On the Origins of Physical Fields

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Abstract

By starting from an infinite dimensional quaternionic separable Hilbert space and its companion Gelfand triple as a base model, the paper uses the capabilities and the restrictions of this model in order to investigate the origins of some basic fields. A special method is introduced that generates normal operators and parameter spaces from existing quaternionic number systems. The same method is used to relate functions that use these parameter spaces with corresponding normal operators. Continuum eigenspaces of operators that reside in the Gelfand triple will represent the investigated basic fields.

The paper exploits all known aspects of the quaternionic number system and it uses quaternionic differential calculus as well as Maxwell based differential calculus. This is done in order to investigate the properties and behavior of the investigated basic fields. The two toolkits offer different views on the basic fields. The views do not affect the behavior of the investigated field.

Maxwell based differential calculus is commonly used to investigate the electromagnetic field, but the application of this set of equations is certainly not restricted to the EM field. The EM field is compared with another basic field that acts as a background embedding continuum. Both basic fields can be investigated by the Maxwell based equations and by the quaternion differential calculus. It will be shown that the fundamental difference between the basic fields is located in the artifacts that vibrate and deform these basic fields.

The paper produces an algorithm that calculates the electric charge and color charge of elementary objects from the symmetry properties of their local parameter spaces. The electric charges generate the electric field. Also the spin of elementary objects is considered.

Via the analysis of the Dirac equation the paper indicates that the wave equation is the result of the coupling between two first order differential equations that hold for different parameter spaces.

The paper reveals that dedicated mechanism control the coherence of the dynamics of the fields.

The behavior of photons and dynamic electromagnetic fields is used in order to investigate the long range behavior of these fields. The paper shows that the usual interpretation of a photon as an electric wave is not correct. In addition the relation between gravitation and inertia is explained.

If the paper introduces new science, then it has fulfilled its purpose.

Keywords

Quantum physics, Mathematical model, Electric charge, Electric field, Gravity, Embedding field, Photon, Wave equation, Quaternionic Hilbert space, Quaternionic differential calculus, Symmetry flavor, Reverse bra-ket method.

Subject areas

Quantum physics, quaternions, field theory, Maxwell equations.

Contents

1 Introduction

Indications suggest that electrical charges are properties of space. The major indication is the fact that quaternionic number systems exist in several versions that differ in their symmetry properties. These symmetry properties are related to the way that these versions are ordered. Ordered number systems are often used as parameter spaces. This paper introduces the reverse bra-ket method, which can relate parameter spaces and functions that use these parameter spaces to operators, which reside in Hilbert spaces. Apart from a countable number of local exceptions, the function is supposed to be continuous. In reverse, not all operators can be related to a pair consisting of a parameter space and a function that uses this parameter space.

As a consequence of the possibility to define operators from a combination of a function and its parameter space, it makes sense to introduce the notion of types of spaces where each type has its own symmetry flavor. An important category of these spaces are symmetry centers. Symmetry centers float on a covering background space that has its own symmetry flavor.

Within a separable Hilbert space such types of spaces can coexist as eigenspaces of corresponding types of quaternionic operators. That is why we will use an infinite dimensional separable quaternionic Hilbert space $\mathfrak{H}$ as part of our base model. Each infinite dimensional separable Hilbert space owns a companion Gelfand triple $\mathcal{H}$, which is a non-separable Hilbert space. In the separable Hilbert space $\mathfrak{H}$ the eigenspaces of operators are countable. In the Gelfand triple $\mathcal{H}$ the eigenspaces of operators can be continuums. Together, the two Hilbert spaces form the base model.

In the separable quaternionic Hilbert space we introduce the concept of ***well-ordered normal reference operators***. A well-ordered normal reference operator offers eigenvalues that have unique real parts. The eigenvalues can then be ordered with respect to the values of their real parts. We will define a well-ordered reference operator $\mathcal{R}$ whose eigenspace $\mathcal{R}$ acts as a model-wide quaternionic parameter space. The well-ordered reference operator $\mathcal{R}$ that provides the countable parameter space $\mathcal{R}$ in the separable Hilbert space $\mathfrak{H}$ owns a companion reference operator $\mathfrak{R}$ in the Gelfand triple $\mathcal{H}$ that provides a quaternionic continuum eigenspace $\mathfrak{R}$. That eigenspace will be used as the natural parameter space of the quaternionic functions that define operators in the Gelfand triple.

Fields will appear as continuum eigenspaces of normal operators that reside in the Gelfand triple. We will show that a category of fields can be defined by using the reverse bra-ket method and quaternionic functions that use the eigenspace of the reference operator $\mathfrak{R}$ as their parameter space. This procedure relates the function and the parameter space to a unique normal operator that resides in the Gelfand triple. The same methodology can also be applied in the separable Hilbert space.

Symmetry centers reside in the separable Hilbert space and are maintained in finite dimensional subspaces. Symmetry centers exist in a small number of types that differ in the corresponding symmetry flavor. Corresponding normal operators $\mathfrak{S}_n^x$ map these subspaces onto themselves. Superscript ^{x} refers to the type dependent properties of the symmetry center. Subscript n enumerates the individual symmetry centers. The center location of the symmetry center corresponds to the value of a quaternionic mapping function of its quaternionic location in the parameter space that is defined via the well-ordered reference operator $\mathcal{R}$ and its companion $\mathfrak{R}$. That value is a location in a background continuum $\mathfrak{C}$. $\mathfrak{R}$ is the parameter space of the quaternionic function $\mathfrak{C}(q)$ that defines continuum $\mathfrak{C}$ and its corresponding operator $\mathfrak{C}$.

The wave equation appears to be the result of the coupling of two first order partial differential equations for which the subjected functions each use a parameter space that differs from the parameter space of the other function.

2 Quaternions

Quaternions can be interpreted as combinations of a real scalar and a three dimensional real vector [1]. The vector forms the imaginary part of the quaternion. The combination supports numeric arithmetic. The vector part introduces a non-commutative multiplication.

We will indicate the real part of quaternion a by subscripted a_0 and the vector part will be put in bold font face $\boldsymbol{a}$.

$$a = a_0 + \boldsymbol{a} \tag{1}$$

a^* is the quaternionic conjugate of a.

$$a^* = a_0 - \boldsymbol{a} \tag{2}$$

The sum of two quaternions is defined by:

$$c = c_0 + \boldsymbol{c} = a + b \tag{3}$$

$$c_0 = a_0 + b_0 \tag{4}$$

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} \tag{5}$$

The product rule is defined by:

$$c = a\ b = (a_0 + \boldsymbol{a})(b_0 + \boldsymbol{b}) = a_0\ b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0\boldsymbol{b} + b_0\boldsymbol{a} \pm \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

$$c_0 = a_0\ b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \tag{7}$$

$$\boldsymbol{c} = a_0\boldsymbol{b} + b_0\boldsymbol{a} \pm \boldsymbol{a} \times \boldsymbol{b} \tag{8}$$

$\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the inner vector product. $\boldsymbol{a} \times \boldsymbol{b}$ is the outer vector product. The $\pm$ sign signalizes the choice between a right handed and a left handed external vector product. This choice indicates that quaternionic number systems exist in multiple versions. Due to the four dimensions of quaternions will quaternionic number system exist in sixteen different symmetry flavors. This is treated in more detail in the section about symmetry flavors. The handedness depends on the symmetry flavor.

$$d = (a\ b)^* = b^*\ a^* = c^* = a_0\ b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle - a_0\boldsymbol{b} - b_0\boldsymbol{a} \pm \boldsymbol{b} \times \boldsymbol{a} \tag{9}$$

The norm of a quaternion is defined by:

$$|a| = \sqrt{aa^*} = \sqrt{a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle} \tag{10}$$

The norm of a quaternionic function is defined by:

$$\|f\| = \int_q f(q) f^*(q)\ dq \tag{11}$$

3 Quaternionic Hilbert spaces

Separable Hilbert spaces are linear vector spaces in which an inner product is defined. This inner product relates each pair of Hilbert vectors. The value of that inner product must be a member of a division ring. Suitable division rings are real numbers, complex numbers and quaternions. This paper uses quaternionic Hilbert spaces [2][3][4].

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits [5].

$$\langle x|y\rangle = \langle y|x\rangle^* \tag{1}$$

$$\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle \tag{2}$$

$$\langle \alpha x|y\rangle = \alpha\, \langle x|y\rangle \tag{3}$$

$$\langle x|\alpha y\rangle = \langle x|y\rangle\, \alpha^* \tag{4}$$

$\langle x|$ is a bra vector. $|y\rangle$ is a ket vector. α is a quaternion.

This paper considers Hilbert spaces as no more and no less than structured storage media for dynamic geometrical data that have an Euclidean signature. Quaternions are ideally suited for the storage of such data. Quaternionic Hilbert spaces are described in "Quaternions and quaternionic Hilbert spaces" [6].

The operators of separable Hilbert spaces have countable eigenspaces. Each infinite dimensional separable Hilbert space owns a Gelfand triple. The Gelfand triple embeds this separable Hilbert space and offers as an extra service operators that feature continuums as eigenspaces. In the corresponding subspaces the definition of dimension loses its sense.

3.1 Tensor products

The tensor product of two quaternionic Hilbert spaces is a real Hilbert space [7]. For that reason this model does not apply tensor products. As a consequence Fock spaces are not applied in this paper.

Instead the paper represents the whole model by a single infinite dimensional separable quaternionic Hilbert space and its companion Gelfand triple. Elementary objects and their composites will be represented by subspaces of the separable Hilbert space. Their local living spaces coexist as eigenspaces of dedicated operators.

3.2 Representing continuums and continuous functions

Operators map Hilbert vectors onto other Hilbert vectors. Via the inner product the operator T may be linked to an adjoint operator $T^\dagger$.

$$\langle Tx|y\rangle \equiv \langle x|T^\dagger y\rangle \tag{1}$$

$$\langle Tx|y\rangle = \langle y|Tx\rangle^* = \langle T^\dagger y|x\rangle^* \tag{2}$$

A linear quaternionic operator T, which owns an adjoint operator $T^\dagger$ is normal when

$$T^\dagger T \;=\; T\, T^\dagger \tag{3}$$

$T_0 = (T + T^\dagger)/2$ is a self adjoint operator and $\boldsymbol{T} = (T - T^\dagger)/2$ is an imaginary normal operator. Self adjoint operators are also Hermitian operators. Imaginary normal operators are also anti-Hermitian operators.

By using what we will call ***reverse bra-ket notation***, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined by starting from an orthonormal base

of vectors. In this base the vectors are normalized and are mutually orthogonal. The vectors span a subspace of the Hilbert space. We will attach eigenvalues to these base vectors via the ***reverse bra-ket notation***. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let $\{q_i\}$ be the set of rational quaternions in a selected quaternionic number system and let $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R} = |q_i\rangle q_i \langle q_i|$. Here we enumerate the base vectors with index i.

$$\mathcal{R} \equiv |q_i\rangle q_i \langle q_i| \tag{4}$$

$\mathcal{R}$ is the configuration parameter space operator.

This notation must not be interpreted as a simple outer product between a ket vector $|q_i\rangle$, a quaternion q_i and a bra vector $\langle q_i|$. It involves a complete set of eigenvalues $\{q_i\}$ and a complete orthomodular set of Hilbert vectors $\{|q_i\rangle\}$. It implies a summation over these constituents, such that for all bra's $\langle x|$ and ket's $|y\rangle$:

$$\langle x|\mathcal{R}\, y\rangle = \sum_i \langle x|q_i\rangle q_i \langle q_i|y\rangle \tag{5}$$

$\mathcal{R}_0 = \left(\mathcal{R} + \mathcal{R}^\dagger\right)/2$ is a self-adjoint operator. Its eigenvalues can be used to arrange the order of the eigenvectors by enumerating them with the eigenvalues. The ordered eigenvalues can be interpreted as ***progression values***.

$\boldsymbol{\mathcal{R}} = \left(\mathcal{R} - \mathcal{R}^\dagger\right)/2$ is an imaginary operator. Its eigenvalues can also be used to order the eigenvectors. The eigenvalues can be interpreted as ***spatial values*** and can be ordered in several ways.

Let $f(q)$ be a mostly continuous quaternionic function. Now the reverse bra-ket notation defines operator f as:

$$f \equiv |q_i\rangle f(q_i) \langle q_i| \tag{6}$$

f defines a new operator that is based on function $f(q)$. Here we suppose that the target values of f belong to the same version of the quaternionic number system as its parameter space does.

Operator f has a countable set of discrete quaternionic eigenvalues.

For this operator the reverse bra-ket notation is a shorthand for

$$\langle x|f\, y\rangle = \sum_i \langle x|q_i\rangle f(q_i) \langle q_i|y\rangle \tag{7}$$

In a non-separable Hilbert space, such as the Gelfand triple, the continuous function $\mathcal{F}(q)$ can be used to define an operator, which features a continuum eigenspace.

$$\mathcal{F} = |q\rangle \mathcal{F}(q) \langle q| \tag{8}$$

Via the continuous quaternionic function $\mathcal{F}(q)$, the operator $\mathcal{F}$ defines a curved continuum $\mathcal{F}$. This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

$$\mathfrak{R} = |q\rangle q\langle q| \tag{9}$$

The function $\mathcal{F}(q)$ uses the eigenspace of the reference operator $\mathfrak{R}$ as a flat parameter space that is spanned by a quaternionic number system $\{q\}$. The continuum $\mathcal{F}$ represents the target space of function $\mathcal{F}(q)$.

Here we no longer enumerate the base vectors with index i. We just use the name of the parameter. If no conflict arises, then we will use the same symbol for the defining function, the defined operator and the continuum that is represented by the eigenspace.

For the shorthand of the reverse bra-ket notation of operator $\mathcal{F}$ the integral over q replaces the summation over q_i.

$$\langle x|\mathcal{F}\ y\rangle = \int_q \langle x|q\rangle\mathcal{F}(q)\langle q|y\rangle\ dq \tag{10}$$

Remember that quaternionic number systems exist in several versions, thus also the operators f and $\mathcal{F}$ exist in these versions. The same holds for the parameter space operators. When relevant, we will use superscripts in order to differentiate between these versions.

Thus, operator $f^x = |q_i^x\rangle f^x(q_i^x)\langle q_i^x|$ is a specific version of operator f. Function $f^x(q_i^x)$ uses parameter space $\mathcal{R}^x$.

Similarly, $\mathcal{F}^x = |q^x\rangle\mathcal{F}^x(q^x)\langle q^x|$ is a specific version of operator $\mathcal{F}$. Function $\mathcal{F}^x(q^x)$ and continuum $\mathcal{F}^x$ use parameter space $\mathfrak{R}^x$. If the operator $\mathcal{F}^x$ that resides in the Gelfand triple $\mathcal{H}$ uses the same defining function as the operator $\mathcal{F}^x$ that resides in the separable Hilbert space, then both operators belong to the same quaternionic ordering version.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space $\mathcal{H}$ can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space $\mathfrak{H}$. Both representations use a flat parameter space $\mathfrak{R}^x$ or $\mathcal{R}^x$ that is spanned by quaternions. $\mathcal{R}^x$ is spanned by rational quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues $\{q_i^x\}$ that occur as eigenvalues of the reference operator $\mathcal{R}^x$ in the separable Hilbert space map onto the rational quaternionic eigenvalues $\{q_i^x\}$ that occur as subset of the quaternionic eigenvalues $\{q^x\}$ of the reference operator $\mathfrak{R}^x$ in the Gelfand triple. In this way the reference operator $\mathcal{R}^x$ in the infinite dimensional separable Hilbert space $\mathfrak{H}$ relates directly to the reference operator $\mathfrak{R}^x$, which resides in the Gelfand triple $\mathcal{H}$.

