The Constant Cavity Pressure Casimir Inaptly Discarded

Steven Kenneth Kauffmann*

Abstract

Casimir's celebrated result that the conducting plates of an unpowered rectangular cavity attract each other with a pressure inversely proportional to the fourth power of their separation entails an unphysical unbounded pressure as the plate separation goes to zero. An unphysical result isn't surprising in light of Casimir's unphysical assumption of perfectly conducting plates that zero out electric fields regardless of their frequency, which he sought to counteract via a physically foundationless discarding of the pressure between the cavity plates when they are sufficiently widely separated. Casimir himself, however, emphasized that real metal plates are transparent to sufficiently high electromagnetic frequencies, which makes removal of the frequency cutoff that he inserted unjustifiable at any stage of his calculation. Therefore his physically groundless discarding of the large-separation pressure isn't even needed, and when it is left out a constant attractive pressure between cavity plates exists when their separation is substantially larger than the cutoff wavelength. The intact cutoff furthermore implies zero pressure between cavity plates when their separation is zero, and also that Casimir's pressure is merely the subsidiary lowest-order correction term to the constant attractive pressure between cavity plates that is dominant when their separation substantially exceeds the cutoff wavelength.

Introduction

H. B. G. Casimir's groundbreaking 1948 presentation "On the attraction between two perfectly conducting plates" [1] is a fascinating chronicle of his strivings to extract theoretical physics sense from the ostensibly infinite electromagnetic-field ground-state energy $\frac{1}{2}\sum\hbar\omega$ that is captured in standing waves within a conducting rectangular cavity whose dimensions are $L_1 \times L_2 \times a$.

The method for "taming" this supposedly infinite energy which gained traction in Casimir's mind was to subtract from $\frac{1}{2}\sum\hbar\omega$ at any arbitrary value of the separation a between the cavity's two $L_1 \times L_2$ plates that sum's value at a sufficiently *large* value of that two-plate separation a. To be sure, the difference between two ostensibly infinite energy sums is ill-defined, but Casimir's plan to overcome that difficulty was to cut off both of the infinite-valued sums that are involved in precisely the same way, and then to *remove* that cutoff *after the subtraction* of the sum having a sufficiently large value of *a from* the sum having an arbitrary value of *a is safely accomplished*. Casimir of course hoped that this recipe would produce a result which is both finite and unique, and it turns out that for "reasonable" cutoffs Casimir's hope is actually fulfilled—we shall have much more to say below about how the criterion for a "reasonable" cutoff was entwined in Casimir's thinking with the response of *real* conducting metals to arbitrarily high-frequency electromagnetic fields, and about how *continuing* to think along those *physical* lines makes it obvious that the ostensibly "infinite" energy sums which bedeviled Casimir *are wholly unphysical*.

Before we delve further into that matter, however, it is important to underline a crucial *elementary consequence* of Casimir's above-described subtraction procedure *which Casimir himself failed to notice*: the results obtained from his subtraction procedure obviously cannot possibly properly describe the ground-state electromagnetic energy content of rectangular cavities which have sufficiently large values of the $L_1 \times L_2$ plate separation distance a because part of that energy content *has, of course, been subtracted away*. As a consequence, Casimir's results are inherently *incapable* of describing the pressure between cavity plates which are separated by a distance a that is sufficiently large. Indeed, Casimir's pressure results necessarily exhibit *short-range character as a function of a that is a pure unphysical artifact*.

To get a feeling for the artificiality which Casimir's subtraction procedure *injects* into his results, we note that his "subtracted energy" $\delta E(L_1, L_2, a)$ for the "perfectly conducting" rectangular cavity whose dimensions are $L_1 \times L_2 \times a$ with L_1 and L_2 sufficiently large and a (ostensibly) arbitrary is [1],

$$\delta E(L_1, L_2, a) = -\hbar c(\pi^2/720)L_1 L_2/a^3, \tag{1a}$$

which exhibits a drastically *different* dependence on a than it has on L_1 and L_2 *in the case where all three of these cavity dimensions are arbitrarily large*. We see that $\delta E(L_1, L_2, a)$ is very long-range in L_1 and L_2 *but short-range in a*, the latter being *an unphysical pure artifact of Casimir's subtraction procedure*.

