Non-stationary helical flows for incompressible 3D Navier-Stokes equations

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In fluid mechanics, a lot of authors have been executing their researches to obtain the analytical solutions of Navier-Stokes equations, even for 3D case of *compressible* gas flow. But there is an essential deficiency of non-stationary solutions indeed.

In our derivation, we explore the case of non-stationary *helical* flow of the Navier-Stokes equations for incompressible fluids at *any* given initial conditions for velocity fields (*it means an open choice for the space part of a solution*).

Such a non-stationary *helical* flow is proved to be decreasing exponentially in regard to the time-parameter, the extent of time-dependent exponential component is given by the coefficient of kinematic viscosity, multiplied by the square of the coefficient of proportionality between the vorticity and velocity field.

Keywords: Navier-Stokes equations, non-stationary helical flow, Arnold-Beltrami-Childress (ABC) flow.

1. Introduction, the Navier-Stokes system of equations.

In accordance with [1-3], the Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (*under the proper initial conditions*):

$$\nabla \cdot \vec{u} \ = 0 \,, \qquad (1.1)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F} \,, \qquad (1.2)$$

- where ***u*** is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and ***F*** represents external force (*per unit of mass in a volume*) acting on the fluid. Let us also choose the Ox axis coincides to the main direction of flow propagation; notation ***u*** or $\vec{u}$ means a vector field.

Besides, we assume here external force ***F*** above to be the force, which has a potential ϕ represented by ***F*** = -∇ ϕ.

2. The originating system of PDE for Navier-Stokes Eqs.

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$, we could present the Navier-Stokes equations (1.1)-(1.2) for incompressible viscous flow $\boldsymbol{u} = \{u_1, u_2, u_3\}$ as below [4-5]:

$$\begin{aligned} &\nabla \cdot \vec{u} \ = 0 \,, \\ &\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{w} + \nu \cdot \nabla^2 \vec{u} - \left(\frac{1}{2} \nabla (\vec{u}^{\,2}) + \frac{\nabla p}{\rho} + \nabla \phi \right) \end{aligned} \qquad (2.1)$$

- here we denote *the curl field* ***w***, a pseudovector *time-dependent* field [6]; besides, let us denote: $-\{(\nabla p/\rho) + \nabla \phi\} = \{f_x, f_y, f_z\}$.

Vorticity, associated with the curl field, is assumed to be arising due to the proper sources of vorticity in the flow of fluids [4-5]. For example, such a sources could be associated with the solid surface or pressure gradient in case of non-barotropic compressible fluids, influence of viscous forces, Coriolis forces (when one's reference frame is rotating rigidly) or curving shock fronts when speed is supersonic.

3. <u>The presentation of time-dependent solution.</u>

Let us search for solutions of the system (2.1) in a form of *helical* flow below:

$$\vec{w} = \alpha \cdot \vec{u} \;\Rightarrow\; \vec{u} \times \vec{w} = \vec{0}, \;\; \nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\alpha^2 \cdot \vec{u} \qquad (3.1)$$

- here α is the constant coefficient, given by the initial conditions ($\alpha \neq 0$).

Then we should obtain from (2.1) and expression for curl [6] the proper system of PDE:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -\nu \cdot \alpha^2 \cdot u_1 - \dfrac{1}{2}\dfrac{\partial}{\partial x}(u_1{}^2 + u_2{}^2 + u_3{}^2) + f_x \,, \\[2ex] \dfrac{\partial u_2}{\partial t} = -\nu \cdot \alpha^2 \cdot u_2 - \dfrac{1}{2}\dfrac{\partial}{\partial y}(u_1{}^2 + u_2{}^2 + u_3{}^2) + f_y \,, \\[2ex] \dfrac{\partial u_3}{\partial t} = -\nu \cdot \alpha^2 \cdot u_3 - \dfrac{1}{2}\dfrac{\partial}{\partial z}(u_1{}^2 + u_2{}^2 + u_3{}^2) + f_z \,, \end{cases} \qquad (3.2)$$