All operators that reside in the Gelfand triple and are defined via a mostly continuous quaternionic function have a representation in the separable Hilbert space.

3.3 Stochastic operators

Stochastic operators do not get their data from a continuous quaternionic function. Instead a stochastic process delivers the eigenvalues. Again these eigenvalues are quaternions and the real parts of these quaternions can be interpreted as progression values. The generated eigenvalues are picked from a selected parameter space.

Stochastic operators only act in a step-wise fashion. Their eigenspace is countable. Stochastic operators may act in a cyclic fashion.

The mechanisms that control the stochastic operator can synchronize the progression values with the model wide progression that is set by a selected reference operator.

Characteristic for stochastic operators is that the imaginary parts of the eigenvalues are not smooth functions of the real values of those eigenvalues.

3.3.1 Density operators

The eigenspace of a stochastic operator may be characterized by a continuous spatial density distribution. In that case the corresponding stochastic process must ensure that this continuous density distribution fits. The density distribution can be constructed afterwards or after each regeneration cycle. Constructing the density distribution involves a reordering of the imaginary parts of the produced eigenvalues. This act will usually randomize the real parts of those eigenvalues. A different operator can then use the continuous density distribution in order to generate its functionality. The old real parts of the eigenvalues may then reflect the reordering. The construction of the density distribution is a pure administrative action that is performed as an aftermath. The constructed density operator represents a continuous function and may reside both in the separable Hilbert space and in the Gelfand triple. The construction of the density function involves a selected parameter space. That parameter space need not be the same as the parameter space from which the stochastic process picked its eigenvalues.

3.4 Notations

The reverse bra-ket notation enables the definition of some special operators that play an unique role in the model. We will reserve special symbols for these operators and we will also use special symbols in order to distinguish separable from non-separable Hilbert spaces.

Symbol	Meaning	Applied in	As
$\mathfrak{H}$	Separable Hilbert space	Model	Structured storage
$\mathcal{H}$	Non-separable Hilbert space, Gelfand triple	Model	Structured storage
$\mathcal{R}$	Reference operator	$\mathfrak{H}$	Parameter space
$\mathfrak{R}$	Reference operator	$\mathcal{H}$	Parameter space
$\mathfrak{C}$	Embedding continuum operator	$\mathcal{H}$	Field, function
$\mathfrak{A}$	Symmetry related field operator	$\mathcal{H}$	Field, function
$\mathfrak{S}$	Symmetry center operator	$\mathfrak{H}$	Floating parameter space
$\mathscr{o}$	Coherent swarm operator	$\mathfrak{H}$	Dynamic location distribution
$\mathscr{b}$	Mapped coherent swarm operator	$\mathfrak{H}$	Dynamic location distribution
ρ	Density operator	$\mathcal{H}$	Density function

The defining function in the reverse bra-ket notation enables the definition of operators in both the separable Hilbert space $\mathfrak{H}$ and in the Gelfand triple $\mathcal{H}$. Still different symbols are used for reference operators $\mathfrak{R}$ and $\mathcal{R}$.

$\mathscr{o}$ is a stochastic operator. $\mathscr{b}$ maps the eigenspace of $\mathscr{o}$ in parameter space $\mathcal{R}$. ρ is the corresponding density operator.

4 Change of base

In quaternionic Hilbert space a change of base can be achieved by:

$$\langle x|\tilde{\mathcal{F}}\ y\rangle = \int_{\tilde{q}} \langle x\ |\tilde{q}\rangle \left\{ \int_{q} \langle \tilde{q}|q\rangle \mathcal{F}(q)\langle q|\tilde{q}\rangle\ dq \right\} \langle \tilde{q}|y\rangle\, d\tilde{q} \tag{1}$$

$$= \int_{\tilde{q}} \langle x|\tilde{q}\rangle \tilde{\mathcal{F}}(\tilde{q})\langle \tilde{q}|y\rangle\ d\tilde{q}$$

$$\tilde{\mathcal{F}}(\tilde{q}) = \int_{q} \langle \tilde{q}|q\rangle \mathcal{F}(q)\langle q|\tilde{q}\rangle\ dq \tag{2}$$

$$\tilde{\mathfrak{R}}(\tilde{q}) = \int_{q} \langle \tilde{q}|q\rangle q\langle q|\tilde{q}\rangle\ dq \tag{3}$$

$$\langle x|\tilde{\mathfrak{R}}\ y\rangle = \int_{\tilde{q}} \langle x|\tilde{q}\rangle \tilde{\mathfrak{R}}(\tilde{q})\langle \tilde{q}|y\rangle\ d\tilde{q} \tag{4}$$

$$\tilde{\mathfrak{R}} = |\tilde{q}\rangle \tilde{q} \langle \tilde{q}| \tag{5}$$

However, as we see in the formulas this method merely achieves a rotation of parameter spaces and functions. In the complex number based Hilbert space it would achieve no change at all.

4.1 Fourier transform

A Fourier transform uses a different approach. It is not a direct transform between parameter spaces, but instead it is a transform between sets of mutually orthogonal functions, which are formed by inner products, which are related to different parameter spaces. The quaternionic

Fourier transform exists in three versions. The first two versions have a reverse Fourier transform.

The left oriented Fourier transform is defined by:

$$\tilde{\mathcal{F}}_L(\tilde{q}_L) = \int_q \langle \tilde{q}_L | q \rangle \, \mathcal{F}(q) \; dq \tag{1}$$

Like the functions $\langle q|q'\rangle$ and $\langle \tilde{q}_L|\tilde{q}'_L\rangle$, the functions $\langle \tilde{q}_L|q\rangle$ and $\langle q|\tilde{q}_L\rangle$ form sets of mutually orthogonal functions, as will be clear from:

$$\langle q|q'\rangle = \delta(q-q') \tag{2}$$

$$\langle \tilde{q}_L|\tilde{q}'_L\rangle = \delta(\tilde{q}_L - \tilde{q}'_L) \tag{3}$$

$$\int_{\tilde{q}_L} \langle q'|\tilde{q}_L\rangle\langle \tilde{q}_L|q\rangle \; d\tilde{q}_L = \delta(q-q') \tag{4}$$

$$\int_q \langle \tilde{q}'_L|q\rangle\langle q|\tilde{q}_L\rangle \; dq = \delta(\tilde{q}_L - \tilde{q}'_L) \tag{5}$$

The reverse transform is:

$$\mathcal{F}(q) = \int_{\tilde{q}_L} \langle q|\tilde{q}_L\rangle \tilde{\mathcal{F}}_L(\tilde{q}_L) \, d\tilde{q}_L = \int_{\tilde{q}_L}\int_{q'} \langle q|\tilde{q}_L\rangle\langle \tilde{q}_L|q'\rangle \mathcal{F}(q') \, d\tilde{q}_L \, dq' \tag{6}$$

$$= \int_{q'} \left\{ \int_{\tilde{q}_L} \langle q|\tilde{q}_L\rangle\langle \tilde{q}_L|q'\rangle \, d\tilde{q}_L \right\} \mathcal{F}(q') \, dq' = \int_{q'} \delta(q-q')\mathcal{F}(q') \, dq'$$

The reverse bra-ket form of the operator $\tilde{\mathcal{F}}_L$ equals:

$$\tilde{\mathcal{F}}_L = |\tilde{q}_L\rangle \tilde{\mathcal{F}}_L(\tilde{q}_L)\langle \tilde{q}_L| \tag{7}$$

Operator $\tilde{\mathfrak{R}}_L$ provides the parameter space for the left oriented Fourier transform $\tilde{\mathcal{F}}_L(\tilde{q}_L)$ of function $\mathcal{F}(q)$ in equations (1) and (6).

$$\tilde{\mathfrak{R}}_L = |\tilde{q}_L\rangle \tilde{q}_L \langle \tilde{q}_L| \tag{8}$$

Similarly the right oriented Fourier transform can be defined.

$$\tilde{\mathcal{F}}_R(\tilde{q}) = \int_q \mathcal{F}(q')\langle q'|\tilde{q}\rangle \; dq' \tag{9}$$

The reverse transform is:

$$\mathcal{F}(q) = \int_{\tilde{q}_R} \tilde{\mathcal{F}}_R(\tilde{q}_R)\langle q|\tilde{q}_R\rangle \, d\tilde{q}_R = \int_{\tilde{q}_R}\int_{q'} \mathcal{F}(q')\langle q'|\tilde{q}_R\rangle\langle \tilde{q}_R|q\rangle \, dq' \, d\tilde{q}_R \tag{10}$$

$$= \int_{q'} \mathcal{F}(q') \left\{ \int_{\tilde{q}_R} \langle q'|\tilde{q}_R\rangle\langle\tilde{q}_R|q\rangle \; d\tilde{q}_R \right\} dq' = \int_{q'} \mathcal{F}(q') \; \delta(q - q') \; dq'$$

Also here the functions $\langle q|q'\rangle$, $\langle\tilde{q}_R|\tilde{q}'_R\rangle$, $\langle\tilde{q}_R|q\rangle$ and $\langle q|\tilde{q}_R\rangle$ form sets of mutually orthogonal functions.

The reverse bra-ket form of the operator $\tilde{\mathcal{F}}_R$ equals:

$$\tilde{\mathcal{F}}_R = |\tilde{q}_R\rangle\tilde{\mathcal{F}}_R(\tilde{q}_R)\langle\tilde{q}_R| \tag{11}$$

Operator $\tilde{\mathfrak{R}}_R$ provides the parameter space for the right oriented Fourier transform $\tilde{\mathcal{F}}_R(\tilde{q}_R)$ of function $\mathcal{F}(q)$ in equations (9) and (10).

$$\tilde{\mathfrak{R}}_R = |\tilde{q}_R\rangle\tilde{q}_R\langle\tilde{q}_R| \tag{12}$$

The third version of the Fourier transform is:

$$\tilde{\mathcal{F}}(\tilde{q}_L, \tilde{q}_R) = \frac{\tilde{\mathcal{F}}_L(\tilde{q}_L) + \tilde{\mathcal{F}}_R(\tilde{q}_R)}{2} = ½ \int_q \{\langle\tilde{q}_L|q\rangle\mathcal{F}(q) + \mathcal{F}(q)\langle q|\tilde{q}_R\rangle\} \; dq \tag{13}$$

In contrast to the right and left version, the third version has no reverse.

5 Well-ordered reference operators

The eigenvalues of a normal operator T that resides in a separable Hilbert space can be ordered with respect to the real part of the eigenvalues. Operator $T_0 = (T + T^\dagger)/2$ is the corresponding self-adjoint operator. If each real value occurs only once, then the operator T and its adjoint $T^\dagger$ can be well-ordered. The imaginary part of the eigenvalues can then still be ordered in different ways. Operator $\boldsymbol{T} = (T - T^\dagger)/2$ is the corresponding anti-Hermitian operator. For example it can be ordered according to Cartesian coordinates or according to spherical coordinates. Also each of these orderings can be done in different ways.

The reference operators $\mathcal{R}^x$ will all be considered to be well-ordered. All reference operators $\mathcal{R}^x$ share the same Hermitian part $\mathcal{R}_0$. The property of being well-ordered is restricted to operators with countable eigenspaces. However, via the defining functions, the well-orderedness can be transferred to the corresponding operator in the Gelfand triple. Thus the reference operators $\mathfrak{R}^x$ that reside in the Gelfand triple will also be well–ordered.

If a normal operator that is defined by using the reverse bra-ket notation and a continuous function that uses a well-ordered parameter space, then we qualify this operator as being indirectly well-ordered. This qualification is independent of the fact that the target values of the function are no longer well-ordered.

5.1.1 Progression ordering

A single self-adjoint reference operator that offers an infinite set of rational eigenvalues can synchronize a ***category of well-ordered normal operators***. We use $\mathcal{R}_0$ for this purpose. The ordered eigenvalues of this self-adjoint operator act as progression values. In this way the infinite dimensional separable Hilbert space owns a model wide clock. With this choice the separable Hilbert space steps with model-wide progression steps.

The selected well-ordered normal reference operator $\mathcal{R}^{⓪}$ that resides in an infinite dimensional separable quaternionic Hilbert space is used in the specification of the companion quaternionic Gelfand triple. There it corresponds to reference operator $\mathfrak{R}^{⓪}$. In that way progression steps in the separable Hilbert space and flows in the companion Gelfand triple. Both reference operators will be used to provide parameter spaces. We will often omit the superscript for the reference operators $\mathcal{R}^{⓪}$ and $\mathfrak{R}^{⓪}$.

The countable set of progression values of the Hermitian part $\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$ of the well-ordered reference operator $\mathcal{R}$ can be used to enumerate other ordered sequences.

5.1.2 Cartesian ordering

The whole separable Hilbert space can at the same time be spanned by the eigenvectors of a reference operator whose eigenvalues are well-ordered with respect to the real parts of the eigenvalues, while the imaginary parts are ordered with respect to a Cartesian coordinate system.

For Cartesian ordering, having an origin is not necessary. In affine Cartesian ordering only the direction of the ordering is relevant. Affine Cartesian ordering exists in eight symmetry flavors.

Affine and normal Cartesian ordering suppose a unique orientation of the Cartesian axes.

The well-ordered reference operator $\mathcal{R}$ is supposed to feature affine Cartesian ordering.

5.1.3 Spherical ordering

Spherical ordering starts with a selected Cartesian set of coordinates. In this case the origin is at a unique center location. Spherical ordering can be done by first ordering the azimuth and after that the polar angle is ordered. Finally, the radial distance from the center can be ordered. Another procedure is to start with the polar angle, then the azimuth and finally the radius. Such, spherical orderings may create a ***symmetry center***. Since the ordering starts with a selected Cartesian coordinate system, spherical ordering will go together with affine Cartesian ordering.

Each symmetry center is described by the eigenspaces of an anti-Hermitian operator $\mathfrak{S}_n^x$ that map a finite dimensional subspace of Hilbert space $\mathfrak{H}$ onto itself. Superscript ^{x} refers to the ordering type of the symmetry center. Subscript n enumerates the symmetry centers. If there is no reason for confusion, then this subscript will be omitted. $\mathfrak{S}_n^x$ has no Hermitian part. Only through its ordering it can synchronize with progression steps.

6 Symmetry flavor

Quaternions can be mapped to Cartesian coordinates along the orthonormal base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; with $\boldsymbol{ij} = \boldsymbol{k}$

Due to the four dimensions of quaternions, quaternionic number systems exist in 16 well-ordered versions $\{q^x\}$ that differ only in their discrete Cartesian symmetry set. The quaternionic number systems $\{q^x\}$ correspond to 16 versions $\{q_i^x\}$ of rational quaternions.

Half of these versions are right handed and the other half are left handed. Thus the handedness is influenced by the symmetry flavor.

The superscript ^{x} can be ^{⓪}, ^{①}, ^{②}, ^{③}, ^{④}, ^{⑤}, ^{⑥}, ^{⑦}, ^{⑧}, ^{⑨}, ^{⑩}, ^{⑪}, ^{⑫}, ^{⑬}, ^{⑭}, or ^{⑮}.

This superscript represents the ***symmetry flavor*** of the superscripted subject.