* Retired, American Physical Society Senior Life Member, E-mail: SKKauffmann@gmail.com

The celebrated Casimir pressure result between the two $L_1 \times L_2$ plates of this "perfectly conducting" $L_1 \times L_2 \times a$ rectangular cavity is of course given by [1],

$$-\left(\frac{\partial\delta E(L_1,L_2,a)}{\partial a}\right)(L_1L_2)^{-1} = -\pi^2\hbar c/(240a^4), \tag{1b}$$

which *not only* is short-range in a, but *also* is unphysically *unbounded* as the separation a between the two $L_1 \times L_2$ plates goes to zero! Likewise, Casimir's "subtracted energy" of Eq. (1a) is unphysically *unbounded* as the separation a between the two $L_1 \times L_2$ plates goes to zero. Thus there is a *second* pathology intrinsic to Casimir's celebrated pressure result, one which is the *consequence* of the wholly unphysical "perfect conductivity" of Casimir's cavity *even notwithstanding* the finite and unique nature of Casimir's "subtracted energy" result $\delta E(L_1, L_2, a)$ of Eq. (1a)—the *mere finite uniqueness* of Casimir's "subtracted energy" result *of course does not per se imply that that result is physically correct or sound*!

This *second* pathology in Casimir's celebrated pressure result focuses our attention on Casimir's *own comment* that any "reasonable" cutoff of $\frac{1}{2}\sum\hbar\omega$ which is to be applied *before* his subtraction procedure and his subsequent *removal* of that cutoff is undertaken *must adequately model the fact that real conducting metals are transparent to sufficiently high-frequency electromagnetic fields*. Casimir's recipe for a "reasonable" cutoff of a sum $\frac{1}{2}\sum\hbar\omega$ incorporates that feature via the *replacement* of such a sum by $\frac{1}{2}\sum\hbar\omega f(\omega/(c\kappa))$, where $f(x)$ has the salient characteristics of e^{-x} or e^{-x^2} for $x > 0$, namely $f(x)$ is positive and decreases monotonically *from its value of unity at* $x = 0$ in such a way that $f(1) = e^{-1}$ and $f(x)$ *tends very strongly to zero as* $x \to +\infty$. Therefore if Casimir had not been so intensely *preoccupied* with actually *carrying through* his programme of cutoff, subtraction and finally *removal* of the cutoff, it surely would have *dawned* on him *that the physical nature of real conducting metals forbids the removal at any stage whatsoever in his calculation of the just-described cutoff which he inserts into it*. Given that Casimir's cutoff *is physically required to be permanently in place*, it *also* would have dawned on Casimir that the entire raison d'être of his (in fact physically counterproductive) subtraction procedure *simply falls away*. (It might *even* then have dawned on Casimir *just how physically counterproductive* the effect of his subtraction procedure on his result *actually is*.)

In the following section we therefore *redo* Casimir's calculation of $\frac{1}{2}\sum\hbar\omega f(\omega/(c\kappa))$, leaving $f(\omega/(c\kappa))$ *permanently in place*—we specifically choose $f(x) = e^{-x}$ because that choice is calculationally advantageous. Of course *we entirely omit Casimir's counterproductive subtraction procedure*.

