# Fractal Chaotic Solitary Wavelets

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## Abstract:
The present work pertains to the generation of a chaotic signal by taking the inverse Fourier Transform of a Fractal Spectral Profile. The presence of chaos is ascertained and characterized using phase portraits, recurrence plots, Lyapunov exponents and Kolmogorov entropies. This signal is modulated by a hyperbolic secant solitary pulse to formulate the "Fractal Chaotic Solitary Wavelet" (FCSW), the analysis of which reveals vanishing higher moments, translating to efficient capabilities of burst and discontinuity detection, apart from the advantage of security owing to the induced unpredictability of chaos. This results in the proposed Fractal Chaotic Solitary Wavelets having potential applications in secure telecommunications and encryption systems.

**Keywords**: Fractals, Chaos Generation, Solitary Wavelet, Signal Processing, Nonlinear Analysis

## 1. Introduction
The study of Nonlinear Dynamics and Chaos Theory, fuelled by the recent advancements in signal processing and computing technologies have enhanced and enriched man's understanding of the complexities and intricacies of nature [1-9]. Among the most beautiful aspects of nature are the concepts of fractals, which essentially are structures where every part of the signal represents the whole [10]. From fern fronds to snail shells and snowflakes, fractal based designs are present in virtually every aspect of nature [10-13]. Mathematically, the most popular known fractal is the Mandelbrot set fractal generator, shown in Fig. 1, which is said to contain within itself, every known fractal configuration, in its infinite iterations [10].

Chaos Theory, with its characteristic signatures of determinism and an extremely sensitive dependence on initial conditions is another flagship of nonlinear dynamics, and has found widespread application in physics, biology and engineering [1-9]. The self-similar fractal nature and the unpredictability of chaos signals, leading to their 'theoretically deterministic yet practically random' nature, have enabled the use of chaotic signals for secure communication applications, where these signals are generated using op-amp based physical realizations of nonlinear partial differential equations [9].

In the present work, recursive iteration is used to construct a fractal spectral profile, whose inverse Fourier Transform yields the 'fractal signal'. The presence of chaos in the generated signal is quantitatively asserted using the Kolmogorov Entropy and Lyapunov Exponent. Finally, the generated signal is modulated by a hyperbolic secant pulse, giving rise to a fractal chaotic solitary wavelet (FCSW), which possesses an efficient burst detection capability, as seen from the vanishing higher moments. This property, along with the property of security enabled by chaos-induced uncertainty, as seen from the

Kolmogorov Entropy, enables the use of fractal chaotic solitary wavelets in secure communications and encryption techniques.

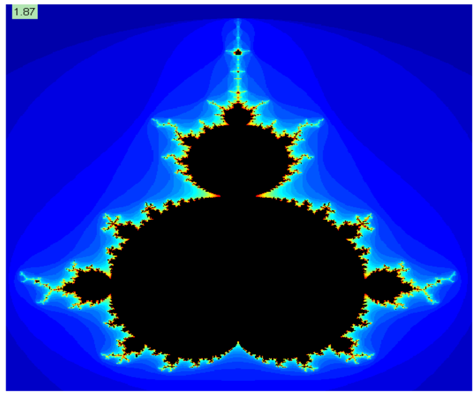


**Figure 1 The Mandelbrot Set Fractal Generator**

## 2. Generation of Fractal Chaotic Signals

The starting step in the proposed generation of fractal chaotic signals, is to define a fractal spectral profile. In the present work, the fractal spectrum is represented as follows:

$$C(f) = \sum_{i=1}^{N} \frac{1}{i}\delta\left(f - \frac{1}{r^i}\right) + \frac{1}{i}\delta\left(f + L - \frac{1}{r^i}\right) \qquad (1)$$

Where L is the length of the spectral profile, set to 500 in the present work, and N is the number of iterations, set to 10 in the present work; r is the control parameter, determining the spectral components generated, and δ(f) is the unit impulse function, which yields a value of 1 at f=0, and 0 everywhere else.

