Attaching one Column

4.1. **Sub-classification: NE value and E rim steps.** Another classification of the terrace forms is by considering the altitude $l \equiv A_{1,m}$ at the NE corner of the rectangle, $0 \le l \le m-1$, and step distribution of the staircase of walking from there along the E rim to the SE corner. These combinations are characterized by the number l and the binary vectors of length $n-1$ of zeros and ones of the altitude differences walking along the eastern rim of the rectangle: Then the values in the last column, m, of the matrix are

$$A_{s,m} = l + \sum_{i=0}^{s-2} d_i, \quad 0 \le l < m, \quad d_i \in \{0,1\}, \tag{11}$$

by accumulating the partial sum of the digits of a number

$$b = \sum_{i\ge 0} d_i 2^i \tag{12}$$

which signifies in its binary representations the locations of up-steps in column m, $0 \le b < 2^{n-1}$.

Definition 3.

$$H(b) \equiv \sum_{i \ge 0} d_i \tag{13}$$

is the non-negative sum of the binary digits d_i *of* $b = \sum_i d_i 2^i$, $0 \le d_i \le 1$.

Then the value at the SE corner of the matrix is

$$h = l + H(b). \tag{14}$$

Definition 4. $T_{n\times m}(l,b)$ *is the number of 2-way monotonic terrace forms with an altitude* l *in the NE corner and a step distribution* b *along the E edge (column* m*).*

The $T_{n\times m}(l,b)$ contribute to $T_{n\times m}(h)$ if the number of up-steps matches (14):

$$T_{n\times m}(h) = \sum_{0\le l<m} \sum_{0\le b\le 2^{n-1}} T_{n\times m}(l,b)\delta_{H(b),h-l}. \tag{15}$$

4.2. **Compatibility of step distributions.** Augmenting the size of the matrix by one column, counting $T_{n\times(1+m)}$, is recursively done by constructing $T_{n\times m}$ and counting the different ways of either copying the l at the NE corner to become the NE corner of the augmented matrix, $l' = l$, or increasing it by one, $l' = l+1$. In both cases we will write $l' = l + a$ with $a \in \{0,1\}$ to simplify the notation. In each of the two cases there are distributions b' of the steps in column $m' = 1+m$ which are compatible with the E rim, b, of the $n \times m$ matrix, in the sense that $A(s,m') = l' + \sum_{i=0}^{s-2} d'_i$ is either $A(s,m)$ or $A(s,m)+1$ for all $1 \le s \le n$ as required by the monotonicity:

$$A(s,m) \le A(s,m') \le 1 + A(s,m); \tag{16}$$

$$\Leftrightarrow l + \sum_{i=0}^{s-2} d_i \le l + a + \sum_{i=0}^{s-2} d'_i \le 1 + l + \sum_{i=1}^{s-2} d_i. \tag{17}$$

$$\Leftrightarrow \sum_{i=0}^{s-2} d_i \le a + \sum_{i=0}^{s-2} d'_i \le 1 + \sum_{i=1}^{s-2} d_i. \tag{18}$$

We construct two $2^{n-1} \times 2^{n-1}$ compatibility matrices $C^{(a)}_{b,b'}$ which have a value of 1 if the vectors of binary digits d_i and d'_i satisfy this pair of inequalities, and a value of 0 if they do not.

Remark 2. $C^{(0)}_{b,b'}$ *and* $C^{(1)}_{b,b'}$ *are actually transposed of each other because the pair of inequalities demands that the partial sum of* d'_i *runs at most 1 ahead of the partial sum of* d_i *if* $a = 0$ *and lags at most 1 behind* d_i *if* $a = 1$.

Example 4. *Keeping the step distribution for the new column results in a valid step distribution independent which of the two* a *is applied:*

$$C^{(0)}_{b,b} = C^{(1)}_{b,b} = 1. \tag{19}$$

Example 5. *If the* $n \times m$ *E edge had no rises the new edge may have at most one rise. If* $a = 1$ *the* b' *must also be flat:*

$$C^{(1)}_{0,b'} = \delta_{b',0}, \tag{20}$$

and if $a = 0$ *the* b' *may have at most a single rise:*

$$C^{(0)}_{0,b'} = \begin{cases} 1, & b' = 0; \\ 1, & b' > 0 \wedge H(b') = 1; \\ 0, & b' > 0 \wedge H(b') > 1. \end{cases} \tag{21}$$

$T_{n\times m}(l,b)$ is a matrix with rows enumerated by $0 \le l < m$ and columns enumerated by $0 \le b < 2^{n-1}$. The two matrix multiplications

$$T^{(0)}_{n\times(m+1)}(l,b') \equiv \sum_{b=0}^{2^{n-1}} T_{n\times m}(l,b) C^{(0)}_{b,b'}, \tag{22}$$

$$T^{(1)}_{n\times(m+1)}(l+1,b') \equiv \sum_{b}^{2^{n-1}} T_{n\times m}(l,b) C^{(1)}_{b,b'}, \tag{23}$$

count the 2-way monotonic terrace forms derived from the two possible values of a. Shifting and adding the two matrices represents the result with $1+m$ columns:

$$T_{n\times(m+1)}(l,b') = T^{(0)}_{n\times(m+1)}(l,b') + T^{(1)}_{n\times(m+1)}(l,b'). \tag{24}$$

The recurrence is started at $m = 1$ as described in Example 2 with a $1 \times 2^{n-1}$ matrix:

$$T_{n\times 1}(l,b) = 1; \quad l = 0; \quad 0 \le b < 2^{n-1}. \tag{25}$$

This describes the numerical implementation via the program in Section A.

4.3. **Transfer Matrix Method.** One can imagine a different implementation where the representation of the E (rightmost) column of the terrace form is not the step distribution, b, but a vector of $n-1$ symbols taken from Figure 1. This is not favorable numerically because the compatibility table would grow in both dimensions as 6^{n-1} and not 2^{n-1}, as there are 6 candidates at each place. (It also has the disadvantage that one must keep track of stacking only symbols downwards the new column where the lines that enter at the top and leave at the bottom stay connected.)

The formal advantage of that method is that it clearly becomes a Transfer Matrix Method because each vector of the $n-1$ symbols at the E rim of the $n \times m$ matrix has a (finite) number of compatible vectors of the $n-1$ symbols of the next column. Following standard arguments of an associated state diagram method [4] this proves that the generating function

$$G_n(z) \equiv \sum_{m\ge 1} T_{n\times m} z^m \tag{26}$$

is a rational polynomial function of z. The bivariate generating function is symmetric as a consequence of (8):

$$G(z,t) \equiv \sum_{n\ge 1}\sum_{m\ge 1} T_{n\times m} t^n z^m = G(t,z). \tag{27}$$

5. Terrace Form Results

5.1. **Tabulation.** The results are represented by a table which has two types of lines.