A simple model of the attraction between two real metal cavity walls

We use Casimir's techniques to model and calculate the standing-wave electromagnetic-field ground-state energy $\frac{1}{2}\sum\hbar\omega\exp(-\omega/(c\kappa))$ captured by an $L_1 \times L_2 \times a$ rectangular real metal cavity under the assumption that $L_1 \gg 1/\kappa$ and $L_2 \gg 1/\kappa$, but without making any assumption about the relation of a to κ. Taking account of the field polarizations in the way that Casimir does [1] produces,

$$\begin{aligned} E(L_1, L_2, a; \kappa) &\stackrel{\text{def}}{=} \tfrac{1}{2}\textstyle\sum\hbar\omega\exp(-\omega/(c\kappa)) = \\ \hbar c\int_0^\infty dm_1\int_0^\infty dm_2 &\Bigg[\tfrac{1}{2}\left(\left(\tfrac{\pi m_1}{L_1}\right)^2+\left(\tfrac{\pi m_2}{L_2}\right)^2\right)^{\frac{1}{2}} e^{-\left(\left(\frac{\pi m_1}{\kappa L_1}\right)^2+\left(\frac{\pi m_2}{\kappa L_2}\right)^2\right)^{\frac{1}{2}}} \\ &+\sum_{n=1}^{\infty}\left(\left(\tfrac{\pi m_1}{L_1}\right)^2+\left(\tfrac{\pi m_2}{L_2}\right)^2+\left(\tfrac{\pi n}{a}\right)^2\right)^{\frac{1}{2}} e^{-\left(\left(\frac{\pi m_1}{\kappa L_1}\right)^2+\left(\frac{\pi m_2}{\kappa L_2}\right)^2+\left(\frac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}}}\Bigg]. \end{aligned} \tag{2a}$$

We change the two integration variables to $u_1 = (\pi m_1)/(\kappa L_1)$ and $u_2 = (\pi m_2)/(\kappa L_2)$, and also take advantage of the fact that the integrand is an even function of those integration variables to obtain,

$$\begin{aligned} E(L_1, L_2, a; \kappa) = \left(\tfrac{\hbar c\kappa^3 L_1 L_2}{(2\pi)^2}\right)\int_{-\infty}^{\infty} du_1\int_{-\infty}^{\infty} du_2 &\Bigg[\tfrac{1}{2}(u_1^2+u_2^2)^{\frac{1}{2}}e^{-(u_1^2+u_2^2)^{\frac{1}{2}}} \\ &+\sum_{n=1}^{\infty}\left(u_1^2+u_2^2+\left(\tfrac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}} e^{-\left(u_1^2+u_2^2+\left(\frac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}}}\Bigg]. \end{aligned} \tag{2b}$$

We switch to polar coordinates, i.e., $u = (u_1^2 + u_2^2)^{\frac{1}{2}}$, and are able to immediately integrate over the polar angle to obtain,

$$E(L_1, L_2, a; \kappa) = \left(\tfrac{\hbar c\kappa^3 L_1 L_2}{4\pi}\right)\int_0^\infty 2u\,du\left[\tfrac{1}{2}ue^{-u}+\sum_{n=1}^{\infty}\left(u^2+\left(\tfrac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}} e^{-\left(u^2+\left(\frac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}}}\right]. \tag{2c}$$

We now carry out one elementary integration, and then under the summation sign we change the integration variable to $x = u^2$ to obtain,

$$E(L_1, L_2, a; \kappa) = \left(\tfrac{\hbar c \kappa^3 L_1 L_2}{4\pi}\right)\left[2 + \sum_{n=1}^{\infty}\int_0^{\infty} dx\, \left(x + \left(\tfrac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}} e^{-\left(x+\left(\frac{\pi n}{\kappa a}\right)^2\right)^{\frac{1}{2}}}\right]. \tag{2d}$$

In order to perform the integration in Eq. (2d) we carry out one last change of integration variable to $w = (x + ((\pi n)/(\kappa a))^2)^{\frac{1}{2}}$, which implies that $dx = 2w dw$ and yields,

$$E(L_1, L_2, a; \kappa) = \left(\tfrac{\hbar c \kappa^3 L_1 L_2}{2\pi}\right)\left[1 + \sum_{n=1}^{\infty}\int_{\left(\frac{\pi n}{\kappa a}\right)}^{\infty} dw\, w^2 e^{-w}\right] = \left(\tfrac{\hbar c \kappa^3 L_1 L_2}{2\pi}\right)\left[1 + \sum_{n=1}^{\infty}\left((n\alpha)^2 + 2(n\alpha) + 2\right)e^{-n\alpha}\right], \tag{2e}$$

where after the second equal sign in Eq. (2e) we have introduced the convenient abbreviation $\alpha \stackrel{\text{def}}{=} (\pi/(\kappa a))$.