- besides, the continuity equation (1.1) should be satisfied due to the special form (3.1) of helical flow solution. Also taking into consideration the expression for curl [6], we obtain:

$$\left\{\alpha\cdot u_1,\, \alpha\cdot u_2,\, \alpha\cdot u_3\right\} \;\equiv\; \left\{\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right),\;\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right),\;\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\right\}.$$

Let us differentiate the 3-rd equation of system (3.2) in regard to variable y, additionaly differentiate the 2-nd equation (3.2) in regard to variable z, then subtract it one from each other:

$$\frac{\partial\left(\dfrac{\partial u_3}{\partial y}-\dfrac{\partial u_2}{\partial z}\right)}{\partial t} \;=\; -\,\nu\cdot\alpha^2\cdot\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right)+\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\,,$$

- where from expression for curl [6] we determine as

$$\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right) \;=\; \alpha\cdot u_1\,,$$

- so, each equations of the system (3.2) should be transformed as below

$$\begin{cases} \alpha\cdot\dfrac{\partial u_1}{\partial t} \;=\; -\,\nu\cdot\alpha^3\cdot u_1 \;+\; \dfrac{\partial f_z}{\partial y}-\dfrac{\partial f_y}{\partial z}\,, \\[2ex] \alpha\cdot\dfrac{\partial u_2}{\partial t} \;=\; -\,\nu\cdot\alpha^3\cdot u_2 \;+\; \dfrac{\partial f_x}{\partial z}-\dfrac{\partial f_z}{\partial x}\,, \\[2ex] \alpha\cdot\dfrac{\partial u_3}{\partial t} \;=\; -\,\nu\cdot\alpha^3\cdot u_3 \;+\; \dfrac{\partial f_y}{\partial x}-\dfrac{\partial f_x}{\partial y}\;. \end{cases} \tag{3.3}$$

System of equations (3.3) could be presented as

$$\alpha\cdot\frac{\partial\vec{u}}{\partial t} \;=\; -\,\nu\cdot\alpha^3\cdot\vec{u} \;+\; \nabla\times\vec{f}\,,$$

- where

$$\vec{f}=-\left(\frac{\nabla p}{\rho}+\nabla\phi\right),$$

- thus, finally we obtain

$$\alpha \cdot \frac{\partial \vec{u}}{\partial t} = -\nu \cdot \alpha^3 \cdot \vec{u} \;, \qquad \Rightarrow$$

$$\vec{u} = \exp(-\nu \cdot \alpha^2 \cdot t) \cdot \vec{u}(t_0) \qquad (3.4)$$

- here $u_1(t_0)$, $u_2(t_0)$, $u_3(t_0)$ are the set of functions, depending on variables $\{x, y, z\}$ (which are given by the initial conditions).

As for the components of pressure gradient field, according to [7] and Eqs. (2.1), it could be presented for such a type of helical flows as below:

$$\left(\frac{1}{2} \nabla (\vec{u}^{\,2}) + \frac{\nabla p}{\rho} + \nabla \phi \right) = 0 \;\Rightarrow\; \frac{\nabla p}{\rho} = -\exp(-2\nu \cdot \alpha^2 \cdot t) \cdot \frac{1}{2} \nabla (\vec{u}^{\,2}(t_0)) - \nabla \phi \qquad (3.5)$$

4. <u>Discussion.</u>

There exists a well known helical steady solution, the Arnold-Beltrami-Childres flow [8-9], which is the particular simple case of *helical* flow (3.1). The last is also known as Beltrami flow, i.e. a fluid motion in which the vorticity vector is parallel to the velocity vector at every point of the fluid. For stationary case (3.4), the space part of such an ABC-flow should be presented as below (A, B, C = const):

$$u_1(t_0) = A \sin z + C \cos y, \quad u_2(t_0) = B \sin x + A \cos z, \quad u_3(t_0) = C \sin y + B \cos x \qquad (4.1)$$