- The first line type displays 4 integers separated by blanks which show n, m, h and $T_{n\times m}(h)$. The values for $2h > n+m-2$ are redundant according to (10) and not listed.

- The second line type displays 3 integers separated by blanks which show n, m, and $T_{n \times m}$. The second type is a kind of check sum according to (5) and redundant.

```
1 1 0 1
1 1 1

2 1 0 1
2 1 2

2 2 0 1
2 2 1 4
2 2 6

3 1 0 1
3 1 1 2
3 1 4

3 2 0 1
3 2 1 8
3 2 18

3 3 0 1
3 3 1 18
3 3 2 44
3 3 82

4 1 0 1
4 1 1 3
4 1 8

4 2 0 1
4 2 1 13
4 2 2 26
4 2 54

4 3 0 1
4 3 1 33
4 3 2 153
4 3 374

4 4 0 1
4 4 1 68
4 4 2 615
4 4 3 1236
4 4 2604

5 1 0 1
5 1 1 4
5 1 2 6
5 1 16

5 2 0 1
5 2 1 19
5 2 2 61
5 2 162

5 3 0 1
5 3 1 54
5 3 2 413
5 3 3 770
5 3 1706

5 4 0 1
5 4 1 124
5 4 2 1953
5 4 3 6997
5 4 18150

5 5 0 1
5 5 1 250
5 5 2 7313
5 5 3 46812
5 5 4 84910
5 5 193662

6 1 0 1
6 1 1 5
6 1 2 10
6 1 32

6 2 0 1
6 2 1 26
6 2 2 120
6 2 3 192
6 2 486

6 3 0 1
6 3 1 82
6 3 2 949
6 3 3 2859
6 3 7782

6 4 0 1
6 4 1 208
6 4 2 5281
6 4 3 30802
6 4 4 53950
6 4 126534

6 5 0 1
6 5 1 460
6 5 2 23203
6 5 3 248182
6 5 4 762227
6 5 2068146

6 6 0 1
6 6 1 922
6 6 2 85801
```

6 6 3 1592348
6 6 4 8241540
6 6 5 14024408
6 6 33865632

7 1 0 1
7 1 1 6
7 1 2 15
7 1 3 20
7 1 64

7 2 0 1
7 2 1 34
7 2 2 211
7 2 3 483
7 2 1458

7 3 0 1
7 3 1 118
7 3 2 1948
7 3 3 8694
7 3 4 13976
7 3 35498

7 4 0 1
7 4 1 328
7 4 2 12686
7 4 3 112877
7 4 4 315198
7 4 882180

7 5 0 1
7 5 1 790
7 5 2 64920
7 5 3 1100210
7 5 4 5385305
7 5 5 8989062
7 5 22091514

7 6 0 1
7 6 1 1714
7 6 2 277585
7 6 3 8528422
7 6 4 71297441
7 6 5 197352882
7 6 554916090

7 7 0 1
7 7 1 3430
7 7 2 1030330
7 7 3 54926890
7 7 4 759337545
7 7 5 3397542544
7 7 6 5530983756
7 7 13956665236

8 1 0 1
8 1 1 7
8 1 2 21
8 1 3 35
8 1 128

8 2 0 1
8 2 1 43
8 2 2 343
8 2 3 1050
8 2 4 1500
8 2 4374

8 3 0 1
8 3 1 163
8 3 2 3676
8 3 3 22924
8 3 4 54199
8 3 161926

8 4 0 1
8 4 1 493
8 4 2 27805
8 4 3 359550
8 4 4 1499394
8 4 5 2376024
8 4 6150510

8 5 0 1
8 5 1 1285
8 5 2 164399
8 5 3 4230324
8 5 4 31454256
8 5 5 82146778
8 5 235994086

8 6 0 1
8 6 1 3001
8 6 2 806347
8 6 3 39423196
8 6 4 512868867
8 6 5 2213802873
8 6 6 3561146170
8 6 9094954740

8 7 0 1
8 7 1 6433
8 7 2 3407823
8 7 3 303382053
8 7 4 6725497344

8 7 5 47269152002
8 7 6 121303742075
8 7 351210375462

8 8 0 1
8 8 1 12868
8 8 2 12742873
8 8 3 1988261908
8 8 4 73117894428
8 8 5 819944490812
8 8 6 3296290368486
8 8 7 5192169001644
8 8 13574876544396

9 1 0 1
9 1 1 8
9 1 2 28
9 1 3 56
9 1 4 70
9 1 256

9 2 0 1
9 2 1 53
9 2 2 526
9 2 3 2058
9 2 4 3923
9 2 13122

9 3 0 1
9 3 1 218
9 3 2 6497
9 3 3 54272
9 3 4 177848
9 3 5 260962
9 3 738634

9 4 0 1
9 4 1 713
9 4 2 56624
9 4 3 1024773
9 4 4 6083808
9 4 5 14274628
9 4 42881094

9 5 0 1
9 5 1 2000
9 5 2 383735
9 5 3 14477724
9 5 4 157376166
9 5 5 612222006
9 5 6 952152558
9 5 2521075822

9 6 0 1
9 6 1 5003
9 6 2 2142634
9 6 3 161160206
9 6 4 3160111147
9 6 5 20525389173
9 6 6 50689903445
9 6 149077423218

9 7 0 1
9 7 1 11438
9 7 2 10237249
9 7 3 1471499970
9 7 4 50869309436
9 7 5 546793506964
9 7 6 2144980290812
9 7 7 3351708981962
9 7 8839958693702

9 8 0 1
9 8 1 24308
9 8 2 42993671
9 8 3 11360377192
9 8 4 675539536773
9 8 5 11837958868428
9 8 6 73048480647796
9 8 7 176885984094690
9 8 524918733085718

9 9 0 1
9 9 1 48618
9 9 2 161937617
9 9 3 75922639116
9 9 4 7578889491370
9 9 5 212879678428784
9 9 6 2038576101280635
9 9 7 7545165464129370
9 9 8 11583105980431652
9 9 31191658416342674

10 1 0 1
10 1 1 9
10 1 2 36
10 1 3 84
10 1 4 126
10 1 512

10 2 0 1
10 2 1 64
10 2 2 771
10 2 3 3732
10 2 4 9069
10 2 5 12092

10 2 39366

10 3 0 1
10 3 1 284
10 3 2 10894
10 3 3 118057
10 3 4 513905
10 3 5 1041518
10 3 3369318

10 4 0 1
10 4 1 999
10 4 2 108549
10 4 3 2667554
10 4 4 21733215
10 4 5 71899369
10 4 6 106145902
10 4 298965276