We now carry out the summation in Eq. (2e) by using the geometric series related formulas $\sum_{n=1}^{\infty} \varepsilon^n = \varepsilon(1-\varepsilon)^{-1}$, $\sum_{n=1}^{\infty} n\varepsilon^n = \varepsilon(1-\varepsilon)^{-2}$ and $\sum_{n=1}^{\infty} n^2\varepsilon^n = \varepsilon(1+\varepsilon)(1-\varepsilon)^{-3}$, which are valid for $|\varepsilon| < 1$, and thereby obtain,

$$E(L_1, L_2, a; \kappa) = \left(\tfrac{\hbar c \kappa^3 L_1 L_2}{2\pi}\right)\left[1 + e^{-\alpha}\left(\tfrac{\alpha^2(1+e^{-\alpha})}{(1-e^{-\alpha})^3} + \tfrac{2\alpha}{(1-e^{-\alpha})^2} + \tfrac{2}{(1-e^{-\alpha})}\right)\right]. \tag{2f}$$

Since we have assumed that $L_1 \gg 1/\kappa$ and $L_2 \gg 1/\kappa$ it is interesting to examine the special case that $a \gg 1/\kappa$, which implies that $\alpha \ll 1$ and thereby causes Eq. (2f) to reduce to,

$$E(L_1, L_2, a; \kappa) \approx \left(\tfrac{\hbar c \kappa^3 L_1 L_2}{2\pi}\right)\left[\tfrac{6}{\alpha}\right] = \left(\tfrac{3\hbar c \kappa^4 L_1 L_2 a}{\pi^2}\right), \tag{2g}$$

which is invariant under the interchange of *any two* of the *three* cavity dimensions L_1, L_2 and a, as of course it *must* be. Casimir's "subtracted energy" $\delta E(L_1, L_2, a)$ of Eq. (1a) however *completely fails to manifest this essential symmetry property*. That, along with the fact that $\delta E(L_1, L_2, a)$ is *unbounded* as $a \to 0$, spotlights the ineluctable *failure* of Casimir's "subtracted energy" attempt to cope with *his inherently unphysical assumption that the cavity is perfectly conducting*.

The pressure $P(a; \kappa)$ between the cavity's two $L_1 \times L_2$ plates can be calculated using Eq. (2f), but less algebraic effort is needed if Eq. (2e) is used in conjunction with the summation formula $\sum_{n=1}^{\infty} n^3\varepsilon^n = \varepsilon(1 + 4\varepsilon + \varepsilon^2)(1-\varepsilon)^{-4}$, which is of course valid for $|\varepsilon| < 1$. In any case the result is,

$$P(a; \kappa) = -\left(\tfrac{\partial E(L_1, L_2, a; \kappa)}{\partial a}\right)(L_1 L_2)^{-1} = -\tfrac{\hbar c \kappa^4 \alpha^4 e^{-\alpha}(1+4e^{-\alpha}+e^{-2\alpha})}{2\pi^2(1-e^{-\alpha})^4} = -\tfrac{\hbar c \kappa^4 (2+\cosh(\alpha))}{\pi^2((2/\alpha)\sinh(\alpha/2))^4}, \tag{2h}$$

where, of course, $\alpha = (\pi/(\kappa a))$.

From Eq. (2h) it is apparent that the pressure $P(a; \kappa)$ between the two $L_1 \times L_2$ plates is *always* attractive, and that when $a \gg 1/\kappa$ (i.e., when $\alpha \ll 1$), $P(a; \kappa) \approx -3\pi^{-2}\hbar c \kappa^4$, which also follows immediately from Eq. (2g). This large-separation attractive *constant* pressure *is completely deleted from Casimir's celebrated pressure result* $-\pi^2\hbar c/(240a^4)$ of Eq. (1b) by his physically counterproductive subtraction procedure.