The reference operator $\mathcal{R}^{⓪} = \left|q_i^{⓪}\right\rangle q_i^{⓪} \left\langle q_i^{⓪}\right|$ in separable Hilbert space $\mathfrak{H}$ maps into the reference operator $\mathfrak{R}^{⓪} = \left|q^{⓪}\right\rangle q^{⓪} \left\langle q^{⓪}\right|$ in Gelfand triple $\mathcal{H}$.

The symmetry flavor of the symmetry center $\mathfrak{S}^x$, which is maintained by operator $\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x|$ is determined by its Cartesian ordering and then compared with the reference symmetry flavor, which is the symmetry flavor of the reference operator $\mathcal{R}^{⓪}$. ⬆⬆⬆⬆

Now the symmetry related charge follows in three steps.

1. Count the difference of the spatial part of the symmetry flavor of $\mathfrak{S}^x$ with the spatial part of the symmetry flavor of reference operator $\mathcal{R}^{⓪}$.
2. If the handedness changes from **R** to **L**, then switch the sign of the count.
3. Switch the sign of the result for anti-particles.

Symmetry flavor					
Ordering x y z τ	Super script	Handedness Right/Left	Color charge	Electric charge * 3	Symmetry center type. Names are taken from the standard model
⬆⬆⬆⬆	⓪	**R**	N	+0	neutrino
⬇⬆⬆⬆	①	**L**	R	−1	down quark
⬆⬇⬆⬆	②	**L**	G	−1	down quark
⬇⬇⬆⬆	③	**L**	B	−1	down quark

	④	**R**	B	+2	up quark
	⑤	**R**	G	+2	up quark
	⑥	**R**	R	+2	up quark
	⑦	**L**	N	−3	electron
	⑧	**R**	N	+3	positron
	⑨	**L**	R	−2	anti-up quark
	⑩	**L**	G	−2	anti-up quark
	⑪	**L**	B	−2	anti-up quark
	⑫	**R**	B	+1	anti-down quark
	⑬	**R**	R	+1	anti-down quark
	⑭	**R**	G	+1	anti-down quark
	⑮	**L**	N	−0	anti-neutrino

Per definition, members of coherent sets $\{a_i^x\}$ of quaternions all feature the same symmetry flavor that is marked by superscript ^{x}.

Also continuous functions and continuums feature a symmetry flavor. Continuous quaternionic functions $\psi^x(q^x)$ and corresponding continuums do not switch to other symmetry flavors ^{y}.

The reference symmetry flavor $\psi^y(q^y)$ of a continuous function $\psi^x(q^y)$ is the symmetry flavor of the parameter space $\{q^y\}$.

If the continuous quaternionic function describes the density distribution of a set $\{a_i^x\}$ of discrete objects a_i^x, then this set must be attributed with the same symmetry flavor ^{x}. The real part describes the location density distribution and the imaginary part describes the displacement density distribution.

7 Symmetry centers

Each symmetry center corresponds to a dedicated subspace of the infinite dimensional separable Hilbert space. That subspace is spanned by the eigenvectors $\{|\mathfrak{s}_i^x\rangle\}$ of a corresponding symmetry center reference operator $\mathfrak{S}_n^x$. Here the superscript ^{x} refers to the type of the symmetry center. The subscript n enumerates the symmetry centers. The type covers more than just the symmetry flavor. We will often omit the subscript.

Symmetry flavors relate to affine Cartesian ordering. Each symmetry center will own a single symmetry flavor. The symmetry flavor of the symmetry center relates to the Cartesian coordinate system that acts as start for the spherical ordering. The combination of affine Cartesian ordering and spherical ordering puts corresponding axes in parallel. Spherical ordering relates to spherical coordinates. Starting spherical ordering with the azimuth corresponds to ***half integer spin***. The azimuth runs from 0 to π radians. Starting spherical ordering with the polar angle corresponds to ***integer spin***. The polar angle runs from 0 to 2π radians. These selections add to the type properties of the symmetry centers.

The model suggests that symmetry centers are maintained by special mechanisms that ensure the spatial and dynamical coherence of the coupling of the symmetry center to the background space. Several types of such mechanisms exist. Each symmetry center type corresponds to a mechanism type. ***These mechanisms are not part of the separable Hilbert space***.

Symmetry centers are ***resources*** where the coherence ensuring mechanisms can take dynamic locations that are stored in quaternionic eigenvalues of dedicated stochastic operators, in order

to generate coherent location swarms that represent point-like objects. The type of the point-like object corresponds to the type of the controlling mechanism.

The basic symmetry center is independent of progression. Once created, a symmetry center persists until it is annihilated. However, during creation its ordering can be synchronized with selected progression steps. Any progression dependence that concerns a symmetry center is handled by a type dependent mechanism. The type depends on the symmetry flavor and on the spin. Further, it depends on other characteristics that will not be treated in this paper, but that will appear as properties of the point-like object that will be supported by the controlling mechanism. An example is the generation flavor of the point-like particle. In this way the same symmetry center type can support electrons, muons and tau particles. Symmetry flavor and spin can be related to ordering of the symmetry center. Generation flavor is a property of the controlling mechanism.

The mechanisms that control the usage of symmetry centers act mostly in a cyclic fashion. When compared to mechanisms that care about particles, the cycles that occur in equivalent mechanisms that care about corresponding anti-particles act in the reverse direction. As a consequence many of the properties of the anti-particles are the opposite of the properties of the corresponding particles. This holds for the sign of the symmetry related charge and it holds for the color charge, but it does not hold for the mass and for the energy of the (anti)particle.

Symmetry centers have a well-defined spatial origin. That origin floats on the eigenspace of the reference operator $\mathcal{R}^{⓪}$. Symmetry centers are formed by a dedicated category of ***compact anti-Hermitian operators*** $\{\mathfrak{S}_n^x\}_n$.

An infinite dimensional separable Hilbert space can house a set of subspaces that each represent such a symmetry center. Each of these subspaces then corresponds to a dedicated spherically ordered reference operator $\mathfrak{S}_n^x$. The superscript ^{x} distinguishes between symmetry flavors and other properties, such as spin and generation flavor. Symmetry centers correspond to dedicated subspaces that are spanned by the eigenvectors $\{|\mathfrak{s}_i^x\rangle\}$ of the symmetry center reference operator $\mathfrak{S}^x$. (Here we omit subscript n).

$$\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x| \tag{1}$$

$$\mathfrak{S}^{x\dagger} = -\mathfrak{S}^x \tag{2}$$

Only the location of the center inside the eigenspace of reference operator $\mathcal{R}^{⓪}$ is a progression dependent value. This value is not eigenvalue of operator $\mathfrak{S}_n^x$. The location of the center inside $\mathcal{R}^{⓪}$ is eigenvalue of a central governance operator $\mathcal{g}$.

Symmetry centers feature a ***symmetry related charge*** that depends on the difference between the symmetry flavor of the symmetry center and the symmetry flavor of the reference operator $\mathcal{R}^{⓪}$, which equals the symmetry flavor of the embedding continuum $\mathfrak{C}$. The symmetry related charges raise a ***symmetry related field*** $\mathfrak{A}$. The symmetry related field $\mathfrak{A}$ influences the position of the center of the symmetry center in parameter space $\mathcal{R}^{⓪}$ and indirectly it influences the position of the map of the symmetry center into the field that represents the embedding continuum $\mathfrak{C}$. Both fields, $\mathfrak{A}$ and $\mathfrak{C}$ use the eigenspace of the reference operator $\mathfrak{R}$ as their parameter space.

The closed subspaces that correspond to a symmetry center have a fixed finite dimension. This dimension is the same for all types of symmetry centers. This ensures that symmetry related charges all appear in the same short list.

7.1 Synchronization via coupling

The basic symmetry center is independent of progression. Any progression dependence that concerns a symmetry center is handled by a type dependent mechanisms that controls the usage of the symmetry center. The type dependent mechanism acts in a progression dependent fashion. On certain progression steps the mechanism selects a location from the symmetry center that will be used to embed a point-like object in the background space.

The background space, is maintained by reference operator $\mathcal{R}$. Embedding the symmetry center into the eigenspace of this operator ensures the synchronization of the symmetry center with the background space. That is why the embedding occurs at instances that are selected from the progression values, which are offered as eigenvalues by $\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$. However, the controlling mechanism does not embed the center location, but instead the mechanism uses a stochastic process in order to select a location somewhere inside the symmetry center. Further, not all eigenvalues $\{\mathfrak{s}_i^x\}$ of $\mathfrak{S}_n^x$ will be used in the embedding process. A special operator $\mathcal{O}_n^x$ that is dedicated to the type of the embedded point-like object describes the selected locations in its eigenvalues. The eigenspace of operator $\mathcal{O}_n^x$ is mapped onto the eigenspace of $\mathcal{R}$. This converts operator $\mathcal{O}_n^x$ into operator $\mathcal{B}_n$. Operator $\mathcal{B}_n$ has an equivalent $\mathfrak{C}(\mathcal{B}_n)$ in the Gelfand triple. Function $\mathfrak{C}(q)$maps eigenvalues of $\mathcal{B}_n$ onto continuum $\mathfrak{C}$.

The embedding location represents a point-like object that resides in the symmetry center. That embedding location is mapped onto the embedding continuum, which resides as the eigenspace of operator $\mathfrak{C}$ in the Gelfand triple $\mathcal{H}$. This continuum is defined as a function $\mathfrak{C}(q)$ over parameter space $\mathfrak{R}$.

The locations in the symmetry center that for the purpose of the embedding are selected, form a coherent location swarm and a hopping path that characterize the dynamic behavior of the point-like object. The embedding process deforms continuum $\mathfrak{C}$.This embedding process is treated in more detail in [8].

8 Central governance

The eigenvalues of the central governance operator $\mathcal{g}$ administer the relative locations of the symmetry centers with respect to the reference operator $\mathcal{R}^{\textcircled{0}}$ which resides in the separable Hilbert space $\mathfrak{H}$ and maps to the reference continuum $\mathfrak{R}^{\textcircled{0}}$ in the Gelfand triple $\mathcal{H}$. A further map projects onto the embedding continuum $\mathfrak{C}$.The central governance operator $\mathcal{g}$ resides in the separable Hilbert space $\mathfrak{H}$. Operator $\mathcal{g}$ has an equivalent $\mathfrak{C}(\mathcal{g})$ in the Gelfand triple. Function $\mathfrak{C}(q)$maps eigenvalues of $\mathcal{g}$ onto continuum $\mathfrak{C}$.

The reference continuum $\mathfrak{R}^{\textcircled{0}}$ acts as a parameter space of the function $\mathfrak{A}(q)$ that specifies the symmetry related field $\mathfrak{A}$, which is eigenspace of the corresponding operator.

Each symmetry center owns a symmetry related charge, which is located at its geometric center. Each symmetry related charge owns an individual field φ that contributes to the overall symmetry related field $\mathfrak{A}$.

The reference continuum $\mathfrak{R}^{\textcircled{0}}$ also acts as a parameter space of the function $\mathfrak{C}(q)$ that specifies the embedding continuum $\mathfrak{C}$, which is eigenspace of the corresponding operator $\mathfrak{C}$.

A fundamental difference exists between field $\mathfrak{A}$ and field $\mathfrak{C}$. However both fields obey the same quaternionic differential calculus. The difference originates from the artifacts that cause the discontinuities of the fields. In the symmetry related field $\mathfrak{A}$ these artifacts are the symmetry related charges. In the embedding continuum $\mathfrak{C}$ these artifacts are the embedding events. What happens in not too violent conditions will be described by the homogeneous quaternionic second order partial differential equation and the Poisson equation of the corresponding field and will be affected by the local and current conditions. Since the elementary point-like objects reside inside their individual symmetry center, the embedding continuum will also be affected by what happens to the symmetry centers.

Double quaternionic differentiation of field $\mathfrak{A}$ shows the relation between $\mathfrak{A}$ and $\mathcal{g}$.

$$\nabla^{*}\nabla\,\mathfrak{A} = \mathcal{g} \tag{1}$$

Function $\mathfrak{C}(q)$ maps both φ and the eigenspace of $\mathcal{g}$ onto continuum $\mathfrak{C}$.

8.1 Embedding symmetry centers

The well-ordered eigenspace of a quaternionic normal operator $\mathcal{R}^{\textcircled{0}}$ that resides in an infinite dimensional separable Hilbert space acts as a reference operator from which the parameter space $\mathfrak{R}^{\textcircled{0}}$ of the embedding continuum $\mathfrak{C}$ will be derived. This parameter space resides as continuum eigenspace of a corresponding operator $\mathfrak{R}^{\textcircled{0}}$ in the Gelfand triple. This parameter space also acts as parameter space of a symmetry related field $\mathfrak{A}$. It is sparsely covered by locations of symmetry centers. The central governance operator $\mathcal{g}$ administers these locations. The symmetry centers contain symmetry related charges. The locations of these charges are influenced by the symmetry related field $\mathfrak{A}$.

It will appear that the defining function of field $\mathfrak{A}$ can be reformulated into a function that uses a different parameter space. The behavior of that new function can be described by Maxwell based differential calculus. This description can show the wave behavior that field $\mathfrak{A}$ does not show via quaternionic differential calculus.

9 Field dynamics

In this chapter we will use a switch $\circledast = \pm 1$ that selects between two different sets of differential calculus. One set concerns low order quaternionic differential calculus. The other set concerns Maxwell based differential calculus. The switch will be used to highlight the great similarity and the significant differences between these sets.

9.1 Differentiation

In the model that we selected, the dynamics of the fields can be described by quaternionic differential calculus. Apart from the eigenspaces of reference operators and the symmetry centers we encountered two fields that are defined by quaternionic functions and corresponding operators. One is the symmetry related field $\mathfrak{A}$ and the other is the embedding field $\mathfrak{C}$.

$\mathfrak{A}$ determines the dynamics of the symmetry centers. $\mathfrak{C}$ gets deformed and vibrated by the recurrent embedding of point-like elementary particles that each reside on an individual symmetry center.

Apart from the way that they are affected by point-like artifacts that disrupt the continuity of the field, both fields obey, under not too violent conditions and over not too large ranges, the same differential calculus. However, especially field $\mathfrak{A}$ is known to show wave behavior that cannot properly be described by quaternionic differential calculus. For that reason we will also investigate what a change of parameter space brings for the defining functions of the basic fields $\mathfrak{A}$ and $\mathfrak{C}$

9.2 Quaternionic differential calculus.

For quaternionic differential calculus the switch $\circledast$ equals 1.

Under rather general conditions the change of a quaternionic function $f(q)$ can be described by:

$$df(q) \approx \sum_{\mu=0\ldots3} \left\{ \frac{\partial f}{\partial q_\mu} + \sum_{\mu=0\ldots3} \frac{\partial}{\partial \nu} \frac{\partial f}{\partial q_\mu} dq^\nu \right\} dq^\mu = c_\mu(q) dq^\mu + c_{\mu\nu}(q) dq^\mu dq^\nu \tag{1}$$

Here the coefficients $c^\mu(q)$ and $c_{\mu\nu}(q)$ are full quaternionic functions. dq^μ are real numbers. e^ν are quaternionic base vectors.

The conditions that are treated by equation (1) still do not require more than second order differentiation. Thus, these conditions cannot be considered as general conditions!