10 5 0 1
10 5 1 3001
10 5 2 836797
10 5 3 44951694
10 5 4 692347393
10 5 5 3861354450
10 5 6 8866654233
10 5 26932295138

10 6 0 1
10 6 1 8006
10 6 2 5281314
10 6 3 593478797
10 6 4 17063990547
10 6 5 161858075052
10 6 6 591078807754
10 6 7 902439668964
10 6 2443638951906

10 7 0 1
10 7 1 19446
10 7 2 28340232
10 7 3 6383377435
10 7 4 335549742230
10 7 5 5385392009638
10 7 6 31432924990292
10 7 7 74101002378750
10 7 222522561716048

10 8 0 1
10 8 1 43756
10 8 2 132872804
10 8 3 57644900961
10 8 4 5411459549576
10 8 5 145256401778490
10 8 6 1349062860032829
10 8 7 4906423896412527
10 8 8 7489452153650928
10 8 20301876944832816

10 9 0 1
10 9 1 92376
10 9 2 555632319
10 9 3 447545856560
10 9 4 73254444131056
10 9 5 3241512720057621
10 9 6 47479351481850465
10 9 7 264568744107280331
10 9 8 611700400839334208
10 9 1854127423388469874

10 10 0 1
10 10 1 184754
10 10 2 2105918045
10 10 3 3044977814280
10 10 4 848729993032718
10 10 5 60950615354247160
10 10 6 1392709034950421612
10 10 7 11778687670391686084
10 10 8 40803422317854501308
10 10 9 61353264333274642456
10 10 169426507164530254380

11 1 0 1
11 1 1 10
11 1 2 45
11 1 3 120
11 1 4 210
11 1 5 252
11 1 1024

11 2 0 1
11 2 1 76
11 2 2 1090
11 2 3 6369
11 2 4 19095
11 2 5 32418
11 2 118098

11 3 0 1
11 3 1 362
11 3 2 17492
11 3 3 239798
11 3 4 1342933
11 3 5 3600608
11 3 6 4966934
11 3 15369322

11 4 0 1
11 4 1 1363
11 4 2 197804
11 4 3 6438457
11 4 4 69921908
11 4 5 314303269
11 4 6 651324265
11 4 2084374134

11 5 0 1
11 5 1 4366
11 5 2 1722394
11 5 3 128484004
11 5 4 2731553014
11 5 5 21180858856
11 5 6 69023191877
11 5 7 101582773162
11 5 287714402186

11 6 0 1
11 6 1 12374
11 6 2 12207123
11 6 3 1997447386
11 6 4 82211677028
11 6 5 1110247570840
11 6 6 5815403108223
11 6 7 13018111367891
11 6 40055966781732

11 7 0 1
11 7 1 31822
11 7 2 73128376
11 7 3 25130419118
11 7 4 1963313662947
11 7 5 46046994751248
11 7 6 390637403100019
11 7 7 1349331496630738
11 7 8 2025629900536780
11 7 5601638723985318

11 8 0 1
11 8 1 75580
11 8 2 380660536
11 8 3 263660741716
11 8 4 38210014068388
11 8 5 1541741349322761
11 8 6 21162752529241540
11 8 7 113212128532755874
11 8 8 256682233387533018
11 8 785274659708798828

11 9 0 1
11 9 1 167958
11 9 2 1758369389
11 9 3 2362795756242
11 9 4 620079831016189
11 9 5 42492215048872994
11 9 6 938336407734666461
11 9 7 7756611567495732130
11 9 8 26522940371769680636
11 9 9 39713000353389852958
11 9 110235006366258376958

11 10 0 1
11 10 1 352714
11 10 2 7327840683
11 10 3 18442485989875
11 10 4 8556080308907065
11 10 5 981228046548465859
11 10 6 34566002291860407546
11 10 7 439143900680902821968
11 10 8 2253047629995480936195
11 10 9 5015491064974601978781
11 10 15486476801039035401374

11 11 0 1
11 11 1 705430
11 11 2 27918122131
11 11 3 127417950149256
11 11 4 102072779621746876
11 11 5 19285216576110954200
11 11 6 1072962724017364414316
11 11 7 20802252828089741539080
11 11 8 159049492340615743488999
11 11 9 521767856595822572685606
11 11 10 771168645274178148758934
11 11 2176592549084872196370724

12 1 0 1
12 1 1 11
12 1 2 55
12 1 3 165
12 1 4 330
12 1 5 462
12 1 2048

12 2 0 1
12 2 1 89
12 2 2 1496
12 2 3 10351
12 2 4 37356
12 2 5 78133
12 2 6 99442
12 2 354294

12 3 0 1
12 3 1 453
12 3 2 27083
12 3 3 460207
12 3 4 3233784
12 3 5 11105458
12 3 6 20227001
12 3 70107974

12 4 0 1
12 4 1 1818
12 4 2 345216
12 4 3 14575914
12 4 4 206052448
12 4 5 1222399174
12 4 6 3423764153
12 4 7 4797897182
12 4 14532174630

12 5 0 1
12 5 1 6186
12 5 2 3373453
12 5 3 341970798
12 5 4 9814551889
12 5 5 103167287216
12 5 6 462448739630
12 5 7 961033692134
12 5 3073619242614

12 6 0 1
12 6 1 18562
12 6 2 26687363
12 6 3 6215980425
12 6 4 358529577294
12 6 5 6743998502448
12 6 6 49402822227475
12 6 7 157036552350535
12 6 8 229501198974954
12 6 656597489663160

12 7 0 1
12 7 1 50386
12 7 2 177500233
12 7 3 90859574359
12 7 4 10332594393296
12 7 5 347325247630750
12 7 6 4196041457531560
12 7 7 20724787692223309
12 7 8 45228611126152887
12 7 141014378310113562

12 8 0 1
12 8 1 125968
12 8 2 1020670112
12 8 3 1100443002833
12 8 4 241118003113096
12 8 5 14367403538888275
12 8 6 286631073865862082
12 8 7 2214220454224794502
12 8 8 7285517200775032301
12 8 9 10773747821800089440
12 8 30375704525543067780

12 9 0 1
12 9 1 293928
12 9 2 5184184484
12 9 3 11314496618377
12 9 4 4660896337017720
12 9 5 486433470221674757
12 9 6 15967739097633126162
12 9 7 193103869242758397524
12 9 8 960624596637727367584
12 9 9 2107063862931741530314
12 9 6554502347192200421702

12 10 0 1
12 10 1 646644
12 10 2 23655220898
12 10 3 100745111116155
12 10 4 76114475049745598
12 10 5 13715397099680564762
12 10 6 735815937034670056882
12 10 7 13895340239793861399550
12 10 8 104394989654669445949065
12 10 9 339077481598614429529693
12 10 10 499540764414851223700240
12 10 1415775602499763032158736