The time series signal C(t) is derived from C(f) by taking the inverse Fourier Transform, and the presence of chaos in C(t) is ascertained using the following nonlinear analysis techniques:

1. Phase Portrait: This is a plot of time derivative of the signal in terms of the signal, illustrating the phase space dynamics and qualitatively serving as a tool to assess sensitivity and ergodicity. The detection of ornamental and rich patterns in a phase portrait is a clear indicator of the presence of chaos underlying the scattering dynamics [9].
2. Recurrence Plot: The main premise in the concept of recurrence plot is that most natural processes possess recurrent behavior in the form of periodicities and irregular cyclicitie, with recurrence is defined as a condition where states in the system are arbitrarily close after some time of divergence [14]. For a discrete signal with N samples denoted by x(n), n<=N, the recurrence between the ith and jth point R(i,j) is given by $R(i,j) = \|x(i) - x(j)\| < T$, T being a threshold. The collection of all the points R(i,j) for all i,j<N form the Recurrence Matrix R, a plot of which is termed the Recurrence Plot (RP) [14].

3. Largest Lyapunov Exponent (LLE): This is a measure of a system's sensitive dependence on initial conditions. In the present work, Rosenstein's algorithm is used to compute the Lyapunov Exponents $\lambda_i$ from the voltage waveform, where the sensitive dependence is characterized by the divergence samples $d_j(i)$ between nearest trajectories represented by i given as follows, $C_j$ being a normalization constant [15,16]: $d_j(i) = C_j e^{\lambda_1 (i\Delta t)}; d(t) = Ce^{\lambda_1 t}$.
4. Kolmogorov Entropy (K): The Kolmogorov entropy ($K$), measured in information units of nats per symbol denotes the entropy and thus the uncertainty present in the signal, and large values indicate more dynamic and unpredictable behavior [15].

The nonlinear analysis for the signal generated using Eq. 1 is performed setting r to values of 2 and 3. The corresponding results are plotted in Fig. 2 and 3.