Under more moderate and sufficiently short range conditions the function behaves more linearly.

$$df(q) \approx \sum_{\mu=0\ldots3} \frac{\partial f}{\partial q_\mu} dq^\mu = c_\mu(q) dq^\mu \tag{2}$$

Under even stricter conditions the functions become real functions $c_0^{\mu}(q)$ attached to quaternionic base vectors:

$$df(q) = c_0^{\tau}\, dq_{\tau} + c_0^{x}\, \boldsymbol{i}\, dq_x + c_0^{y}\, \boldsymbol{j}\, dq_y + c_0^{z}\, \boldsymbol{k}\, dq_z = c_0^{\mu}(q)\, e_{\mu}\, dq_{\mu} \tag{3}$$

$$= \sum_{\mu=0}^{3} \left(\sum_{\varsigma=0}^{3} \frac{\partial f^{\varsigma}}{\partial q_{\mu}} e_{\varsigma} \right) e_{\mu} dq^{\mu} = \sum_{\mu=0\ldots3}^{3} \Phi_{\mu} e_{\mu} dq^{\mu}$$

$$\Phi_{\mu} = c_0^{\mu} = \sum_{\varsigma=0}^{3} \frac{\partial f^{\varsigma}}{\partial q_{\mu}} e_{\varsigma} = \frac{\partial f^{\varsigma}}{\partial q_{\mu}} e_{\varsigma} = \frac{\partial f}{\partial q_{\mu}} \tag{4}$$

Thus, in a rather flat continuum we can use the quaternionic nabla ∇.

$$\nabla = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \frac{\partial}{\partial \tau} + \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} = \sum_{\mu=0}^{3} \frac{\partial f}{\partial q_{\mu}} e_{\mu} \tag{5}$$

$$\Phi = \Phi_0 + \boldsymbol{\Phi} = \nabla\psi = (\nabla_0 + \boldsymbol{\nabla})(\psi_0 + \boldsymbol{\psi}) \tag{5}$$

Thus, in a rather flat continuum we can use the quaternionic nabla ∇.

$$\nabla = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \frac{\partial}{\partial \tau} + \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{6}$$

$$\Phi = \Phi_0 + \boldsymbol{\Phi} = \nabla\psi = (\nabla_0 + \boldsymbol{\nabla})(\psi_0 + \boldsymbol{\psi}) \tag{7}$$

This form of the partial differential equation highlights the fact that in first order and second order partial differential equations ***the nabla operator can be applied as a multiplier***.

$$\Phi_0 = \nabla_0 \psi_0 - \circledast \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle \tag{8}$$

$$\boldsymbol{\Phi} = \nabla_0 \boldsymbol{\psi} + \boldsymbol{\nabla} \psi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{9}$$

These equations represent only low order partial differential equations. In this form the equations can still describe point-like disruptions of the continuity of the field.

$$\Phi^* = (\nabla\psi)^* = \nabla^*\psi^* - 2\,\boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{10}$$

$$\nabla^*(\nabla^*\psi^*)^* = \nabla^*\Phi = \nabla^*\nabla\psi \tag{11}$$

Double partial differentiation will then result in the quaternionic non-homogeneous second order partial differentiation equation:

$$\begin{aligned}\xi = \xi_0 + \boldsymbol{\xi} = \nabla^*\nabla\psi &= (\nabla_0 - \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})(\psi_0 + \boldsymbol{\psi}) = \{\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\}\psi \\ &= \frac{\partial^2\psi}{\partial\tau^2} + \circledast\,\frac{\partial^2\psi}{\partial x^2} + \circledast\,\frac{\partial^2\psi}{\partial y^2} + \circledast\,\frac{\partial^2\psi}{\partial z^2}\end{aligned} \tag{10}$$

$$\begin{aligned}\zeta_0 &= \nabla_0\phi_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\phi}\rangle \\ &= \nabla_0\nabla_0\varphi_0 - \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi_0 + \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \pm \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\times\boldsymbol{\varphi}\rangle \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0\end{aligned} \tag{11}$$

$$\begin{aligned}\boldsymbol{\zeta} &= -\boldsymbol{\nabla}\phi_0 + \nabla_0\boldsymbol{\phi} \mp \boldsymbol{\nabla}\times\boldsymbol{\phi} \\ &= -\boldsymbol{\nabla}\nabla_0\varphi_0 + \circledast\,\boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle + \nabla_0\boldsymbol{\nabla}\varphi_0 + \nabla_0\nabla_0\boldsymbol{\varphi} \pm \nabla_0\boldsymbol{\nabla}\times\boldsymbol{\varphi} \\ &\qquad \mp\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi_0 \mp \boldsymbol{\nabla}\times\nabla_0\boldsymbol{\varphi} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} \\ &= -\boldsymbol{\nabla}\nabla_0\varphi_0 + \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\varphi} + \nabla_0\boldsymbol{\nabla}\varphi_0 + \nabla_0\nabla_0\boldsymbol{\varphi} \pm \nabla_0\boldsymbol{\nabla}\times\boldsymbol{\varphi} \\ &\qquad \mp\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi_0 \mp \boldsymbol{\nabla}\times\nabla_0\boldsymbol{\varphi} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\boldsymbol{\varphi}\end{aligned} \tag{12}$$

Here ξ is a quaternionic function that for a part ρ describes the density distribution of a set of point-like artifacts that disrupt the continuity of function $\psi(q)$.

$$\rho = \rho_0 + \boldsymbol{\rho} = \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{13}$$

$$\xi - \rho = \nabla_0\nabla_0\psi \tag{14}$$

In case of a single static point-like artifact, the solution ψ will describe the corresponding Green's function.

Function $\psi(q)$ describes the mostly continuous field ψ.

The second order partial differential equation can be split into two continuity equations:

$$\Phi = \nabla\psi \tag{15}$$

$$\rho = \nabla^*\Phi \tag{16}$$

If ψ and Φ are normalizable functions and $\|\psi\| = 1$, then with real m and $\|\zeta\| = 1$

$$\nabla\psi = m\,\zeta \tag{17}$$

9.3 Fourier equivalents

In this quaternionic differential calculus, differentiation is implemented as multiplication.

This is revealed by the Fourier equivalents of the equations (4) through (10) in the previous paragraph:

$$\tilde{\Phi} = \tilde{\Phi}_0 + \tilde{\boldsymbol{\Phi}} = p\,\tilde{\psi} = (p_0 + \boldsymbol{p})(\tilde{\psi}_0 + \tilde{\boldsymbol{\psi}}) \tag{1}$$

The nabla ∇ is replaced by operator p. $\tilde{\Phi}$ is the Fourier transform of Φ.

$$\tilde{\Phi}_0 = p_0\tilde{\psi}_0 - \langle \boldsymbol{p}, \tilde{\boldsymbol{\psi}} \rangle \tag{2}$$

$$\tilde{\boldsymbol{\Phi}} = p_0\tilde{\boldsymbol{\psi}} + \boldsymbol{p}\tilde{\psi}_0 \pm \boldsymbol{p} \times \tilde{\boldsymbol{\psi}} \tag{3}$$

The equivalent of the quaternionic second order partial differential equation is:

$$\tilde{\xi} = \tilde{\xi}_0 + \tilde{\boldsymbol{\xi}} = p^*p\,\tilde{\psi} = \{p_0 p_0 + \langle \boldsymbol{p}, \boldsymbol{p} \rangle\}\tilde{\psi} \tag{4}$$

$$\tilde{\rho} = \tilde{\rho}_0 + \tilde{\boldsymbol{\rho}} = \langle \boldsymbol{p}, \boldsymbol{p} \rangle\tilde{\psi} \tag{5}$$

The continuity equations result in:

$$\tilde{\Phi} = p\tilde{\psi} \tag{6}$$

$$\tilde{\rho} = p^*\tilde{\Phi} \tag{7}$$

9.4 Poisson equations

The screened Poisson equation is a special condition of the non-homogeneous second order partial differential equation in which some terms are zero or have a special value.

$$\nabla^*\nabla\psi = \nabla_0\nabla_0\psi + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi = \xi \tag{1}$$

$$\circledast\,\nabla_0\nabla_0\psi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi = \circledast\,\xi \tag{2}$$

$$\circledast\,\nabla_0\nabla_0\psi = -\lambda^2\,\psi \tag{3}$$

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi - \lambda^2\psi = \circledast\,\xi \tag{4}$$

The 3D solution of this equation is determined by the screened Green's function $G(r)$.

Green functions represent solutions for point sources.

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{5}$$

$$\psi = \iiint G(\boldsymbol{r} - \boldsymbol{r}')\,\rho(\boldsymbol{r}')\,d^3\boldsymbol{r}\,' \tag{6}$$

G(r) has the shape of the Yukawa potential [9]

In case of $\lambda = 0$ it resembles the Coulomb or gravitation potential of a point source.

If $\lambda \neq 0$ and $\circledast = 1$, then a solution of equation (3) is:

$$\psi = a(\boldsymbol{x})\,\exp(\pm\, i\,\omega\,\tau);\, \lambda = \pm\, i\,\omega \tag{7}$$

9.5 Solutions of the homogeneous second order partial differential equation

Solutions of the quaternionic homogeneous second order partial differential equation are of special interest because for odd numbers of participating dimensions this equation has solutions in the form of shape keeping fronts.

This homogeneous partial differential equation is given by:

$$\nabla^*\nabla\psi = \nabla_0\nabla_0\psi + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi = \frac{\partial^2\psi}{\partial\tau^2} + \circledast\,\frac{\partial^2\psi}{\partial x^2} + \circledast\,\frac{\partial^2\psi}{\partial y^2} + \circledast\,\frac{\partial^2\psi}{\partial z^2} = 0 \tag{1}$$

Let us start with:

$$\nabla^*\nabla\psi_0 = 0 \tag{2}$$

Equation (2) has three-dimensional spherical shape keeping fronts as its solutions. ψ_0 is a scalar function. By changing to polar coordinates it can be deduced that a solution is given by:

$$\psi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{3}$$

Where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function. It keeps its shape during its travel through the field. Its amplitude quickly diminishes as $1/r$ with distance r from the trigger point.

Next we investigate:

$$\nabla^*\nabla\boldsymbol{\psi} = 0 \tag{4}$$

Here $\boldsymbol{\psi}$ is a vector function.

Equation (4) has one-dimensional shape keeping fronts as its solutions:

$$\boldsymbol{\psi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{5}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one-dimensional shape keeping front.

9.6 Special formulas

$$\boldsymbol{\nabla}\langle \boldsymbol{k}, \boldsymbol{x}\rangle = \boldsymbol{k} \tag{1}$$

$\boldsymbol{k}$ is constant.

$$\langle \boldsymbol{\nabla}, \mathbf{x}\rangle = \mathbf{3} \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \mathbf{0} \tag{3}$$

$$\boldsymbol{\nabla}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \tag{4}$$

$$\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{5}$$

$$\langle\boldsymbol{\nabla},\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rangle = \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\rangle = 4\pi\,\delta(\mathbf{x}-\mathbf{x}') \tag{6}$$

Similar formulas apply to the quaternionic nabla and parameter values.

$$\nabla x = 1 - 3\,;\ \nabla^* x = 1 + 3;\ \nabla x^* = 1 + 3$$

$$\nabla(x^* x) = x$$

$$\nabla|x| = \nabla\sqrt{(x^* x)} = \frac{x}{|x|} \tag{7}$$

$$\nabla\frac{1}{|x-x'|} = -\frac{x-x'}{|x-x'|^3} \tag{8}$$

$$\nabla^*\frac{x-x'}{|x-x'|^3} = \nabla\,\nabla^*\frac{1}{|x-x'|} = \left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau} + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\right)\frac{1}{|x-x'|} \neq 4\pi\,\delta(x-x') \tag{9}$$

Instead:

$$(\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\frac{1}{|x|} = \frac{3\tau^2}{|x|^5} - \frac{1}{|x|^3} + \frac{3\tau^2}{|x|^5} = \frac{6\tau^2 - |x|^2}{|x|^5} = \frac{5\tau^2 - |\boldsymbol{x}|^2}{|x|^5} \tag{10}$$

$$(\nabla_0\nabla_0 - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\frac{1}{|x|} = -\frac{1}{|x|^3} \tag{11}$$

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\frac{1}{|\boldsymbol{x}|} = 4\pi\,\delta(\boldsymbol{x}) \tag{12}$$

Thus, with spherical boundary conditions, $\frac{1}{4\pi\,|\boldsymbol{x}-\boldsymbol{x}'|}$ is suitable as the Green's function for the Poisson equation, but $\frac{1}{4\pi\,|x-x'|}$ does not represent a Green's function for the quaternionic operator $(\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)$!

For a homogeneous second order partial differential equation a Green's function is not required. Thus, the deficit of a green's function does not forbid the existence of a quaternionic homogeneous second order partial differential equation. Still equation (6) forms the base of the Poisson equation.

9.7 Field equations

In this section, we will compare two sets of differential equations. Both sets use pure space as part of the parameter space.

- Quaternionic differential equations
  - These equations use progression as one of its parameters.
- Maxwell based differential equations
  - These equations use quaternionic distance as one of its parameters.

By introducing new symbols $\mathfrak{E}$ and $\mathfrak{B}$ we will turn the quaternionic differential equations into Maxwell-like quaternionic differential equations. We introduced a simple switch $\circledast = \pm 1$ that apart from the difference between the parameter spaces, will turn one set into the other set.

Maxwell based differential equations split quaternionic functions into a scalar function and a vector function. Instead of the quaternionic nabla $\nabla = \nabla_0 + \boldsymbol{\nabla}$ the Maxwell based equations use the scalar operator $\nabla_0 = \frac{\partial}{\partial t}$ and the vector nabla $\boldsymbol{\nabla}$ as separate operators. Maxwell equations use a switch α that controls the structure of a gauge equation.

$$\varkappa = \alpha\frac{\partial}{\partial t}\,\varphi_0 + \langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \tag{1}$$

For Maxwell based differential calculus is $\alpha = +1$ and $\nabla_0 = \frac{\partial}{\partial t}$. The switch value is $\circledast -1$.

For quaternionic differential calculus is $\alpha = -1$ and $\nabla_0 = \frac{\partial}{\partial \tau}$. The switch value is $\circledast = +1$.

In EMFT the scalar field $\varkappa$ is taken as a gauge with

$\alpha = 1$; Lorentz gauge

$\alpha = 0$; Coulomb gauge

$\alpha = -1$; Kirchhoff gauge.

$$\varkappa \equiv \alpha\,\nabla_t\varphi_0 + \langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \Leftrightarrow \phi_0 = \nabla_\tau\varphi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \tag{2}$$

In Maxwell based differential calculus the scalar field $\varkappa$ is ignored or it is taken equal to zero. As will be shown, zeroing $\varkappa$ is not necessary for the derivation of the Maxwell based wave equation.