12 11 0 1
12 11 1 1352076
12 11 2 98294699619
12 11 3 789803841264154
12 11 4 1068072868567758231
12 11 5 327143752965630756509
12 11 6 28431142089139448593070
12 11 7 834423888014977641237967
12 11 8 9436725440027381542948380
12 11 9 45207025228418594727144321
12 11 10 97489555036763562527894022
12 11 305992977895858784447296700

12 12 0 1
12 12 1 2704154
12 12 2 376111502977
12 12 3 5527784646050032
12 12 4 13069744040275286234

12 12 5 6694602919255051129290
12 12 6 932838729305746546704344
12 12 7 42282866765417676439220168
12 12 8 716531088773724361992845891
12 12 9 5037333300372101287507691250
12 12 10 15785287021876749977675523035
12 12 11 22993716372161876140519972488
12 12 66158464020552857153017287240

13 1 0 1
13 1 1 12
13 1 2 66
13 1 3 220
13 1 4 495
13 1 5 792
13 1 6 924
13 1 4096

13 2 0 1
13 2 1 103
13 2 2 2003
13 2 3 16159
13 2 4 68860
13 2 5 173096
13 2 6 271219
13 2 1062882

13 3 0 1
13 3 1 558
13 3 2 40653
13 3 3 841930
13 3 4 7275049
13 3 5 31205972
13 3 6 72626791
13 3 7 95819318
13 3 319801226

13 4 0 1
13 4 1 2378
13 4 2 580452
13 4 3 31230978
13 4 4 563440284
13 4 5 4308844750
13 4 6 15837386451
13 4 7 29917390364
13 4 101317751316

13 5 0 1
13 5 1 8566
13 5 2 6327844
13 5 3 855386126
13 5 4 32510650689
13 5 5 453513842996
13 5 6 2726211879605
13 5 7 7746134324326
13 5 8 10916654365356
13 5 32835119205662

13 6 0 1
13 6 1 27130
13 6 2 55571337
13 6 3 18056320047
13 6 4 1432148475494
13 6 5 36807838023747
13 6 6 369101096848883
13 6 7 1623179464940823
13 6 8 3350943226373403
13 6 10762963773161730

13 7 0 1
13 7 1 77518
13 7 2 408299906
13 7 3 304675148476
13 7 4 49485574097413
13 7 5 2342320800448438
13 7 6 39575079234866364
13 7 7 273765711319068764
13 7 8 846124592931150914
13 7 9 1226173859370396872
13 7 3549888849256712460

13 8 0 1
13 8 1 203488
13 8 2 2581677939
13 8 3 4234173659694
13 8 4 1375790096860222
13 8 5 119061685394284377
13 8 6 3399037213104854233
13 8 7 37299340789656410097
13 8 8 174623785454963865845
13 8 9 372060608711876717467
13 8 1175006427763697066726

13 9 0 1
13 9 1 497418
13 9 2 14357874868
13 9 3 49668339233596
13 9 4 31481767510001471
13 9 5 4923648634193312638
13 9 6 236993849251858941231
13 9 7 4138488747561268565230
13 9 8 29514735244133433892690
13 9 9 93071701579208862546726
13 9 10 135811172414979343192072
13 9 389744921615458992923810

13 10 0 1
13 10 1 1144064
13 10 2 71455794722
13 10 3 501861936920134
13 10 4 604818065845548015
13 10 5 168521704094349195122
13 10 6 13598847946872011042827
13 10 7 377712682023196269052955
13 10 8 4110990219216556786791985
13 10 9 19227464153067231166980308
13 10 10 40990987151608907765963189
13 10 129441844361773715176866644

13 11 0 1
13 11 1 2496142
13 11 2 322690754075
13 11 3 4442597051110232
13 11 4 9925817269445621527
13 11 5 4849663091633098446604
13 11 6 650732604369992235989927
13 11 7 28660167142100430349766304
13 11 8 475733287603627407036839887
13 11 9 3298775535824693455206015368
13 11 10 10256719485642363168472735731
13 11 11 14902728229163978240219007562
13 11 43023816365984990791398559158

13 12 0 1
13 12 1 5200298
13 12 2 1337479334867
13 12 3 34948346397969290
13 12 4 141265130251479290004
13 12 5 119028098551971023058282
13 12 6 26294832239612973514600017
13 12 7 1827148061509699281961762204
13 12 8 46143619730750209695479975288
13 12 9 473457276235766474729015234120
13 12 10 2140173210362933056634269505843
13 12 11 4491954671722504294001852100017
13 12 14307164680230203958444956060462

13 13 0 1
13 13 1 10400598
13 13 2 5136280530303
13 13 3 247288999212565632
13 13 4 1766655687081261435255
13 13 5 2523848597039597597099530
13 13 6 907772027432543083350018589
13 13 7 98869426283121852802260410036
13 13 8 3784646341402418915357243459244
13 13 9 57295805290634579622708250632956
13 13 10 3751118411159962995980492536301[illegible]
13 13 11 1130287455674174264660527058645[illegible]
13 13 12 1625987621134180524757435478838056
13 13 4759146677426447759184119036493676

14 1 0 1
14 1 1 13
14 1 2 78
14 1 3 286
14 1 4 715
14 1 5 1287
14 1 6 1716
14 1 8192

14 2 0 1
14 2 1 118
14 2 2 2626
14 2 3 24388
14 2 4 120835
14 2 5 358072
14 2 6 673699
14 2 7 829168
14 2 3188646

14 3 0 1
14 3 1 678
14 3 2 59411
14 3 3 1478451
14 3 4 15451358
14 3 5 81130329
14 3 6 235217241
14 3 7 396057622
14 3 1458790182

14 4 0 1
14 4 1 3058
14 4 2 944778
14 4 3 63789948
14 4 4 1444285973
14 4 5 13963135688
14 4 6 65772003524
14 4 7 162521024666
14 4 8 218853011926
14 4 706383387198

14 5 0 1
14 5 1 11626
14 5 2 11427851
14 5 3 2025957453
14 5 4 100275435009
14 5 5 1822789071353
14 5 6 14387152855881
14 5 7 54462323883180
14 5 8 104612321338519
14 5 350773799961746

14 6 0 1
14 6 1 38758
14 6 2 110851465
14 6 3 49345966862
14 6 4 5291678799968
14 6 5 182687605821870
14 6 6 2461777263619508
14 6 7 14656103666137736
14 6 8 41592397163617082
14 6 9 58630276427146132
14 6 176426890096852632

14 7 0 1
14 7 1 116278
14 7 2 895580310
14 7 3 955390407363
14 7 4 217813226273441
14 7 5 14284722536228472
14 7 6 332098878305115721
14 7 7 3160419720615784455
14 7 8 13536731385381240415
14 7 9 27638740441225455631
14 7 89364987835152404174