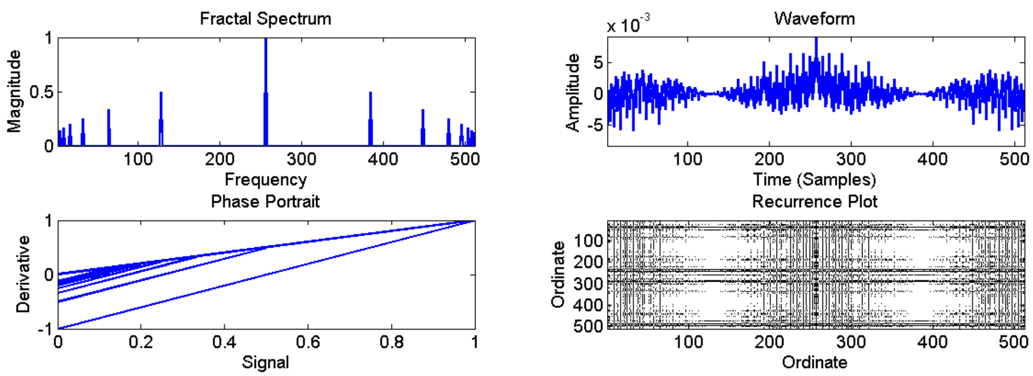


**Figure 2 Nonlinear Analysis of C(t) with r=2**

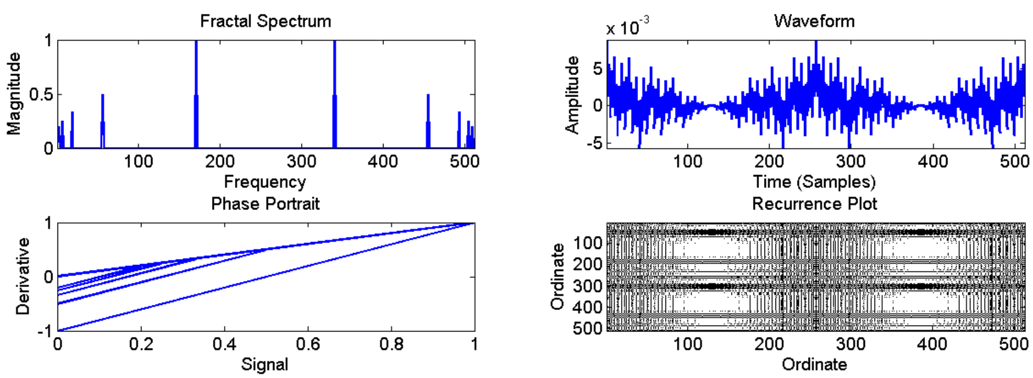


**Figure 3 Nonlinear Analysis of C(t) with r=3**

It is seen that in both the cases, the spectral profile shows a self-similar fractal nature, with every harmonic under L/r containing the similar distribution of sub-harmonics. The recurrence plot shows a darker trend, and more periodicity seen for r=3. A similar trend is also seen in the LLE values, obtained as 9.6205 and 5.3212 for r=2 and r=3 respectively. The corresponding K values are obtained as 5.2503 bits/symbol and 5.3257 bits/symbol respectively, indicating the chaotic and unpredictable nature of the generated signal.

Similar analysis is performed for a non-integer value of r, 1.1. The results are shown in Fig. 4, with LLE and K values obtained as 11.3598 and 6.0023 bits/symbol respectively, showing much higher chaoticity for non-integer r values.

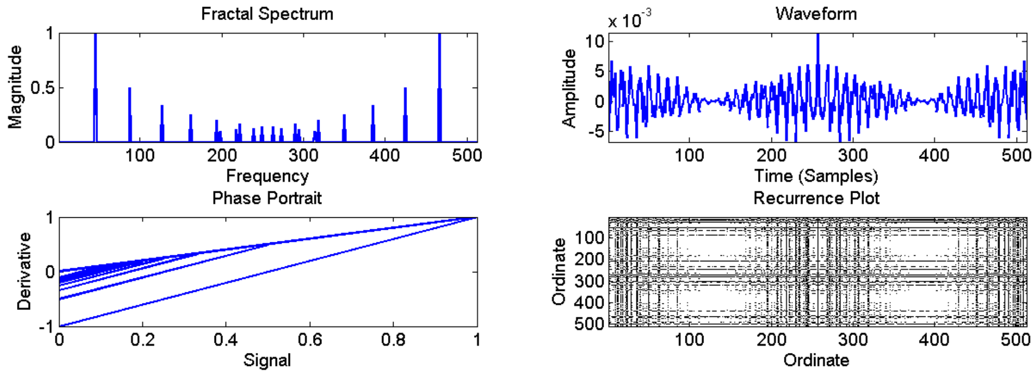


**Figure 4 Nonlinear Analysis of C(t) with r=1.1**

## 3. The Fractal Chaotic Solitary Wavelet

The next step involves the formulation of a wavelet using the generated signal C(t). The hyperbolic secant signal (sech) is used to modulate C(t), with the resultant signal ϕ(t) defined as the father wavelet, or the scaling function. The Father Wavelet ϕ thus defined is used as the basis to form the 'Mother Wavelet' ψ, such that the following criteria are satisfied [17-24]:

1. ψ(t) belongs to a subspace of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the space of absolutely and square integrable measurable functions.
2. φ(t) and ψ(t) are orthogonal to each other.
3. ψ(t) has zero mean, i.e. the following holds: $\int_{-\infty}^{\infty} \psi(t) dt = 0$.
4. ψ(t) has unity square norm, as per the following equation: $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$.
5. It is preferable, but not a mandatory criterion to ensure that ψ(t) possesses a higher number M vanishing moments. In other words, for all m<M, $\int_{-\infty}^{\infty} t^m \psi(t) dt = 0$.

The Mother Wavelet ψ is used to define the daughter wavelets $\psi_{(a,b)}(t)$ in the following fashion with a > 0 denoting the 'scale' and b ϵ ℝ denoting the 'shift': $\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$.

Based on the above procedure, the father and mother 'Fractal Chaotic Solitary Wavelets' (FCSW) have been formed using the MATLAB Wavelet Toolbox. The Father and Mother Wavelet Signals are plotted in Fig. 5, along with the decomposition and reconstruction low/high pass filter coefficients.