Maxwell equations split the considered functions in scalar functions and vector functions. The differential operators are also split and cannot be treated as multipliers.

$$\phi = \{\phi_0, \boldsymbol{\phi}\} = \{\nabla_0, \boldsymbol{\nabla}\}\{\varphi_0, \boldsymbol{\varphi}\} \tag{3}$$

$$\phi_0 = \nabla_0\, \varphi_0 - \circledast\, \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{4}$$

$$\boldsymbol{\phi} = \nabla_0 \boldsymbol{\varphi} + \boldsymbol{\nabla}\varphi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{5}$$

Equations (4) and (5) are not genuine Maxwell equations. We introduce them here as extra Maxwell equations. Choice $\circledast = -1$ conforms to the Lorenz gauge.

$$\mathfrak{E} \equiv -\nabla_0 \boldsymbol{\varphi} - \boldsymbol{\nabla}\varphi_0 \tag{6}$$

$$\nabla_0 \mathfrak{E} = -\nabla_0 \nabla_0\, \boldsymbol{\varphi} - \nabla_0 \boldsymbol{\nabla}\varphi_0 \tag{7}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{E} \rangle = -\nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 \tag{8}$$

$$\mathfrak{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{9}$$

These definitions imply:

$$\langle \mathfrak{E}, \mathfrak{B} \rangle = 0 \tag{10}$$

$$\nabla_0 \mathfrak{B} = -\boldsymbol{\nabla} \times \mathfrak{E} \tag{11}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{B} \rangle = 0 \tag{12}$$

$$\boldsymbol{\nabla} \times \mathfrak{B} = \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} \tag{13}$$

Also the following two equations are not genuine Maxwell equations, but they relate to the gauge equation.

$$\nabla_0\phi_0 = \nabla_0\nabla_0\,\varphi_0 - \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \tag{14}$$

$$\boldsymbol{\nabla}\phi_0 = \nabla_0\,\boldsymbol{\nabla}\varphi_0 - \circledast\,\boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle = \nabla_0\,\boldsymbol{\nabla}\varphi_0 - \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\,\boldsymbol{\varphi} - \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\,\boldsymbol{\varphi} \tag{15}$$

$$\zeta = (\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi = \zeta_0 + \boldsymbol{\zeta} \Leftrightarrow \{\zeta_0,\boldsymbol{\zeta}\} = \{\nabla_0,-\boldsymbol{\nabla}\}\{\phi_0,\phi\} \tag{16}$$

$$\zeta_0 = (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0 = \nabla_0\,\phi_0 - \circledast\,\langle\boldsymbol{\nabla},\mathfrak{E}\rangle \tag{17}$$

$$\boldsymbol{\zeta} = (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\boldsymbol{\varphi} = -\boldsymbol{\nabla}\phi_0 - \nabla_0\mathfrak{E} - \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} \tag{18}$$

More in detail the equations mean:

$$\begin{aligned}\zeta_0 &= \nabla_0\phi_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\phi}\rangle \\ &= \{\nabla_0\nabla_0\varphi_0 - \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle\} + \{\circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi_0 + \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \pm \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\times\boldsymbol{\varphi}\rangle\} \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0\end{aligned} \tag{19}$$

$$\begin{aligned}\zeta_0 &= \nabla_0\,\phi_0 - \circledast\,\langle\boldsymbol{\nabla},\mathfrak{E}\rangle \\ &= \{\nabla_0\nabla_0\,\varphi_0 - \circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle\} + \{\circledast\,\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi_0\} \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0\end{aligned} \tag{20}$$

$$\begin{aligned}\boldsymbol{\zeta} &= -\boldsymbol{\nabla}\phi_0 + \nabla_0\boldsymbol{\phi} \mp \boldsymbol{\nabla}\times\boldsymbol{\phi} \\ &= \{-\boldsymbol{\nabla}\nabla_0\varphi_0 + \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\varphi}\} + \{\nabla_0\boldsymbol{\nabla}\varphi_0 + \nabla_0\nabla_0\boldsymbol{\varphi} \pm \nabla_0\boldsymbol{\nabla}\times\boldsymbol{\varphi}\} \\ &\qquad \{\mp\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi_0 \mp \boldsymbol{\nabla}\times\nabla_0\boldsymbol{\varphi} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi}\} \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\boldsymbol{\varphi} + \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi}\end{aligned} \tag{21}$$

$$\begin{aligned}\boldsymbol{\zeta} &= -\boldsymbol{\nabla}\phi_0 - \nabla_0\mathfrak{E} - \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} \\ &= \{-\boldsymbol{\nabla}\nabla_0\varphi_0 + \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\varphi}\} + \{\nabla_0\nabla_0\,\boldsymbol{\varphi} + \nabla_0\boldsymbol{\nabla}\varphi_0\} - \circledast\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\varphi} \\ &= (\nabla_0\nabla_0 + \circledast\,\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\boldsymbol{\varphi}\end{aligned} \tag{22}$$

Equation (21) reveals why Maxwell based differential equations use the gauge $\varkappa$ rather than accept equation (4) as a genuine Maxwell equation.

$$\rho_0 =\circledast \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 = \zeta_0 - \nabla_0 \nabla_0 \varphi_0 \tag{23}$$

$$\boldsymbol{\rho} =\circledast \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} = \boldsymbol{\zeta} - \nabla_0 \nabla_0 \boldsymbol{\varphi} \tag{24}$$

Thus a simple change of a parameter and the control switch $\circledast$ turn quaternionic differential equations into equivalent Maxwell differential equations and vice versa. This makes clear that both sets represent two different views from the same subject, which is a field that can be stored in the eigenspace of an operator that resides in the Gelfand triple.

Still the comparison shows an anomaly in equation (21) that represents a significant difference between the two sets of differential equations that goes beyond the difference between the parameter spaces. A possible clue will be given in the section on the Dirac equation. This clue comes down to the conclusion that equations do not represent a coupling of a first and a second order partial differential equation, but instead they represent a coupling between two first order differential equations that each represent either particle behavior or antiparticle behavior.

9.8 Genuine Maxwell wave equations

The scalar part of the genuine Maxwell based differential equals zero. This is oppressed by the Lorenz gauge.

The genuine Maxwell differential equations deliver different inhomogeneous wave equations:

$$\mathfrak{E} \equiv -\nabla_0 \boldsymbol{\varphi} - \boldsymbol{\nabla} \varphi_0 \tag{1}$$

$$\mathfrak{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{2}$$

The following definitions follow from the definitions of $\mathfrak{E}$ and $\mathfrak{B}$.

$$\nabla_0 \mathfrak{E} \equiv -\nabla_0 \nabla_0 \, \boldsymbol{\varphi} - \nabla_0 \boldsymbol{\nabla} \varphi_0 \tag{3}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \equiv -\nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 \tag{4}$$

$$\nabla_0 \mathfrak{B} \equiv -\boldsymbol{\nabla} \times \mathfrak{E} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \mathfrak{B} \rangle \equiv 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \mathfrak{B} \equiv \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} \tag{7}$$

The Lorenz gauge means:

$$\nabla_0\varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle = 0 \tag{8}$$

The genuine Maxwell based wave equations are:

$$(\nabla_0\nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\varphi_0 = \rho_0 = \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \tag{9}$$

$$(\nabla_0\nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\boldsymbol{\varphi} = J = \boldsymbol{\nabla} \times \mathfrak{B} - \nabla_0\mathfrak{E} \tag{10}$$

9.9 Poynting vector

The definitions invite the definition of the Poynting vector:

$$\boldsymbol{S} = \mathfrak{E} \times \boldsymbol{\mathcal{B}} \tag{1}$$

$$u = \tfrac{1}{2}(\langle \mathfrak{E}, \mathfrak{E} \rangle + \langle \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{B}} \rangle) \tag{2}$$

$$\frac{\partial u}{\partial \tau} = \langle \boldsymbol{\nabla}, \boldsymbol{S} \rangle + \langle \boldsymbol{J}, \mathfrak{E} \rangle \tag{3}$$

9.10 Solutions of the wave equation

The Maxwell based differential calculus offers second order partial differential equations in the form of the wave equations:

$$(\nabla_0\nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\varphi_0 = \frac{\partial^2\varphi_0}{\partial\tau^2} - \frac{\partial^2\varphi_0}{\partial x^2} - \frac{\partial^2\varphi_0}{\partial y^2} - \frac{\partial^2\varphi_0}{\partial z^2} = \rho_0 \tag{1}$$

$$(\nabla_0\nabla_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\boldsymbol{\varphi} = \frac{\partial^2\boldsymbol{\varphi}}{\partial\tau^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial x^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial y^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial z^2} = \boldsymbol{\rho} \tag{2}$$

9.10.1 Shape keeping fronts

Like the quaternionic second order partial differential equation this wave equation offers solutions that represent shape keeping fronts.

For isotropic conditions in three participating dimensions the shape keeping front solution runs:

$$\varphi_0 = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{3}$$

This follows from

$$\langle \nabla, \nabla \rangle\varphi_0 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_0}{\partial r}\right)\right) = \frac{f''(r - ct)}{r} = \frac{1}{c^2}\frac{\partial^2\varphi_0}{\partial t^2} \tag{4}$$

In a single participating dimension the shape keeping front solution runs:

$$\varphi_0 = f(x - ct), \text{where } c = \pm 1; f \text{ is real} \tag{5}$$

The same solutions hold for vector function $\boldsymbol{\varphi}$.

9.10.2 Other solutions of the homogenous wave equation

Apart from the shape keeping solutions the homogeneous wave equation offers wave form solutions. Some of these solutions are obtained by starting with:

$$\nabla_0 \nabla_0 f = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle f = -\omega^2 f \tag{1}$$

$$f(t, \boldsymbol{x}) = a \exp\big(i\omega(ct - |\boldsymbol{x} - \boldsymbol{x}'|)\big); c = \pm 1 \tag{2}$$

This leads to a category of solutions that are known as solutions of the Helmholtz equation.

9.10.3 The Maxwell based Poisson equations

The screened Poisson equation in Maxwell based differential calculus runs:

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \lambda^2)\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \lambda\varphi = -\rho \tag{1}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \lambda^2 \varphi \tag{2}$$

$$\varphi = a(\boldsymbol{x}) \exp(\pm \lambda t) \tag{3}$$

This differs significantly from the quaternion differential calculus version of the screened Poisson equation.

10 Dirac equation

10.1 The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors, matrices and partial derivatives.

Instead of the usual $\left\{\frac{\partial f}{\partial t}, \boldsymbol{i}\frac{\partial f}{\partial x}, \boldsymbol{j}\frac{\partial f}{\partial y}, \boldsymbol{k}\frac{\partial f}{\partial z}\right\}$ we want to use operators $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$

The operator ∇ relates to the applied parameter space. This means that the parameter space is also configured of combinations $x = \{t, \boldsymbol{x}\}$ of a scalar t and a vector $\boldsymbol{x}$. Also the functions can be split in scalar functions and vector functions. The subscript $_0$ indicates the scalar part. Bold face indicates the vector part.

Here t represents a local scalar, which is defined as the scalar part of the applied parameter space.

The original Dirac equation uses 4x4 matrices $\boldsymbol{\alpha}$ and β. [10] [11]:

$\boldsymbol{\alpha}$ and β are matrices that implement the quaternion arithmetic behavior including the possible symmetry flavors of quaternionic number systems and continuums.

$$\alpha_1 = \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} \leftrightarrow -i \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 = \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} \leftrightarrow -i \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 = \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \leftrightarrow -i \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta = \gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

The unity matrix I and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are given by [12]:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

For one of the potential orderings of the quaternionic number system, the Pauli matrices together with the unity matrix I relate to the quaternionic base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

$$1 \mapsto I, \quad \boldsymbol{i} \mapsto i\,\sigma_1, \quad \boldsymbol{j} \mapsto i\,\sigma_2, \quad \boldsymbol{k} \mapsto i\,\sigma_3 \tag{6}$$

$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2\,i\,\sigma_3;\ \sigma_2\sigma_3 - \sigma_3\sigma_2 = 2\,i\,\sigma_1;\ \sigma_3\sigma_1 - \sigma_1\sigma_3 = 2\,i\,\sigma_2 \tag{7}$$

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \beta\beta = I \tag{8}$$

Together with the $\boldsymbol{\alpha}$ matrices, the matrix β represents quaternionic conjugation. As a consequence, it switches the handedness of the external vector product.

The interpretation of the Pauli matrices as representation of a special kind of angular momentum has led to the half integer eigenvalue of the corresponding spin operator.

10.2 Splitting into two equations

One interpretations of the Dirac equation is [13]:

$$\left(\gamma_0 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x} - \gamma_2 \frac{\partial}{\partial y} - \gamma_3 \frac{\partial}{\partial z} - \frac{m}{i\hbar}\right)\{\psi\} = 0 \tag{1}$$

This invites splitting of the four component spinor equation into two equations for two component spinors:

$$i\frac{\partial\varphi_A}{\partial t} - i\sigma_1\frac{\partial\varphi_A}{\partial x} - i\sigma_2\frac{\partial\varphi_A}{\partial y} - i\sigma_3\frac{\partial\varphi_A}{\partial z} = \frac{m}{\hbar}\,\varphi_B \tag{2}$$

$$i\frac{\partial\varphi_B}{\partial t} + i\sigma_1\frac{\partial\varphi_B}{\partial x} + i\sigma_2\frac{\partial\varphi_B}{\partial y} + i\sigma_3\frac{\partial\varphi_B}{\partial z} = \frac{m}{\hbar}\,\varphi_A \tag{3}$$

$$i\frac{\partial\varphi_A}{\partial t} - \boldsymbol{i}\frac{\partial\varphi_A}{\partial x} - \boldsymbol{j}\frac{\partial\varphi_A}{\partial y} - \boldsymbol{k}\frac{\partial\varphi_A}{\partial z} = (i\nabla_0 - \boldsymbol{\nabla})\varphi_A = \frac{m}{\hbar}\,\varphi_B \tag{4}$$

$$i\frac{\partial\varphi_B}{\partial t} + \boldsymbol{i}\frac{\partial\varphi_B}{\partial x} + \boldsymbol{j}\frac{\partial\varphi_B}{\partial y} + \boldsymbol{k}\frac{\partial\varphi_B}{\partial z} = (i\nabla_0 + \boldsymbol{\nabla})\varphi_B = \frac{m}{\hbar}\,\varphi_A \tag{5}$$

$$\begin{aligned}(i\nabla_0 + \boldsymbol{\nabla})(i\nabla_0 - \boldsymbol{\nabla})\varphi_A &= (-\nabla_0\nabla_0 - \boldsymbol{\nabla}\boldsymbol{\nabla})\varphi_A = (\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_A \\ &= \frac{m}{\hbar}(i\nabla_0 + \boldsymbol{\nabla})\,\varphi_B = \frac{m^2}{\hbar^2}\,\varphi_A\end{aligned} \tag{6}$$

$$(\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_A = \frac{m^2}{\hbar^2}\,\varphi_A \tag{7}$$

$$\begin{aligned}(i\nabla_0 - \boldsymbol{\nabla})(i\nabla_0 + \boldsymbol{\nabla})\varphi_B &= (-\nabla_0\nabla_0 - \boldsymbol{\nabla}\boldsymbol{\nabla})\varphi_B = (\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_B \\ &= \frac{m}{\hbar}(i\nabla_0 - \boldsymbol{\nabla})\,\varphi_A = \frac{m^2}{\hbar^2}\,\varphi_B\end{aligned} \tag{8}$$

$$(\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle - \nabla_0\nabla_0)\varphi_B = \frac{m^2}{\hbar^2}\,\varphi_B \tag{9}$$

Thus the four component spinors $\{\psi\}$ can be converted in two component spinors $\{\,\varphi_A\}$ and $\{\,\varphi_B\}$. Quaternionic functions are not complex number based spinors, but the form of equations (7) and (9) offer sufficient info for the conversion. With respect to second order differentiation, the two component spinors and the quaternionic functions show similar behavior.

Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic functions χ_A and χ_B that replace the spinors $\{\,\varphi_A\}$ and $\{\,\varphi_B\}$. These functions have

different parameter spaces. As a consequence the nabla operators act differently onto χ_A and χ_B. This results into two coupled first order partial differential equations.

$$(\nabla_0 - \boldsymbol{\nabla})\chi_A = \nabla^* \chi_A = \frac{m}{\hbar}\,\chi_B \tag{10}$$

$$(\nabla_0 + \boldsymbol{\nabla})\chi_B = \nabla \chi_B = \frac{m}{\hbar}\,\chi_A \tag{11}$$

This also corresponds to two quite similar second order partial differential equations:

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0)\chi_A = \frac{m^2}{\hbar^2}\chi_A \tag{12}$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0)\chi_B = \frac{m^2}{\hbar^2}\chi_B \tag{13}$$

And one homogeneous second order partial differential equation

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle - \nabla_0 \nabla_0)\chi = 0 \tag{14}$$

This equation is a wave equation. The set of its solutions includes waves.

Thus, the functions χ_A and χ_B describe two different solutions of the same Maxwell-like second order partial differential equation.

According to the Dirac matrices the natural parameter spaces of functions χ_A and χ_B concern two different quaternionic number systems that differ in the handedness of their external vector product.

One of these natural parameter spaces is right handed and the other natural parameter space is left handed.

The factor m couples χ_A and χ_B.

Since both χ_A and χ_B are quaternionic functions, they also obey other second order partial wave equations.

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle + \nabla_0 \nabla_0)\chi_A = \xi_A \tag{15}$$

$$(\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle + \nabla_0 \nabla_0)\chi_B = \xi_B \tag{16}$$

$$\left(2\,\nabla_0\nabla_0 + \frac{m^2}{\hbar^2}\right)\chi_A = \xi_A \tag{17}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\, \chi_A = \xi_A + \frac{m^2}{\hbar^2}\chi_A \tag{18}$$

Natural parameter spaces are spanned by a version of the quaternionic number system. Due to the four dimensions of quaternions, these natural parameter spaces represent two different sign flavors of one and the same quaternionic field that exists in 16 versions that only differ in their discrete symmetry set.

The fields that represent the two natural parameter spaces can be considered to be each other's quaternionic conjugate. As a consequence, they differ in the handedness of the external vector product. These fields relate to the symmetry centers from which the particle generating mechanisms take their resources.

In the direct environment of the free particle, apparently two kinds of potential embedding exist. Alongside the particle embedding with solution χ_A exists an antiparticle embedding with solution χ_B.

In the quaternionic Dirac equations the function χ is a local representative of the locally untouched particle embedding field $\mathfrak{C}$. Function χ represents a curl free field.

$$\nabla\chi = \nabla^*\chi^* \tag{19}$$

This short range equation also holds locally for the locally untouched embedding field $\mathfrak{C}$.

$$\nabla\mathfrak{C} = \nabla^*\mathfrak{C}^* \tag{20}$$

10.3 Alternatives

10.3.1 Minkowski parameter space

In quaternionic differential calculus the local quaternionic distance can represent a scalar that is independent of the direction of progression. It corresponds to the notion of coordinate time. That means that a small coordinate time step Δt equals the sum of a small proper time step $\Delta\tau$ and a small pure space step $\Delta\boldsymbol{x}$. In quaternionic format the step $\Delta\tau$ is a real number. The space step $\Delta\boldsymbol{x}$ is an imaginary quaternionic number. The original Dirac equation does not pay attention to the difference between coordinate time and proper time, but the quaternionic presentation of these equations show that a progression independent scalar can be useful as the scalar part of the parameter space. This holds especially for solutions of the homogeneous wave equation.

10.3.2 Other natural parameter spaces

The Dirac equation in quaternionic format treats a coupling of parameter spaces that are each other's quaternionic conjugate. This can also be applied when anisotropic conjugation is applied. This concerns conjugations in which only one or two dimensions get a reverse ordering. In that case the equations handle the dynamic behavior of anisotropic particles such as quarks.

11 Double differentiation

The partial differential equations hide that they are part of a differential equation.

$$\nabla'\nabla f = \xi = \sum_{\nu=0}^{3} e'_\nu \frac{\partial}{\partial q'_\nu}\left(\sum_{\mu=0}^{3} e_\mu \frac{\partial f}{\partial q_\mu}\right) = \left(e'_\nu e_\mu \frac{\partial^2}{\partial q_\mu \partial q'_\nu}\right) f \tag{1}$$

Single difference is defined by

$$df(q) = \sum_{\mu=0}^{3} \sum_{\varsigma=0}^{3} \frac{\partial f^\varsigma}{\partial q_\mu} e_\mu e_\varsigma \; dq^\mu \; = \sum_{\nu=0}^{3} \phi_\nu e_\nu dq^\nu \tag{2}$$

$$\frac{\partial f^\varsigma}{\partial q_\mu} e_\mu e_\varsigma = \begin{bmatrix} \frac{\partial f^0}{\partial q_0} & \frac{\partial f^1}{\partial q_0}\boldsymbol{i} & \frac{\partial f^2}{\partial q_0}\boldsymbol{j} & \frac{\partial f^3}{\partial q_0}\boldsymbol{k} \\ \frac{\partial f^0}{\partial q_1}\boldsymbol{i} & \frac{\partial f^1}{\partial q_1} & \frac{\partial f^2}{\partial q_1}\boldsymbol{k} & -\frac{\partial f^3}{\partial q_1}\boldsymbol{j} \\ \frac{\partial f^0}{\partial q_2}\boldsymbol{j} & -\frac{\partial f^1}{\partial q_2}\boldsymbol{k} & \frac{\partial f^2}{\partial q_2} & \frac{\partial f^3}{\partial q_2}\boldsymbol{i} \\ \frac{\partial f^0}{\partial q_3}\boldsymbol{k} & \frac{\partial f^1}{\partial q_3}\boldsymbol{j} & -\frac{\partial f^2}{\partial q_3}\boldsymbol{i} & \frac{\partial f^3}{\partial q_3} \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} \frac{\partial f^0}{\partial q_0} & -\mathcal{E}_x\boldsymbol{i} & -\mathcal{E}_y\boldsymbol{j} & -\mathcal{E}_z\boldsymbol{k} \\ \mathcal{E}_x\boldsymbol{i} & \frac{\partial f^1}{\partial q_1} & -\mathcal{B}_{z1}\boldsymbol{k} & -\mathcal{B}_{y2}\boldsymbol{j} \\ \mathcal{E}_y\boldsymbol{j} & -\mathcal{B}_{z2}\boldsymbol{k} & \frac{\partial f^2}{\partial q_2} & -\mathcal{B}_{x1}\boldsymbol{i} \\ \mathcal{E}_z\boldsymbol{k} & -\mathcal{B}_{y1}\boldsymbol{j} & -\mathcal{B}_{x2}\boldsymbol{i} & \frac{\partial f^3}{\partial q_3} \end{bmatrix}$$

Here

$$\mathcal{B}_x = \mathcal{B}_{x1} - \mathcal{B}_{x2};\; \mathcal{B}_y = \mathcal{B}_{y1} - \mathcal{B}_{y2};\; \mathcal{B}_z = \mathcal{B}_{z1} - \mathcal{B}_{z2} \tag{4}$$

$$\dot{f} = \frac{df}{d\lambda} = \sum_{\mu=0}^{3} \phi_\mu e_\mu \frac{dq^\mu}{d\lambda} = \sum_{\mu=0}^{3} \phi_\mu e_\mu \dot{q}^\mu \tag{5}$$

The scalar λ is can be a linear function of τ or a scalar function of q.

$$\dot{q} \equiv \frac{dq}{d\lambda} = e_\mu \frac{dq^\mu}{d\lambda} = e_\mu \dot{q}^\mu \tag{6}$$

Double difference is defined by:

$$d^2 f(q) = \sum_{\nu=0}^{3} e'_\nu \left(\sum_{\mu=0}^{3} \frac{\partial^2 f^\varsigma}{\partial q_\mu \partial q'_\nu} e_\mu dq^\mu \right) e_\varsigma dq'^\nu \tag{7}$$

$$\ddot{f} \equiv \frac{d^2 f(q)}{d\lambda^2} = e_\varrho \ddot{f}^\varrho = \sum_{\nu=0}^{3} e'_\nu \left(\sum_{\mu=0}^{3} \frac{\partial^2 f^\varsigma}{\partial q_\mu \partial q'_\nu} e_\mu \frac{dq^\mu}{d\lambda} \right) e_\varsigma \frac{dq'^\nu}{d\lambda} \tag{8}$$

$$= \sum_{\nu=0}^{3} e'_\nu \left(\sum_{\mu=0}^{3} \frac{\partial^2 f^\varsigma}{\partial q_\mu \partial q'_\nu} e_\mu \dot{q}^\mu \right) e_\varsigma \dot{q}'^\nu = \left(\dot{q}^\mu \dot{q}'^\nu \frac{\partial^2}{\partial q_\mu \partial q'_\nu} e'_\nu e_\mu \right) f = \zeta_{\nu\mu} f$$

$$\zeta_{\nu\mu} = e'_\nu e_\mu \, \dot{q}'^\nu \, \dot{q}^\mu \frac{\partial^2}{\partial q_\mu \partial q'_\nu} = e'_\nu e_\mu \Upsilon_{\nu\mu} \tag{9}$$

$$\Upsilon_{\nu\mu} = \dot{q}'^\nu \, \dot{q}^\mu \frac{\partial^2}{\partial q_\mu \partial q'_\nu} \tag{10}$$

If we apply $\phi = \nabla f$ as the first differential operation and $\xi = \nabla^* \phi$ as the second differential operation, then $e = \{1, +\boldsymbol{i}, +\boldsymbol{j}, +\boldsymbol{k}\}$ and $e' = \{1 - \boldsymbol{i}, -\boldsymbol{j}, -\boldsymbol{k}\}$ and

$$\Upsilon_{\nu\mu} = \begin{bmatrix} +\Upsilon_{00} & +\Upsilon_{01}\boldsymbol{i} & +\Upsilon_{02}\boldsymbol{j} & +\Upsilon_{03}\boldsymbol{k} \\ -\Upsilon_{10}\boldsymbol{i} & \circledast\, \Upsilon_{11} & +\Upsilon_{12}\boldsymbol{k} & +\Upsilon_{13}\boldsymbol{j} \\ -\Upsilon_{20}\boldsymbol{j} & -\Upsilon_{21}\boldsymbol{k} & \circledast\, \Upsilon_{22} & -\Upsilon_{23}\boldsymbol{i} \\ -\Upsilon_{30}\boldsymbol{k} & -\Upsilon_{31}\boldsymbol{j} & +\Upsilon_{32}\boldsymbol{i} & \circledast\, \Upsilon_{33} \end{bmatrix} \tag{11}$$

Here again the switch $\circledast$ distinguishes between quaternionic differential calculus and Maxwell based differential calculus.

11.1 Deformed space

If the investigated field represents deformed space $\mathfrak{C}$, then the field $\mathfrak{R}$, which represents the parameter space of function $\mathfrak{C}(q)$ represents the virgin state of that deformed space.

Further, the equation $\frac{d^2\mathfrak{C}(q)}{d\lambda^2} = 0$ represents a local condition in which $\mathfrak{C}$ is not affected by external influences. Here λ can be any linear combination of progression τ or is can represent the equivalent of local quaternionic distance:

$$\lambda = a\, q_0 + b$$

or

$$\lambda = |q|$$

12 Quaternionic wave equation

A quaternionic wave equation

$$\nabla\nabla\psi = (\nabla_0\nabla_0 - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\psi = 0 \tag{1}$$

is achieved for fields φ and conditions ξ that obey:

$$(\nabla_0 + \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})\psi_0 \equiv \nabla_0\nabla_0\psi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\psi_0 + 2\nabla_0\boldsymbol{\nabla}\psi_0 = \xi_0 \tag{2}$$

$$2\nabla_0\boldsymbol{\nabla}\psi_0 = \xi_0 \tag{3}$$

$$(\nabla_0 + \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})\boldsymbol{\psi} \equiv \nabla_0\nabla_0\boldsymbol{\psi} - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\psi} - 2\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle + 2\nabla_0\boldsymbol{\nabla}\times\boldsymbol{\psi} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi} = \boldsymbol{\xi} \tag{4}$$

$$-2\nabla_0\langle\boldsymbol{\nabla},\boldsymbol{\psi}\rangle + 2\nabla_0\boldsymbol{\nabla}\times\boldsymbol{\psi} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi} = \boldsymbol{\xi} \tag{5}$$

13 Asymmetric tensor

The Maxwell-based equation

$$\phi \Leftrightarrow \{\phi_0,\boldsymbol{\phi}\} \Leftrightarrow \{\nabla_0,\boldsymbol{\nabla}\}\{\varphi_0,\boldsymbol{\varphi}\} = \{\nabla_0,-\boldsymbol{\nabla}\}\{A_0,\boldsymbol{A}\} \tag{1}$$

$$\boldsymbol{\phi} = -\mathfrak{E} \pm \mathfrak{B} \tag{2}$$

$$\mathfrak{E}_\nu \equiv -\left(\frac{\partial\varphi_0}{\partial x_\nu} + \frac{\partial\varphi_\nu}{\partial t}\right) = -F_{0\nu} = \partial_0 A_\nu - \partial_\nu A_0; \nu = 1..3 \tag{3}$$

$$\mathfrak{B}_{\mu\nu} = (\boldsymbol{\nabla}\times\boldsymbol{\varphi})_{\mu\nu} = \left(\frac{\partial\varphi_\mu}{\partial x_\nu} - \frac{\partial\varphi_\nu}{\partial x_\mu}\right) = \partial_\mu A_\nu - \partial_\nu A_\mu; \mu = 1..3; \nu = 1..3; \tag{4}$$

corresponds with the asymmetric tensor $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & -\mathfrak{E}_x & -\mathfrak{E}_y & -\mathfrak{E}_z \\ \mathfrak{E}_x & 0 & +\mathfrak{B}_z & -\mathfrak{B}_y \\ \mathfrak{E}_y & -\mathfrak{B}_z & 0 & +\mathfrak{B}_x \\ \mathfrak{E}_y & +\mathfrak{B}_y & -\mathfrak{B}_x & 0 \end{bmatrix} \tag{5}$$

$$\nabla_\mu F^{\nu\mu} = 4\pi J^\nu$$

13.1 Critics on the design of the antisymmetric tensor

The design of this tensor gives the false impression that the design is enforced by physical reality. In fact the design is based on a series of arbitrary choices.

For the quaternionic differential calculus the same tensor can be generated. However, the tensor elements are not necessarily asymmetric by nature. That only holds for the terms that belong to the magnetic field $\mathfrak{B}_{\mu\nu}$. The tensor does not show the nature of the partial derivatives that are contained in the $\mathfrak{E}_{\mu\nu}$ terms. The tensor hides the real parts of the differential.