14 8 0 1
14 8 1 319768
14 8 2 6200574002
14 8 3 15149489664932
14 8 4 7169675537911497
14 8 5 887116432872276851
14 8 6 35720078525420913480
14 8 7 548625253884800248060
14 8 8 3592940420432009232493
14 8 9 10794659210759867910051
14 8 10 15506887223221394326924
14 8 4545256575295379242919 4

14 9 0 1
14 9 1 817188
14 9 2 37612542564
14 9 3 201684696550100
14 9 4 193052564598623987
14 9 5 44548656263775166754
14 9 6 3102862444405706829681
14 9 7 77278993630721480330546
14 9 8 779742790603829321019545
14 9 9 3478353006606043123821563
14 9 10 7249296785798427345737170
14 9 2317563836198795532287819 8

14 10 0 1
14 10 1 1961254
14 10 2 203439245065
14 10 3 2301293336017498
14 10 4 4338439132822873045
14 10 5 1840201966073971313695
14 10 6 220592775664478172419187
14 10 7 8917625708185015852528338
14 10 8 139178533886934317348576390
14 10 9 926343236235184337197454795
14 10 10 2813974200963496802769454282
14 10 11 4057937716254648647889236966
14 10 11835209784478991957712924066

14 11 0 1
14 11 1 4457398
14 11 2 995183131129
14 11 3 2289839168625218 6
14 11 4 828193034237063361 87
14 11 5 635279816961190430522 49
14 11 6 130039498461086069441122 21
14 11 7 851629768159250425148469676
14 11 8 2058542101439274333217989199 1
14 11 9 2048852339441386027027192518 22
14 11 10 9086839524340662838631143302 93
14 11 11 1889892880572884405433971163097
14 11 6049824370588622586587400896500

14 12 0 1
14 12 1 9657698
14 12 2 4454478957997
14 12 3 201607648015744292
14 12 4 136387154935017941235 4
14 12 5 185966664795381112574748 2
14 12 6 643646119014860196711794669
14 12 7 67999124529961089644598044569
14 12 8 2543584974261757838343113406315
14 12 9 37874379873257035544447660311780
14 12 10 245242696258812327996900700173152
14 12 11 734347431458865169791797144576231
14 12 12 1054260552491195131798560365468132
14 12 3094414026884947108271747841121212

14 13 0 1
14 13 1 20058298
14 13 2 18421452918813
14 13 3 1590238459349270018
14 13 4 19639394369841159785004
14 13 5 46772970975347073258631796
14 13 6 27066300640090690577068822371
14 13 7 4585081169707941556795908009930
14 13 8 264749288756650882820648742703643
14 13 9 5892459630972278966942625805764391
14 13 10 55648424630454980592574874721027094
14 13 11 239288484041873307333738371268471547

```
14 13 12 490593281165662062018547294744927295
14 13 1583384021903964479573689967000780402

14 14 0 1
14 14 1 40116598
14 14 2 70961632168027
14 14 3 11357451257590359840
14 14 4 250232365919081938956865
14 14 5 1022857577249059021514748742
14 14 6 977702068950765256879289304431
14 14 7 263612023260531856801214095233768
14 14 8 23404197624934775760703541434286424
14 14 9 777053429918809607908553532496030304
14 14 10 10684975270153684612120404751235035986
14 14 11 65789460797825068518108569300698730840
14 14 12 191604921933626366955120969400733752168
14 14 13 272649922373702777601949547955204462396
14 14 810410082813497381147177065840601910384
```

Many of these values are not new because $T_{n\times m}(h)$ where $h = n+m-4$, $h = n+m-5$, or $h = n+m-6$, have been published by Hardin in the Online Encyclopedia of Integer Sequence (OEIS) in Sequences A252876, A252976 and A252930 [6]. Sequences where $h = 1$ are A166810 ($n = 6$), A166812 ($n = 7$) and A166813 ($n = 8$).

5.2. **3-colorings.** The $T_{n\times m}$ are also the number of 3-colorings of the $n\times m$ grid [6, A078099], where the rules are that the color of the NW point is fixed and that none of the four points which is an immediate neighbor to the N, E, S or W from a point has the same color as the point [3]. The reason for this match is that the tiling with 3 colors (enumerated 0 to 2) allows the 6 color combinations

$$\boxed{\begin{matrix}0&2\\2&1\end{matrix}},\boxed{\begin{matrix}0&2\\1&0\end{matrix}},\boxed{\begin{matrix}0&1\\2&0\end{matrix}},\boxed{\begin{matrix}0&1\\1&0\end{matrix}},\boxed{\begin{matrix}0&2\\2&0\end{matrix}},\boxed{\begin{matrix}0&1\\1&2\end{matrix}}, \tag{28}$$

(and what follows by adding 1 or 2 modulo 3) at each of the 4-way crossings. If these are mapped in that order onto the 6 cards of Figure 1, the compatibility rules of attaching a symbol at any of the four sides of another symbol are actually the same. [This bijective map is established if an up-step in Figure 1 is replaced by increasing the color number by 1 (mod 3) and if a flat step in Figure 1 is replaced by increasing the color number by 2 (mod 3).] The first symbol of Figure 1 can be stacked on the top of itself, on top of the second symbol and on top of the fifth symbol, for example; in the same manner the first symbol of (28) can be stacked on top of itself, on top of the second and on top of the fifth symbol of (28).

5.3. **Generating Functions.** Generating functions (26) are easily extracted from the tables:

$$G_{1,z} = \frac{x}{1-2x} \tag{29}$$

is equivalent to (6).

$$G_{2,z} = \frac{2x}{1-3x} \tag{30}$$

has a simple interpretation of counting single lines of cards where the factor 3 means that each card has three candidates (possibly including itself) with matching half-edge for attachment at its E rim.

$$G_{3,z} = \frac{2x(2-x)}{1-5x+2x^2}. \tag{31}$$

[6, A078100] gives

$$G_{4,z} = \frac{2x(4-9x+4x^2)}{1-9x+15x^2-6x^3}. \tag{32}$$

Apart from a factor two there are [6, A207994-A207996]

$$G_{5,z} = \frac{2x(8-47x+77x^2-44x^3+8x^4)}{1-16x+65x^2-92x^3+48x^4-8x^5}. \tag{33}$$

$$G_{6,z} = \frac{2x(16-237x+1257x^2-3198x^3+4206x^4-2736x^5+688x^6}{1-30x+291x^2-1278x^3+2901x^4-3519x^5+2152x^6-516x^7}. \tag{34}$$

$$G_{7,z} = \frac{2xp_7(z)}{q_7(z)}, \tag{35}$$

where

$$p_7 \equiv 32 - 1031x + 13142x^2 - 89204x^3 + 360470x^4 - 909704x^5 + 1454814x^6 - 1461492x^7 + 896144x^8 - 320568x^9 + 63472x^{10} - 6400x^{11} + 256x^{12}; \tag{36}$$

$$q_7 \equiv 1 - 55x + 1109x^2 - 11330x^3 + 67206x^4 - 247404x^5 + 582440x^6 - 881876x^7 + 846764x^8 - 499200x^9 + 172400x^{10} - 33152x^{11} + 3264x^{12} - 128x^{13}. \tag{37}$$