Furthermore, it is clear from Eq. (2h) that as the plate separation a goes to zero (i.e., as $\alpha \to +\infty$), $P(a; \kappa) \to 0$. That physically sensible result stands in stark contrast to the fact that the magnitude of Casimir's pressure $-\pi^2\hbar c/(240a^4)$ increases rapidly and without bound as $a \to 0$.

Notwithstanding these devastating observations about Casimir's pressure $-\pi^2\hbar c/(240a^4)$, it *does* in fact play a *subsidiary* physical role: it turns out to be *the lowest-order correction* in powers of α to the large-separation *constant* attractive pressure $-3\pi^{-2}\hbar c \kappa^4$ between the two $L_1 \times L_2$ plates.

To expand the pressure $P(a; \kappa)$ of Eq. (2h) in powers of α, we note that,

$$(2 + \cosh(\alpha)) = 3(1 + \alpha^2/6 + \alpha^4/72 + \alpha^6/2160 + \cdots),$$

and also that,

$$\begin{aligned}((2/\alpha)\sinh(\alpha/2))^4 &= ((2/\alpha^2)(\cosh(\alpha) - 1))^2 = (2/\alpha^4)(\cosh(2\alpha) - 4\cosh(\alpha) + 3) \\ &= (1 + \alpha^2/6 + \alpha^4/80 + 17\alpha^6/30240 + \cdots),\end{aligned}$$

from which we obtain,

$$\tfrac{2+\cosh(\alpha)}{((2/\alpha)\sinh(\alpha/2))^4} = 3(1 + \alpha^4/720 - \alpha^6/3024 + \cdots).$$

From this expansion and Eq. (2h) the first two corrections in powers of $\alpha = (\pi/(\kappa a))$ to the large-separation constant attractive pressure $-3\pi^{-2}\hbar c\kappa^4$ come out as follows,

$$P(a;\kappa) = -3\pi^{-2}\hbar c\kappa^4 - \pi^2\hbar c/(240a^4) + \left(\pi^2\hbar c/(1008a^4)\right)(\pi/(\kappa a))^2 + \cdots. \tag{2i}$$

Thus the attractive Casimir pressure $-\pi^2\hbar c/(240a^4)$ is *the lowest-order correction in powers of* $\alpha = (\pi/(\kappa a))$ *to the large-separation constant attractive pressure* $-3\pi^{-2}\hbar c\kappa^4$. This Casimir-pressure correction term under no circumstance dominates $P(a;\kappa)$, however. Numerical study shows that the exact $P(a;\kappa)$ of Eq. (2h) attains its minimum value of approximately $-3\pi^{-2}\hbar c\kappa^4 \times (1.00723)$ at approximately $\alpha = 2.144$ (i.e., at approximately $a = 1.4653/\kappa$), and from that minimum at $a = 1.4653/\kappa$, $P(a;\kappa)$ increases monotonically to zero as $a \to 0$ (i.e., from that minimum at $\alpha = 2.144$, $P(a;\kappa)$ increases monotonically toward zero as $\alpha \to +\infty$). In other words, the Casimir-pressure correction term $-\pi^2\hbar c/(240a^4)$ of Eq. (2i) *never perturbs* $P(a;\kappa)$ *by as much as three quarters of a percent*.

However, because the Casimir-pressure correction term $-\pi^2\hbar c/(240a^4)$ varies rapidly with the plate separation a and corrects the *constant* pressure term $-3\pi^{-2}\hbar c\kappa^4$ *which doesn't vary at all with the plate separation* a, the Casimir-pressure correction term ought to be *discernible* even notwithstanding that it is a decidedly *small* correction.

References

[1] H. B. G. Casimir, "On the attraction between two perfectly conducting plates", www.dwc.knaw.nl/DL/publications/PU00018547.pdf (1948).