Quaternionic differentiation has the advantage that the differential operator acts as a multiplier. In terms of the quaternionic differential calculus the tensor corresponds to the equation:

$$\phi = \phi_0 + \boldsymbol{\phi} = (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla\varphi = (\nabla_0 - \boldsymbol{\nabla})(A_0 + \boldsymbol{A}) = \nabla^* A \tag{1}$$

$$\boldsymbol{\phi} = -\mathfrak{E} \mp \mathfrak{B} = (\nabla_0\boldsymbol{\varphi} + \boldsymbol{\nabla}\varphi_0) \mp \boldsymbol{\nabla} \times \boldsymbol{\varphi} = (\nabla_0\boldsymbol{A} - \boldsymbol{\nabla}A_0) \pm \boldsymbol{\nabla} \times \boldsymbol{A} \tag{2}$$

$$\mathfrak{E} = -\nabla_0\boldsymbol{\varphi} - \boldsymbol{\nabla}\varphi_0 = -\nabla_0\boldsymbol{A} + \boldsymbol{\nabla}A_0 \tag{3}$$

$$\mathfrak{B} = \boldsymbol{\nabla} \times \boldsymbol{\varphi} = -\boldsymbol{\nabla} \times \boldsymbol{A} \tag{4}$$

The tensor hides the real parts of the differential.

$$\phi_0 = \nabla_0\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle = \nabla_0 A_0 - \langle \boldsymbol{\nabla}, \boldsymbol{A} \rangle \tag{5}$$

The tensor calculus used by current physical theories uses the terms of equation (5) as contents of a gauge that is used to construct the wave equation.

$$\nabla_0\phi_0 = \nabla_0\nabla_0\varphi_0 - \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle = \nabla_0\nabla_0 A_0 - \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{A} \rangle \tag{6}$$

$$\boldsymbol{\nabla}\phi_0 = \boldsymbol{\nabla}\nabla_0\varphi_0 - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle = \boldsymbol{\nabla}\nabla_0 A_0 - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{A} \rangle \tag{7}$$

$$\begin{aligned} \nabla_0\boldsymbol{\phi} &= -\nabla_0\mathfrak{E} \mp \nabla_0\mathfrak{B} = \nabla_0\nabla_0\boldsymbol{\varphi} + \nabla_0\boldsymbol{\nabla}\varphi_0 \pm \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} \\ &= \nabla_0\nabla_0\boldsymbol{A} - \nabla_0\boldsymbol{\nabla}A_0 \mp \nabla_0\boldsymbol{\nabla} \times \boldsymbol{A} \end{aligned} \tag{8}$$

$$\boldsymbol{\nabla}\boldsymbol{\phi} = -\langle \boldsymbol{\nabla}, \boldsymbol{\phi} \rangle \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} = \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \pm \langle \boldsymbol{\nabla}, \mathfrak{B} \rangle \mp \boldsymbol{\nabla} \times \mathfrak{E} - \boldsymbol{\nabla} \times \mathfrak{B} \tag{9}$$

Thus, in terms of the φ field the tensor is not naturally asymmetric. It is only naturally asymmetric in terms of the A field. However, selection between φ and A is arbitrary.

14 The space-progression model

This paper supports two space progression models. Quaternions, quaternionic functions and quaternionic differential equations support parameter spaces that have an Euclidean signature and correspond to a metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Elements of this model can directly be stored as eigenvalues of operators that reside in quaternionic Hilbert spaces.

The Maxwell based equations and the parameter space of these equations support a space-time model with Minkowski signature and correspond to a metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2)$$

Elements of this model must first be dismantled into their real components before they can be stored as eigenvalues of Hermitian operators that reside in real, complex or quaternionic Hilbert spaces

The fact that the quaternionic field can be stored in the eigenspace of an operator that resides in a non-separable quaternionic Hilbert space and that after dismantling into real components the same can be done for a Maxwell based field means that the stored fields can represent one and the same object. It also means that both differential equation sets can investigate the same field and offer different views onto that field that reveal different aspects of the behavior of that field.

It also means that both space-progression models represent different views of the same reality.

15 Tensor differential calculus

We restrict to 3+1 D parameter spaces.

Parameter spaces can differ in the way they are ordered and in the way the scalar part relates to the spatial part.

Fields are functions that have values, which are independent of the selected parameter space. Fields exist in scalar fields, vector fields and combined scalar and vector fields.

Combined fields exist as continuum eigenspaces of normal operators that reside in quaternionic non-separable Hilbert spaces. These combined fields can be represented by quaternionic functions of quaternionic parameter spaces. However, the same field can also be interpreted as the eigenspaces of the Hermitian and anti-Hermitian parts of the normal operator. The quaternionic parameter space can be represented by a normal quaternionic reference operator that features a flat continuum eigenspace. This reference operator can be split in a Hermitian and an anti-Hermitian part.

The eigenspace of a normal quaternionic number system corresponds to a quaternionic number system. Due to the four dimensions of quaternions, the quaternionic number systems exist in 16 versions that differ in their Cartesian ordering. If spherical ordering is pursued, then for each Cartesian start orderings two extra orderings are possible. All these choices correspond to different parameter spaces.

Further it is possible to select a scalar part of the parameter space that is a scalar function of the quaternionic scalar part and the quaternionic vector part. For example it is possible to use quaternionic distance as the scalar part of the new parameter space.

Tensor differential calculus relates components of differentials with corresponding parameter spaces.

Components of differentials are terms of the corresponding differential equation. These terms can be split in scalar functions and in vector functions. Tensor differential calculus treats scalar functions different from vector functions.

Quaternionic fields are special because the differential operators of their defining functions can be treated as multipliers.

15.1 The metric tensor

The metric tensor determines the local “distance”.

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{1}$$

The consequences of coordinate transformations $dx^\nu \Rightarrow dX^\nu$ define the elements $g_{\mu\nu}$ as

$$g_{\mu\nu} = \frac{dX^\mu}{dx^\nu} \tag{2}$$

15.2 Geodesic equation

The geodesic equation describes the situation of a non-accelerated object. In terms of proper time this means:

(1)

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

In terms of coordinate time this means:

$$\frac{\partial^2 x^\mu}{\partial t^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\mu}{dt} \tag{2}$$

15.2.1 Derivation:

We start with the double differential. Let us investigate a function X that has a parameter space existing of scalar τ and a three dimensional vector $\boldsymbol{x} = \{x^1, x^2, x^3\}$. The function X represents three dimensional curved space. The geodesic conditions are:

$$\frac{\partial^2 X^\lambda}{\partial \tau^2} = 0\ ;\ \lambda = 1,2,3 \tag{1}$$

First we derive the first order differential.

$$dX^\lambda = \sum_{\beta=1}^{3}\frac{\partial X^\lambda}{\partial x^\beta}dx^\beta \tag{2}$$

We can use the summation convention for subscripts and superscripts. This avoids the requirement for summation symbols.

$$\frac{dX^\lambda}{d\tau} = \frac{\partial X^\lambda}{\partial x^\beta}\frac{dx^\beta}{d\tau} \tag{3}$$

$$d^2X^\lambda = \sum_{\beta=1}^{3}\left(\frac{\partial X^\lambda}{\partial x^\beta}d^2x^\beta + dx^\beta\sum_{\alpha=1}^{3}\frac{\partial^2 X^\lambda}{\partial x^\beta \partial x^\alpha}dx^\alpha\right) \tag{4}$$

Now we obtained the double differential equation.

$$\frac{d^2X^\lambda}{d\tau^2} = \frac{\partial X^\lambda}{\partial x^\beta}\frac{d^2x^\beta}{d\tau^2} + \frac{\partial^2 X^\lambda}{\partial x^\beta \partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{5}$$

The geodesic requirement results in:

$$\frac{\partial X^\lambda}{\partial x^\beta}\frac{d^2x^\beta}{d\tau^2} = -\frac{\partial^2 X^\lambda}{\partial x^\beta \partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{6}$$

If we use summation signs:

$$\sum_{\beta=1}^{3}\frac{\partial X^\lambda}{\partial x^\beta}d^2x^\beta = -\sum_{\beta=1}^{3}\left(dx^\beta\sum_{\alpha=1}^{3}\left(\frac{\partial^2 X^\lambda}{\partial x^\beta \partial x^\alpha}dx^\alpha\right)\right) \tag{7}$$

Next we multiply both sides with $\frac{\partial X^\lambda}{\partial x^\beta}$ and sum again:

$$\sum_{\lambda=1}^{3}\left(\frac{\partial x^\lambda}{\partial X^\mu}\left(\sum_{\beta=1}^{3}\frac{\partial X^\lambda}{\partial x^\beta}d^2x^\beta\right)\right) = -\sum_{\lambda=1}^{3}\left(\frac{\partial x^\lambda}{\partial X^\mu}\sum_{\beta=1}^{3}\left(dx^\beta\sum_{\alpha=1}^{3}\left(\frac{\partial^2 X^\lambda}{\partial x^\beta \partial x^\alpha}dx^\alpha\right)\right)\right) \tag{8}$$

We apply the fact:

(9)

$$\sum_{\lambda=1}^{3}\left(\frac{\partial x^{\lambda}}{\partial X^{\mu}}\frac{\partial X^{\lambda}}{\partial x^{\beta}}\right)=\delta_{\beta}^{\mu}$$

This results into:

$$d^2x^{\mu}=\sum_{\lambda=1}^{3}\left(\frac{\partial x^{\lambda}}{\partial X^{\mu}}\sum_{\beta=1}^{3}\left(dx^{\beta}\sum_{\alpha=1}^{3}\left(\frac{\partial^2 X^{\lambda}}{\partial x^{\beta}\partial x^{\alpha}}dx^{\alpha}\right)\right)\right)=\Gamma_{\alpha\beta}^{\mu}dx^{\alpha}dx^{\beta} \tag{10}$$

Without summation signs:

$$\Gamma_{\alpha\beta}^{\mu}dx^{\alpha}dx^{\beta}\equiv\left(\frac{\partial x^{\mu}}{\partial X^{\lambda}}\frac{\partial^2 X^{\lambda}}{\partial x^{\alpha}\partial x^{\beta}}\right)dx^{\alpha}dx^{\beta} \tag{11}$$

$$\frac{d^2x^{\mu}}{d\tau^2}=-\Gamma_{\alpha\beta}^{\mu}\frac{dx^{\beta}}{d\tau}\frac{dx^{\alpha}}{d\tau} \tag{12}$$

$$\frac{d^2x^{\mu}}{d\tau^2}=-\left(\frac{\partial x^{\mu}}{\partial X^{\lambda}}\frac{\partial^2 X^{\lambda}}{\partial x^{\alpha}\partial x^{\beta}}\right)\frac{dx^{\beta}}{d\tau}\frac{dx^{\alpha}}{d\tau} \tag{13}$$

$$\frac{d^2x^{\mu}}{dt^2}=-\left(\frac{\partial x^{\mu}}{\partial X^{\lambda}}\frac{\partial^2 X^{\lambda}}{\partial x^{\alpha}\partial x^{\beta}}\right)\frac{dx^{\beta}}{dt}\frac{dx^{\alpha}}{dt}+\left(\frac{\partial x^{0}}{\partial X^{\lambda}}\frac{\partial^2 X^{\lambda}}{\partial x^{\alpha}\partial x^{\beta}}\right)\frac{dx^{\beta}}{dt}\frac{dx^{\alpha}}{dt}\frac{dx^{\mu}}{dt} \tag{14}$$

15.3 Toolbox

Coordinate transformations:

$$S_{\nu'\rho'}^{\mu'}=\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}S_{\nu\rho}^{\mu} \tag{1}$$

The Christoffel symbol plays an important role:

$$2\,g_{\alpha\delta}\,\Gamma_{\beta\alpha}^{\delta}=\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}+\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} \tag{2}$$

$$\Gamma_{\alpha\beta}^{\mu}\equiv\frac{\partial x^{\mu}}{\partial X^{\lambda}}\frac{\partial^2 X^{\lambda}}{\partial x^{\alpha}\partial x^{\beta}} \tag{3}$$

$$\Gamma_{\beta\alpha}^{\delta}=\Gamma_{\alpha\beta}^{\delta} \tag{4}$$

Covariant derivative $\nabla_{\mu}\alpha$ and partial derivative $\partial_{\mu}\alpha$ of scalars

$$\partial_{\mu'}\alpha=\frac{\partial x^{\mu'}}{\partial x^{\mu}}\partial_{\mu}\alpha \tag{5}$$

Covariant derivative $\nabla_{\mu}V^{\nu}$ and partial derivative $\partial_{\mu}V^{\nu}$ of vectors

$$\nabla_{\mu}V^{\nu}=\partial_{\mu}V^{\nu}+\Gamma_{\mu\lambda}^{\nu}V^{\lambda} \tag{6}$$

$$\nabla_{\mu}\varphi_{\nu}=\partial_{\mu}\varphi_{\nu}-\Gamma_{\mu\nu}^{\lambda}\varphi_{\lambda} \tag{7}$$

$$\nabla_{\mu}g_{\alpha\beta}=0 \tag{8}$$

$$\nabla_\mu g^{\alpha\beta} = 0 \tag{9}$$

$$g^{\nu\mu} g_{\nu\mu} = \delta^\mu_\nu \tag{10}$$

$$g = \det(g_{\nu\mu}) \tag{11}$$

$$g' = \left(\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right)^{-2} g \tag{12}$$

$$\det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \text{ is Jacobian} \tag{13}$$

$$d^4x \equiv dx^0 dx^1 dx^2 dx^3 \tag{14}$$

$$d^4x' = \det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) d^4x \tag{15}$$

16 Regeneration and detection

The regeneration of an elementary particle by the controlling mechanism involves the creation of a new embedding location. Detection stops this regeneration process. At detection, the set $\{a_i^x\}$ is no longer filled by taking locations from the members of the set $\{\mathfrak{s}_i^x\}$. No more elements of the set are stored in the separable Hilbert space. With other words afterwards detection occurred at a precisely known location. However, that location was not known beforehand.

A virtual map images the completely regenerated set $\{a_i^x\}$ onto parameter space $\mathcal{R}^{⓪}$. This involves the reordering from the stochastic generation order to the ordering of this new parameter space. This first map turns the location swarm into the eigenspace of a virtual operator $\mathscr{b}$. A continuous location density distribution $\xi(q)$ describes the virtual map of the swarm into parameter space $\mathfrak{R}^{⓪}$. Actually each element of the original swarm is embedded into the deformable embedding continuum $\mathfrak{C}$ where that element is blurred with the Green's function of this embedding continuum.

This indirectly converts the operator $\mathscr{o}$, which describes the regeneration in the symmetry center $\mathfrak{S}_n^x$ into a differently ordered operator ξ that resides in the Gelfand triple $\mathcal{H}$. The defining function $\xi(q)$ of operator ξ describes the triggers in the non-homogeneous quaternionic second order partial differential equation, which describes the embedding behavior of $\mathfrak{C}$.

$$\xi = \nabla^*\nabla\psi = \{\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\psi = \frac{\partial^2\psi}{\partial\tau^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \qquad (1)$$

Function $\xi(q)$ uses $\mathfrak{R}^{⓪}$ as its parameter space. $\mathscr{o}$ describes the hopping of the point-like object, while $\xi(q)$ describes the density distribution of the corresponding location swarm.

Stochastic operator $\mathscr{o}$ describes the hopping of the point-like object, while ξ describes the density distribution of the image of the corresponding location swarm.

17 Photons

Photons are configured by solutions of the quaternionic second order partial differential equation. For odd numbers of participating dimensions the solutions of the homogeneous second order partial differential equation are combinations of shape keeping fronts. In three dimensions the spherical shape keeping fronts diminish their amplitude as $1/r$ with distance r of the trigger point. One-dimensional wave fronts keep their amplitude. As a consequence these shape keeping fronts can travel huge distances through the field that supports them. Each shape keeping front can carry a bit of information and/or energy. In order to reach these distances the carrying field must exist long enough and it must reach far enough.