6. Cut Sets

6.1. **Set of Inner Edges.** In views like (3) the $n \times m$ rectangle has $n(m-1)$ vertical edges where pairs of unit squares meet and $m(n-1)$ horizontal edges where pairs of unit square meet, $2mn-n-m$ in total. There are $2^{2mn-n-m}$ combinations of selecting subsets of these. The display in (2) for example shows a selection which contains 8 vertical edges and 4 horizontal edges. Each of the terrace forms documented above is such a subset of the inner edges, and therefore $2^{2m-n-m} \ge T_{n,m}$ is an upper bound for the number of the terrace forms.

6.2. **Special Sets.** A simple manner of cutting the $n \times m$ rectangle into smaller pieces are so-called guillotine cuts—cuts that run vertically completely through any of the $m-1$ sets of collinear edges or completely horizontally through any of the $n-1$ sets of collinear edges. There are 2^{m-1} sets of vertical guillotine cuts and 2^{n-1} sets of horizontal guillotine cuts. Independent combination of horizontal and vertical cuts gives $2^{n-1}2^{m-1} = 2^{n+m-2}$ simple ways of dissecting the $n \times m$ rectangle into a regular set of smaller rectangles. (Note that using no cut also counts as one way.)

A larger variety of dissections into rectangles could perform c vertical guillotine cuts, $0 \le c < m$, in a first step. There are $\binom{m-1}{c}$ ways of doing this; the result is $c+1$ rectangles of variable width and always the same height n. Within each of these sub-rectangles we can perform horizontal cuts in 2^{n-1} different ways, which are independent and not necessarily aligned or correlated with the horizontal cuts in the neighboring sub-rectangles, illustrated in Figure 3. The total count of that manner of subdivision of the $n \times m$ rectangle into smaller rectangles is [6, A264872]

$$\sum_{c=0}^{m-1} \binom{m-1}{c} 2^{(n-1)(c+1)} = 2^{n-1}(1+2^{n-1})^{m-1}. \tag{38}$$

This formula is unsatisfactory because it is not a symmetric function of n and m. This is improved by noticing that one might as well start in a first step with horizontal guillotine cuts and add the vertical cuts latter. The total number of ways of these semi-regular subdivisions of the rectangle becomes

$$2^{n-1}(1+2^{n-1})^{m-1} + 2^{m-1}(1+2^{m-1})^{n-1} - 2^{m+n-2}, \tag{39}$$

where the last term on the right hand side corrects for a double counting of the aforementioned regular divisions with crossing guillotine cuts.

Remark 3. *The number of subwindows within the $n \times m$ area specified by lower left x, lower left y coordinate, width $w > 0$ and height $h > 0$ is* [6, A085780]*:* $\sum_{x=1}^{m}\sum_{y=1}^{n}\sum_{w=1}^{m-x+1}\sum_{h=1}^{n-y+1} 1 = \sum_{x=1}^{m}\sum_{y=1}^{n}\sum_{w=1}^{m-x+1}(n-y+1) = \sum_{x=1}^{m}\sum_{y=1}^{n}(m-x+1)(n-y+1) = \sum_{x=1}^{m} n(n+1)(1+m-x)/2 = nm(m+1)(n+1)/4$, *a product of two triangular numbers.*

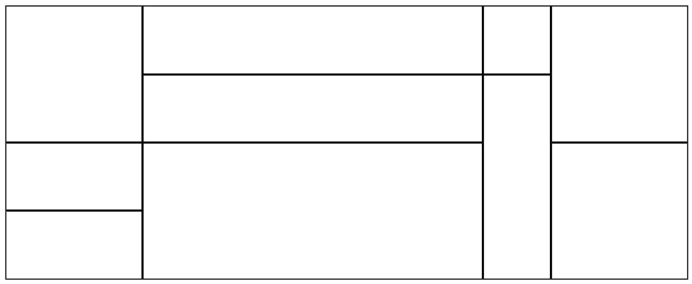

FIGURE 3. An example of hybrid guillotine cuts of the 10×4 rectangle counted by (38). Start with $c = 3$ vertical guillotine cuts which create $c + 1 = 4$ sections of width 2, 5, 1 and 2. Then add 2, 2, 1 and 1 horizontal cuts in the sections.

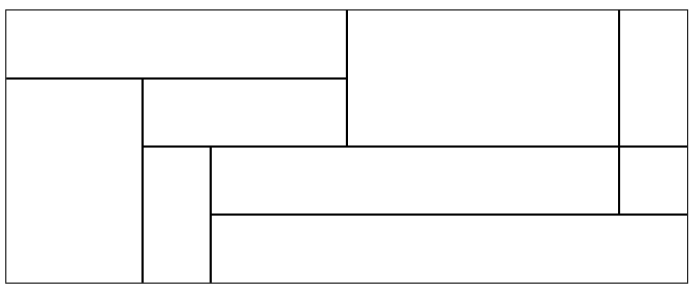

FIGURE 4. An example of a dissection of a rectangle into smaller rectangles not included in the count (39).

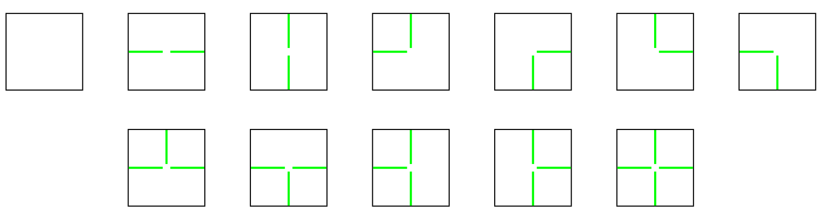

FIGURE 5. The 12 symbols on the cards that cover the $(n-1) \times (m-1)$ cut sets. In a binary hex-encoding for the presence or absence of the fins S, E, N and W these can be labeled 0, 10 (or a), 5, 12 (or c), 3, 6, 9, 14 (or e), 11 (or b), 13 (or d), 7, and 15 (or f) in the order shown.

There are more complicated interlaced and spiraling ways of dissecting the $n \times m$ rectangle into smaller rectangles like in Figure 4 [1, 2]; these counts are tabulated in the Online Encyclopedia of Integer Sequences [6, A116694].