If we remember the originating of denotation for the components of velocity field:

$$u_1(t_0) = \frac{dx}{dt}, \quad u_2(t_0) = \frac{dy}{dt}, \quad u_3(t_0) = \frac{dz}{dt},$$

- it should yield a system of ordinary differential equations as below

$$\begin{cases} \dfrac{d\,x}{d\,t} = A\sin z + C\cos y, \\[2ex] \dfrac{d\,y}{d\,t} = B\sin x + A\cos z, \\[2ex] \dfrac{d\,z}{d\,t} = C\sin y + B\cos x, \end{cases} \qquad (4.2)$$

- which is proved to have not an analytical solutions, but moreover it reveals a dynamical chaos among the trajectories of appropriate solutions of such a system [8-9].

The ansatz in this derivation let us generalize the idea of extending such a steady helical solutions to the viscous unsteady case at *any* given initial conditions for velocity fields (*it means an open choice for the space part of a solution*); so, it let us obtain for the space part, corresponding to the steady case of ABC-flow (4.1)-(4.2), as below:

$$\begin{cases} \dfrac{d\,x}{d\,t} = \exp(-\,\nu\cdot\alpha^2\cdot t)\cdot(A\sin z + C\cos y), \\[2ex] \dfrac{d\,y}{d\,t} = \exp(-\,\nu\cdot\alpha^2\cdot t)\cdot(B\sin x + A\cos z), \\[2ex] \dfrac{d\,z}{d\,t} = \exp(-\,\nu\cdot\alpha^2\cdot t)\cdot(C\sin y + B\cos x), \end{cases} \qquad (4.3)$$

Should the system (4.3) also yield a dynamical chaos for the trajectories of the non-stationary solutions?

To answer, we should comment first this important particular case, i.e. the Arnold-Beltrami-Childres flow.
The ABC flow, a three-parameter velocity field that provides a simple stationary solution of Euler's equations in three dimensions for incompressible, inviscid fluid flows, can be considered to be a prototype for the study of turbulence - the ABC-flow provides a simple example of dynamical chaos, in spite of the simple analytical expression for each of the components of a solution [8].

In our case (3.4)-(3.5), it means that we need some boundary conditions that preserve helical the solution inside of the limited domain: indeed, the space part of the components of pressure gradient field (3.5) should strongly depend on the space part of the initial conditions for the components (3.4) of velocity $u_1(t_0)$, $u_2(t_0)$, $u_3(t_0)$.
Nevertheless, the possible existence of a dynamical chaos for the solutions of (4.3) could mean that such a solutions could be considered as unstable with respect to a small perturbation at the meanings of the boundary conditions.
Also we should note that since the fluid is incompressible for the development above, there is a strong link between boundary conditions and the solution inside. Due to this link, at least for the small values of time-parameter t, the space part of a solution (3.4) or (4.3) should strongly depend on the boundary conditions.

5. Conclusion.

In fluid mechanics, a lot of authors have been executing their researches to obtain the analytical solutions of Navier-Stokes equations [10], even for 3D case of *compressible* gas flow [11]. But there is an essential deficiency of non-stationary solutions indeed.
In our presentation, we explore the case of non-stationary *helical* flow (where vorticity is proportional to the flow velocity) of the Navier-Stokes equations for incompressible fluids at *any* given initial conditions for velocity fields (*it means an open choice for the space part of a solution*).

Such a non-stationary *helical* flow is proved to be decreasing exponentially in regard to the time-parameter (3.4), the extent of time-dependent exponential component is given by the coefficient of kinematic viscosity, multiplied by the square of the coefficient of proportionality between the vorticity and velocity field.

As we know, ABC-ansatz [8-9] was published as a solution of the steady problem. In their work, they never realized that its extension is possible also to the non-stationary problem. It was extended for the first time in [12] to the non-stationary problem as a time-kinematic viscosity decaying solution; also we should mention the comprehensive article [7] in regard to the non-stationary *helical* flows with the special kind of the space part for the velocity fields as well as the appropriate pressure gradient field.

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