The symmetry related field $\mathfrak{A}$ does not fulfil the requirements for long distance travel. It depends on the nearby existence of symmetry related charges and its amplitude also diminishes as 1/r with distance from the charge.

The embedding field $\mathfrak{C}$ is a better candidate for long distance transfer of energy and information. $\mathfrak{C}$ exists always and everywhere. One-dimensional shape keeping fronts vibrate the $\mathfrak{C}$ field, but do not deform this field. They just follow existing deformations.

Creating a string of one-dimensional shape keeping fronts requires a recurrent shape keeping front generation process. Such processes do not underlay the generation of symmetry related

charges that support the $\mathfrak{A}$ field. However, such processes exist during the recurrent embedding of artifacts that occurs in the $\mathfrak{C}$ field.

Recurrent generation of spherical shape keeping fronts is capable to deform the corresponding field. It has similar effects as a stationary deformation by a point-like artifact has.

The fixed speed of shape keeping fronts translates in the same fixed speed for the photons. A string of one-dimensional shape keeping fronts can carry a quantized amount of energy. The relation $E = h\,\nu$ and the fixed speed of photons indicate that at least at relative short range the string of shape keeping fronts takes a fixed amount of progression steps for its creation, for its passage and for its absorption.

However, observations of long range effects over cosmological distances reveal that these relations do not hold over huge distances. Red-shift of patterns of "old" photons that are emitted by atoms and arrive from distant galaxies indicate that the spatial part of field $\mathfrak{C}$ is extending as a function of progression.

With the interpretation of photons as strings of shape keeping fronts this means that the duration of emission and the duration of absorption are also functions of progression. As a consequence, some of the emitted wave fronts are "missed" at later absorption. The detected photon corresponds to a lower energy and a lower frequency than the emitted photon has. According to relation $E = h\,\nu$ that holds locally, the detected photon appears to be red-shifted. The energy of the "missed" shape keeping fronts is converted into other kinds of energy or the missed shape keeping fronts keep proceeding as lower energy photons. Spurious shape keeping fronts may stay undetected.

Thus, the quaternionic second order partial differential equation may be valid in the vicinity of the images of symmetry centers inside $\mathfrak{C}$, but does not properly describe the long range behavior of $\mathfrak{C}$. Due to its restricted range and the non-recurrent generation of its charges, the $\mathfrak{A}$ field does not show the equivalents of photons and red-shift phenomena.

The long range phenomena of photons indicate that the parameter space $\mathfrak{R}^{⓪}$ of $\mathfrak{C}$ may actually own an origin. For higher progression values and for most of the spatial reach of field $\mathfrak{C}$, that origin is located at huge distances. Information coming from low progression values arrives with photons that have travelled huge distances. They report about a situation in which symmetry centers were located on average at much smaller inter-distances.

Instead of photons the $\mathfrak{A}$ field may support waves, such as radio waves and microwaves. These waves are solutions of the wave equation, which is part of Maxwell based differential calculus.

On the other hand the wave equation also has shape keeping fronts as its solutions.

18 Inertia

18.1 Field corresponding to symmetry center

Dedicated mechanisms use symmetry centers as resource for the generation of the locations of elementary particles. Symmetry centers are interesting as a subject for studying inertia. They have a spherical shape and a finite active radius. The activity of the mechanisms can be characterized by a normalized continuous density distribution. As an example we apply a Gaussian density distribution.

$$\rho(\boldsymbol{r}) = \frac{Q}{2\pi\,\sigma^3\sqrt{2\pi}}\,\exp\left(-\frac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{2\sigma^2}\right) \tag{1}$$

Here $\boldsymbol{r}'$ is the location of the center of the symmetry center. The produced distribution moves together with the symmetry center.

The potential of a Gaussian density distribution $\rho(r)$ equals:

$$\varphi_0(\boldsymbol{r}-\boldsymbol{r}') = \frac{Q}{4\pi r^2}\operatorname{erf}\left(\frac{r^2}{\sigma\sqrt{2}}\right)(\boldsymbol{r}-\boldsymbol{r}') \approx \frac{Q(\boldsymbol{r}-\boldsymbol{r}')}{4\pi r^2} \text{ for large } r = |\boldsymbol{r}-\boldsymbol{r}'|. \tag{2}$$

Here r stands for $|\boldsymbol{r}-\boldsymbol{r}'|$.

This is not the electric potential. This potential is generated in a background embedding field $\mathfrak{C}$ due to the recurrent temporary embedding of artifacts that are taken from the symmetry center. This can be shown by computing the double differential of $\varphi_0(\boldsymbol{r})$:

$$\frac{\partial \operatorname{erf}(ar)}{\partial r} = \frac{2a}{\sqrt{\pi}}\exp(-a^2r^2) = \frac{2}{\sigma\sqrt{2\pi}}\exp\left(-\frac{r^2}{2\sigma^2}\right);\ a = \frac{1}{\sigma\sqrt{2}} \tag{3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\left(\frac{\operatorname{erf}(ar)}{r}\right)}{\partial r}\right) = \frac{2a^2}{\sqrt{\pi}}\exp(-a^2r^2) = \frac{1}{\sigma^2\sqrt{\pi}}\exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{4}$$

The plot of the potential proves that this potential has no singularity. It is smooth near the center point.

The gradient of the potential equals:

$$\begin{aligned}\boldsymbol{\nabla}\varphi_0 &= \frac{\partial\varphi_0}{\partial r}\frac{\boldsymbol{r}-\boldsymbol{r}'}{r} \\ &= -\frac{Q}{4\pi r^2}\operatorname{erf}\left(\frac{r}{\sigma\sqrt{2}}\right)\frac{\boldsymbol{r}-\boldsymbol{r}'}{r} + \frac{Q}{2\pi r\sigma\sqrt{2\pi}}\exp\left(-\frac{r^2}{2\sigma^2}\right)\frac{\boldsymbol{r}-\boldsymbol{r}'}{r}\end{aligned} \tag{5}$$

The potential φ_0 adds on top of the average value of the embedding field $\mathfrak{C}$. If the observer position $\boldsymbol{r}$ moves with speed $\boldsymbol{v}$ relative to the embedding continuum $\mathfrak{C}$ then as a consequence a corresponding contribution to the vector potential:

$$\mathfrak{C}(\boldsymbol{r}) = \overline{\mathfrak{C}_0}(\boldsymbol{r})\,\boldsymbol{v}$$

appears to exist. $\overline{\mathfrak{C}_0}(\boldsymbol{r})$ is the average scalar part of the embedding field $\mathfrak{C}(\boldsymbol{r})$. Thus, locally:

$$\mathfrak{C}_0(\boldsymbol{r}) = \overline{\mathfrak{C}_0}(\boldsymbol{r}) + \varphi_0(\boldsymbol{r} - \boldsymbol{r}')$$

$$\boldsymbol{\mathfrak{C}}(\boldsymbol{r}) = \overline{\mathfrak{C}_0}(\boldsymbol{r})\,\dot{\boldsymbol{r}}$$

$$\boldsymbol{\nabla}\overline{\mathfrak{C}_0} \approx \mathbf{0}$$

At the observer point the embedding continuum equals:

$$\mathfrak{C} = \overline{\mathfrak{C}_0}(\boldsymbol{r}) + \varphi_0(\boldsymbol{r} - \boldsymbol{r}') + \overline{\mathfrak{C}_0}(\boldsymbol{r})\,\dot{\boldsymbol{r}} \tag{6}$$

The scalar and vector potentials go together with a field $\boldsymbol{\mathfrak{E}}$:

$$\boldsymbol{\mathfrak{E}}(\boldsymbol{r}) \equiv -\frac{\partial}{\partial\tau}\boldsymbol{\mathfrak{C}} - \boldsymbol{\nabla}\mathfrak{C}(\boldsymbol{r}) = -\overline{\mathfrak{C}_0}(\boldsymbol{r})\,\ddot{\boldsymbol{r}} - \boldsymbol{\nabla}\,\varphi_0(\boldsymbol{r} - \boldsymbol{r}')$$

$$= -\overline{\mathfrak{C}_0}(\boldsymbol{r})\,\ddot{\boldsymbol{r}} + \frac{Q}{4\pi r^2}\operatorname{erf}\left(\frac{r}{\sigma\sqrt{2}}\right)\frac{\boldsymbol{r}}{r} - \frac{Q}{2\pi r\sigma\sqrt{2\pi}}\exp\left(-\frac{r^2}{2\sigma^2}\right)\frac{\boldsymbol{r}}{r}$$

For large $r = |\boldsymbol{r} - \boldsymbol{r}'|$

$$\boldsymbol{\mathfrak{E}}(\boldsymbol{r} - \boldsymbol{r}') \approx -\overline{\mathfrak{C}_0}(\boldsymbol{r})\,\ddot{\boldsymbol{r}} + \frac{Q}{4\pi}\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) = -\overline{\mathfrak{C}_0}(\boldsymbol{r})\,\ddot{\boldsymbol{r}} + \frac{Q}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|^3}(\boldsymbol{r} - \boldsymbol{r}') \tag{8}$$

Here again $\boldsymbol{r}'$ is the geometric center of the symmetry center. Both the acceleration $\ddot{\boldsymbol{r}}$ and the nearness of the artifact with strength Q determine the extra field $\boldsymbol{\mathfrak{E}}$. The first term on the left represents what is usually is experienced as inertia. The second term represents what is usually is experienced as gravitation.

In his paper "On the Origin of Inertia", Denis Sciama used the idea of Mach in order to construct the rather flat field that results from uniformly distributed charges [14]. He then uses the constructed field in order to generate the vector potential, which is experienced by the uniformly moving observer. Here we use the embedding field as the rather flat background field.

18.2 Forces between symmetry centers

Two different symmetry centers represent two different contributions to field $\boldsymbol{\mathfrak{E}}$.

The forces between two symmetry centers are specified by.

$$\boldsymbol{F_{12}} = -\boldsymbol{F_{21}} = \frac{Q_1\, Q_2}{4\pi|\boldsymbol{r_1} - \boldsymbol{r_2}|^3}(\boldsymbol{r_1} - \boldsymbol{r_2}) \tag{9}$$

18.3 Rotational inertia

If the observer rotates with respect to the embedding field, then the observer experiences a curl that is defined by

$$\mathfrak{B} = \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{1}$$

If the rotation changes, then this goes together with a rotation of the $\mathfrak{E}$ field, which counteracts the increase of the rotation.

$$\frac{\partial}{\partial \tau}\mathfrak{B} = -\boldsymbol{\nabla} \times \mathfrak{E} \tag{2}$$

In this case the observer experiences rotational inertia.

19 The dynamic picture

The paper reveals an intimate relation between basic fields and Hilbert spaces. The dynamics of this relation is controlled by mechanisms that are ignored by mainstream physical theories. Via the Hilbert spaces and a category of operators that reside in these Hilbert spaces, the basic fields get related to pairs of flat parameter spaces and functions that use these parameter spaces.

Symmetry centers that float relative to a reference parameter space are characterized by electric charges that relate to the symmetry properties of these spherical symmetric spatial parameter spaces. Each of these charges generates a contribution to a corresponding basic field $\mathfrak{A}$. The basic field and the charges interact dynamically. This basic field uses the reference parameter space on which the symmetry centers float as its natural parameter space.

The natural reference parameter space can be interpreted as the virgin state of the second basic field $\mathfrak{C}$, which acts as a model wide embedding continuum. Controlling mechanisms recurrently pick artifacts from the symmetry centers. The artifacts get subsequently embedded in the continuum $\mathfrak{C}$. This embedding is immediately released.

The controlling mechanisms operate on the rim between past and future. The past is left untouched and is kept stored in the eigenspaces of dedicated operators, which reside in the Hilbert spaces.

The corresponding geometrical data are stored together with a progression stamp in the form of sets of discrete quaternions or as quaternionic fields in the eigenspaces of these dedicated operators.

The controlling mechanisms and the restrictions that are set by the properties of the Hilbert spaces take care that the dynamical coherence of the model-wide step-wise embedding of the separable Hilbert space into the non-separable Hilbert space is ensured.

The dynamics is enforced by the point-like artifacts, which are generated by the actions of the controlling mechanisms. The embedding of these artifacts cause discontinuities and vibrations in the affected continuum.

In this way the sets of differential equations can describe the evolutions of the basic fields that play a role in this model.

This paints the dynamic picture that is described by the differential equations, which describe the behavior of the mentioned basic fields, which indirectly describe the behavior of the recurrently embedded artifacts.

In this interpretation the history of the model does not start with a big bang, but instead it starts with the virgin states of the basic fields. These virgin states are represented by their natural parameter space. These fields share the same natural parameter space.

20 Conclusion

By introducing a background space and a set of symmetry center types, this paper exploits the way in which quaternionic number systems can be ordered. This distinguishes between Cartesian ordering and spherical ordering and it reveals that these ordered versions of the number systems exist in several distinct symmetry flavors. Locally, the background space needs no origin and as a consequence it does not feature spin. The coupling of symmetry centers onto the background space offers the possibility to define an algorithm that computes corresponding symmetry related charges that are in agreement with the short list of electric charges and other discrete properties of elementary particles. For example, also the diversity of color charge and spin can be explained in this way. This indicates that elementary particles inherit these properties from the space in which they reside.

An important role is played by controlling mechanisms that are not part of the Hilbert spaces, but that make use of the Hilbert spaces as a structured storage medium. The elementary particles inherit their properties both from the Hilbert space and from these controlling mechanisms.

This paper also considers the embedding field $\mathfrak{C}$. It uses the same parameter space $\mathfrak{R}$ as the symmetry related field $\mathfrak{A}$ does. The embedding field obeys the same quaternionic differential calculus as the symmetry related field, but the triggers that cause discontinuities differ fundamentally between these fields. That is why these fields behave differently. Still both fields determine the kinematics of elementary particles. This is treated in more detail in [8].

It appears that the behavior of the symmetry related field $\mathfrak{A}$ can be investigated better by using Maxwell based differential calculus, while the behavior of the embedding field $\mathfrak{C}$ can better be comprehended by applying quaternionic differential calculus. Both fields are stored in quaternionic format in the non-separable Hilbert space.

The section about photons indicates that in contrast to what is usually suggested photons are not waves of the electric field $\mathfrak{A}$. Instead they vibrate the embedding field $\mathfrak{C}$ and follow its deformations. They do not themselves deform this field.

The behavior of “old” photons indicates that the validity range of the second order partial differential equations is restricted.

References

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http://en.wikipedia.org/wiki/History_of_quaternions

[2] Quantum logic was introduced by Garret Birkhoff and John von Neumann in their paper: G. Birkhoff and J. von Neumann, (1936) The Logic of Quantum Mechanics, *Annals of Mathematics*, **37**, 823–843

This paper also indicates the relation between this orthomodular lattice and separable Hilbert spaces.

[3] The Hilbert space was discovered in the first decades of the 20-th century by David Hilbert and others. See http://en.wikipedia.org/wiki/Hilbert_space.

[4] In the sixties Israel Gelfand and Georgyi Shilov introduced a way to model continuums via an extension of the separable Hilbert space into a so called Gelfand triple. The Gelfand triple often gets the name rigged Hilbert space. It is a non-separable Hilbert space. http://www.encyclopediaofmath.org/index.php?title=Rigged_Hilbert_space .

[5] Paul Dirac introduced the bra-ket notation, which popularized the usage of Hilbert spaces. Dirac also introduced its delta function, which is a generalized function. Spaces of generalized functions offered continuums before the Gelfand triple arrived.

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