6.3. **Cut Sets.**

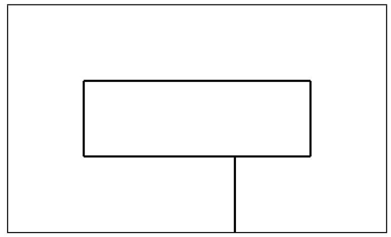

FIGURE 6. A tiling of the 5×3 rectangle which is a cut set but not a polyomino tiling.

Definition 5. *A cut set of a* $n \times m$ *rectangle is a line set which tiles the* $(n-1) \times (m-1)$ *rectangle shifted by* $(1/2, 1/2)$ *along the diagonal with card sets of Figure 5 such that each fin (half edge, in green) either ends at the perimeter of the rectangle or is continued by a fin of the adjacent card.*

The four cards which are not admitted in the layout are those that show one single fin. The definition means that any set of cuts zig-zagging through the rectangle along the inner edges is a cut set if each cut starts and ends either at one of the $2(m-1) + 2(n-1)$ points at the perimeter or at the same or another cut.

The terrace forms discussed in Sections 1 to 5, the tilings with rectangles of Section 6.2, and the more general tilings with polyominos are cut sets according to this definition.

Remark 4. *Polyomino tilings are counted in the OEIS sequence A078469 for* $2 \times m$ *rectangles, in A108808 for* $3 \times m$ *rectangles, and in A221157 for* $4 \times m$ *rectangles* [5].

An example of a cut set which is *not* a polyomino tiling is shown in Figure 6. It contains a straight 3×1 tromino in the center, but the edge that reaches from that tromino to the lower edge of the rectangle prevents the remaining area from being a 3×5-omino with a center hole. That type of extra edges that cut through otherwise connected areas mean that the number of cut sets $C_{n \times m}$ of a rectangle is an upper bound of the number of polyomino tilings of the rectangle.

The number of cut sets is symmetric if the row and column counts are swapped: $C_{n \times m} = C_{m \times n}$. The generating function is defined as usual:

$$C_n(z) \equiv \sum_{m \geq 1} C_{n \times m} z^m. \tag{40}$$

The number of cut sets for the $1 \times m$ rectangle is the number of ways of selecting any of the $m-1$ vertical lines.

$$C_{1 \times m} = 2^{m-1}. \tag{41}$$

The number of cut sets of the $2 \times m$ rectangle is the number of $1 \times (m-1)$ left-right stacks of the 12 symbols. This is

$$2, 127, 74, 456, 2810, 17316, 10606, \ldots$$

for $m \geq 1$ with [6, A078469]

$$C_2(z) = \frac{2x}{1 - 6x - x^2}. \tag{42}$$

This is the same as the number of polyomino tilings because the "narrow" width of 2 in that case prevents the creation of "back door" connections of the type shown in Figure 6.

Remark 5. *The associated recurrence $C_{2\times m} = 6C_{2\times(m-1)} + C_{2\times(m-2)}$ signals with the factor 6 that each of the 12 cards of Figure 5 can be stacked at each side by either 5 or 7 cards (possibly including itself). The other ones are prohibited because they would produce unpaired fins.*

The number of cut sets of the $3 \times m$ rectangle is

$$4, 74, 1442, 28028, 544844, 10591310, \ldots$$

for $m \ge 1$ with

$$C_3(z) = \frac{2x(2-x)}{1-19x-9x^2+9x^3}. \tag{43}$$

The number of cut sets of the $4 \times m$ rectangle is

$$8, 456, 28028, 1716098, 105093828, \ldots$$

for $m \ge 1$ with

$$C_4(z) = \frac{2x(1-x)(x^4+7x^3-26x^2+8x+4)}{1-56x-320x^2-68x^3+952x^4-362x^5-3x^6} \tag{44}$$

The number of cut sets of the $5 \times m$ rectangle is

$$16, 2810, 544844, 105093828, 20276816980, \ldots$$

for $m \ge 1$ with

$$C_5(z) = \frac{2x(8-35x-742x^2+5465x^3-12292x^4+9466x^5-948x^6-1068x^7+142x^8+36x^9)}{1-180x-2533x^2+6672x^3+86813x^4-350346x^5+422201x^6-172374x^7-50x^8+11988x^9-1296x^{10}} \tag{45}$$

The number of cut sets of the $6 \times m$ rectangle is

$$32, 17316, 10591310, 6435880414, 3912156203494, 2378025136264102, 1445496758320387318, \ldots$$

for $m \ge 1$ with

$$C_6(z) = \frac{2xp_6(z)}{q_6(z)}, \tag{46}$$

where

$$\begin{aligned} p_6(z) \equiv{}& 16+146x-34337x^2-43833x^3+15710973x^4-190477581x^5 \\ &+824010172x^6-1226865129x^7-768435521x^8+3422992650x^9-997100979x^{10} \\ &-2955536736x^{11}+1290228493x^{12}+1103100279x^{13}-408174258x^{14}-163645766x^{15} \\ &+45480873x^{16}+6835086x^{17}-1500592x^{18}+54016x^{19}; \end{aligned} \tag{47}$$

$$\begin{aligned} q_6(z) \equiv{}& 1-532x-45246x^2-559732x^3+20947821x^4+155965996x^5 \\ &-5061059637x^6+33040180348x^7-87567881677x^8+72678698622x^9+95753372255x^{10} \\ &-200759997582x^{11}+35720360478x^{12}+117191191846x^{13}-50636219273x^{14} \\ &-21557686612x^{15}+11939533861x^{16}+278945786x^{17}-600818263x^{18}+64383392x^{19}-1613568x^{20}. \end{aligned} \tag{48}$$