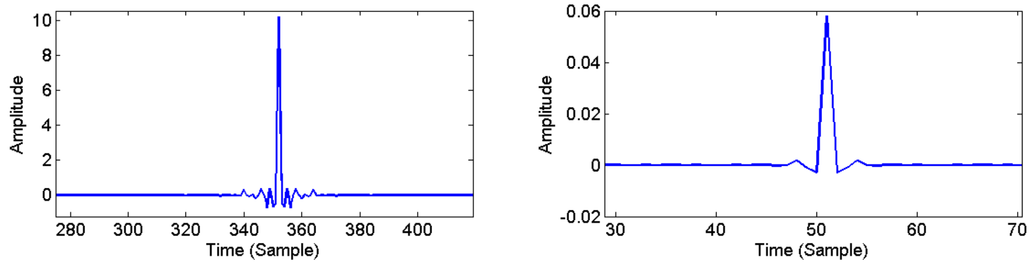


**Figure 5 Father and Mother FCSW Wavelets**

One of the preferable but not mandatory criteria mentioned above in the mother wavelet formulation is the presence of vanishing higher moments, where the ‘m’th moment of the mother wavelet ψ is given by Eq. 4 [17-24]. Physically, the existence of vanishing higher moments signifies that the wavelet has a compact, continuous, smooth structure, and that the analysis of bursts in signals with such wavelets can be carried out with minimal filtering [17-24].

In order to investigate and characterize the performance of the FCSW, the moments upto the tenth order of the wavelet are computed and compared with the corresponding moments of established wavelets, namely Daubechies 4 (DB4), Biorthogonal 4.4 (BIOR4.4), Coiflet 4 (COIF4) and the Discrete Meyer Wavelet (DMEY) [10]. The moments are tabulated in Table 1.

**Table 1 Moments of Various Wavelets upto the Tenth Order**

| Moments | DB4 | BIOR4.4 | COIF4 | DMEY | FCSW |
|---|---|---|---|---|---|
| First | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 |
| Second | 1.33E-01 | 1.09E-01 | 4.26E-02 | 9.90E-03 | 1.10E-02 |
| Third | 2.05E-02 | 5.41E-02 | 1.96E-02 | 3.30E-03 | 1.10E-03 |
| Fourth | 1.13E-01 | 9.95E-02 | 3.74E-02 | 7.60E-03 | 2.20E-03 |
| Fifth | 3.25E-02 | 8.78E-02 | 3.19E-02 | 5.40E-03 | 9.50E-04 |
| Sixth | 1.05E-01 | 1.12E-01 | 4.10E-02 | 7.30E-03 | 9.38E-04 |
| Seventh | 4.19E-02 | 1.16E-01 | 4.19E-02 | 6.60E-03 | 4.47E-04 |
| Eighth | 9.95E-02 | 3.45E-01 | 4.85E-02 | 7.60E-03 | 4.10E-04 |
| Ninth | 4.96E-02 | 1.47E-01 | 5.24E-02 | 7.60E-03 | 3.10E-04 |
| Tenth | 9.56E-02 | 1.66E-01 | 5.87E-02 | 8.30E-03 | 2.26E-04 |

From Table 1, it is seen that the higher moments of the FCSW tend toward zero. From this trend, it is seen that even the Meyer wavelet moments increase after a certain order (sixth). This gives the formulated solitary wavelet the exclusive advantages of smoothness, compactness and effective detection of bursts as explained earlier.

## 4. Conclusion

Taking cue from the successful advancement of nonlinear dynamics in recent times, the present work proposes, designs and characterizes a chaotic signal generated using a fractal spectral profile. Specifically, the signals are generated by taking the inverse Fourier Transform of a fractal spectral profile generated using Eq. 1, and the presence of chaos is ascertained and quantitatively characterized using phase portraits, recurrence plots, Lyapunov exponents and Kolmogorov entropies. Following this, the generated chaotic signal is modulated by a hyperbolic secant signal, to form a “Fractal Chaotic Solitary Wavelet”, which is seen to possess vanishing higher moments, resulting in efficient detection of bursts and discontinuities, apart from the security aspect introduced by the chaos induced uncertainty. This results in the proposed Fractal Chaotic Solitary Wavelets having applications in secure telecommunications and encryption systems.

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