Appendix A. Java Program Counting Terrace Forms

The source code of the JAVA program that generated the table of the results in Section 5.1 is a single file `TWayMono.java` that is reproduced here:

```java
import java.util.* ;
import java.math.* ;
public class TWayMono {
    /** Number of rows
    */
    int n ;

    /** Number of columns
    */
    int m ;

    /** row and column dimension in the compatibility table, 2^(n-1)
    */
    int binn ;

    /** The number of terrasses T[l][b] given l in the range 0 to m-1
    * and b in the range 0 to 2^(n-1)-1.
    */
    BigInteger[][] T ;

    /** The two compatibility tables C[a][b][bprime] for a=0 or a=1.
    * b and b prime in the range 0 to 2^(n-1)-1. a denotes whether
    * the step in the upper right NE corner is an up-step of levelled step.
    */
    byte[][][] compat ;

    /** Ctor with a specified number of rows and a single column.
    * @param nrows The number of rows. A value larger than 0.
    */
    TWayMono(int nrows)
    {
        n = nrows;
        m = 1;
        binn = 1 << (n-1) ;
        initCompat() ;
        T = new BigInteger[1][binn] ;
        for(int b=0 ; b < T[0].length; b++)
            T[0][b] = BigInteger.ONE ;
    } /* ctor */

    /** Ctor by recurrence from m to m+1.
    * @param colless The distribution with one column less.
    * @param ncols The number of rows in colless as well as the one to be constructed.
    */
    TWayMono(TWayMono colless)
    {
        n = colless.n ;
        /* this matrix has one column more than the old one */
        m = colless.m+1;
```

```
        binn = colless.binn ;
        /* compatibility table does not change because n does not change
        */
        compat = colless.compat ;

        /* T01[a] are the two matrices obtained by multiplying
        * the old T matrix with one of the two compatibility matrices
        * of a=0 or a=1.
        */
        BigInteger[][][] T01 = new BigInteger[2][colless.m][binn] ;
        for(int a=0 ; a <=1 ; a++)
            for(int l=0 ; l < T01[a].length ; l++)
            {
                for(int bprime=0 ; bprime < T01[a][l].length ; bprime++)
                {
                    T01[a][l][bprime] = BigInteger.ZERO ;
                    for(int b=0 ; b < T01[a][l].length ; b++)
                    {
                        if ( compat[a][b][bprime] == 1)
                            T01[a][l][bprime] = T01[a][l][bprime].add(colless.T[l][b]) ;
                    }
                }
            }

        /* construct the table T[l,b] by the shift-and-add
        * superposition of T01[0] and T01[1].
        */
        T = new BigInteger[m][binn] ;
        for(int l=0 ; l < T.length ; l++)
        {
            for(int b=0 ; b <T[l].length ; b++)
            {
                if ( l == 0 )
                    T[l][b] = T01[0][l][b] ;
                else if ( l == m-1 )
                    T[l][b] = T01[1][l-1][b] ;
                else
                    T[l][b] = T01[0][l][b].add(T01[1][l-1][b]) ;
            }
        }
    } /* ctor */

    /** Total number of all 2-way monotonic terrace forms.
    * @return T_{n x m}
    */
    public BigInteger count()
    {
        BigInteger val= BigInteger.ZERO ;
        /* sum over all 0<=h<=m+n-2.
        * Note that it would be faster to add simply all elements
        * of T without filtering for distinct values of h. But
        * here it's more convenient to print the table of T(h) on the
        * fly.
```

```
        */
        for(int h=0 ; h < m+n-1 ; h++)
        {
            final BigInteger Tofh = count(h) ;
            /* print results if not redundant according
            * to the conjugacy. Consider h<=n+m-2-h.
            */
            if ( 2*h+2 <= n+m)
                System.out.println(n + " " + m + " " + h + " " +Tofh) ;
            val = val.add(Tofh) ;
        }
        return val;
    } /* count */

    /** Number of bits set in the binary representation of b.
    * @return The number of +1 digits. The Hamming weight of b.
    */
    public static int bitcount(int b)
    {
        int H=0 ;
        /* shift b by one bit to the right and accumulate the LSB */
        while ( b > 0 )
        {
            H += b& 1;
            b >>= 1 ;
        }
        return H;
    } /* bitcount */

    /** Count number of matrices with fixed SE value
    * @param h The maximum value in the matrix. The hill height in the SE.
    * @return T_{n X m)(h)
    */
    public BigInteger count(final int h)
    {
        BigInteger val= BigInteger.ZERO ;
        /* sum over all rows and columns of T obeying H(b)=h-l
        */
        for(int l =0 ; l < T.length ; l++)
        {
            for(int b=0 ; b < T[l].length ; b++)
                if ( l+bitcount(b) == h)
                    val = val.add(T[l][b]) ;
        }
        return val;
    } /* count */

    /** Generate the two compatibility tables for a=0 and a=1.
    */
    private void initCompat()
    {
        compat = new byte[2][][] ;
        for(int a=0 ; a <= 1 ; a++)
```

```
        {
            compat[a] = new byte[binn][binn] ;
            for(int b= 0 ; b < binn ; b++)
            for(int bprime=0 ; bprime < binn ; bprime++)
            {
                /* Initialize the array assuming the compatibility criterion is met.
                */
                compat[a][b][bprime] = 1 ;
                /* the partial sums of the two binary sequences */
                int disum =0 ;
                int diprimesum =0 ;
                for(int s=1 ; s <= n ; s++)
                {
                    /* update the two partial digit sums */
                    if ( (b & (1 << (s-1))) != 0 )
                        disum++ ;
                    if ( (bprime & (1 << (s-1))) != 0 )
                        diprimesum++ ;

                    if ( a+diprimesum < disum || a+diprimesum > 1+disum)
                    {
                        compat[a][b][bprime] =0 ;
                        break;
                    }
                }
            }
        }
    } /* initCompat */

    /** Build a table of the results for n,m>=1
    *  Usage: java -cp . TWayMono
    * @param args
    *
    */
    static public void main(String[] args)
    {
        TWayMono t2 =null;
        /* strict limit of n is internally given by our internal
        * representation of 2^(n-1) as integers, so n<=32. This constraint
        * is not enforced here, because the byte table of the
        * compatibility matrix has 2^(2n-2) entries and that would
        * likely set a lower RAM limit to what can be computed here.
        */
        for(int n=1 ; n < 15 ; n++)
        {
            for(int m=1 ; m <= n ; m++)
            {
                if ( m == 1 )
                    /* all T[0][b]=1 */
                    t2 = new TWayMono(n) ;
                else
                    /* recurrence m->m+1 */
                    t2 = new TWayMono(t2) ;
```

```
                    System.out.println(n + " " + m + " " + t2.count()) ;
                    System.out.println() ;
                }
            }
        } /* main */
} /* TWayMono */
```

References

1. F. R. K. Chung, E. N. Gilbert, R. L. Graham, J. B. Shearer, and J. H. van Lint, *Tiling rectangles with rectangles*, Math. Mag. **55** (1982), no. 5, 286–291. MR 0678194
2. M. Dehn, *über zerlegung von rechtecken in rechtecke*, Math. Ann. **57** (1903), 314–322.
3. Guillaume Fertin, Emmanuel Godard, and André Raspaud, *Acyclic and k-distance coloring of the grid*, Inform. Proc. Lett. **87** (2003), no. 1, 51–58. MR 1979860
4. Richard J. Mathar, *Tilings of rectangular regions by rectangular tiles: counts derived from transfer matrices*, arXiv:1406.7788 [math.CO] (2014).
5. Frank Simon, *Algebraic methods for computing the reliability of networks*, Ph.D. thesis, TU Dresden, 2012.
6. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)

URL: `http://www.mpia.de/~mathar`

Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany