Genesis of the Solitary Wavelet

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Abstract: A solitary wavelet, based on the hyperbolic secant function is proposed, characterized and applied to real-time data. Numerical analysis of the solitary wavelet reveals that it has a huge number of vanishing higher order moments, tending rapidly towards zero with a negative logarithmic slope. It is seen that the wavelet has a very low number of oscillatory sub-lobes, thus making it the ideal candidate to perform signal analysis of burst-type phenomena without undergoing multiple levels of filtering and approximation, and this concept is illustrated by effectively detecting the QRS complex of an ECG cycle without undergoing multiple filtering levels. The ability of the proposed wavelet to perform analysis of a diverse variety of real time data without multiple levels of decomposition and reconstruction forms the novelty of the present work.

One Sentence Summary: A Solitary Wavelet with vanishing higher moments yielding effective detection of bursts with minimal filtering is proposed and investigated.

Introduction:

The efficient understanding of the rich dynamics of most real-time signals and waveforms, both natural and man-made, requires thorough analysis at multiple levels of temporal and spectral resolution [1,2]. It is along these lines of thought that the past couple of decades have witnessed extensive research and application of the wavelet transform – a comprehensive extension to the frequency-localized Fourier Transform by adding time localization as well [2,3,4].

One of the most significant applications of the wavelet based analysis is in detecting, identifying, characterizing and predicting trends and features in real-time signals [2,3]. The key clues in such feature detections are discontinuities and bursts in the time series data, which usually tend to be extremely compact and localized in time [2,3]. It has been a constant challenge to formulate various mathematical functions as bases for wavelets, which are able to capture such bursts with the least possible level of decomposition, reconstruction and filtering [1,2,3,4]. The ability of a wavelet to capture such bursts effectively translates analytically to the wavelet function, also called the 'Mother Wavelet' having the most possible number of zero higher order central moments [3,4,5]. In, literature, many wavelet families have been formulated such as the Daubechies, Biorthogonal, Reverse Biorthogonal and Coiflets, to try and achieve this criteria of maximum number of vanishing moments [4,5,6,7,8,9,10]. Of particular mention are the Daubechies wavelets, which exhibit the highest number A of vanishing moments for a given 'support' N=2A [9]. Also of significant mention is the Discrete Meyer Wavelet which exhibits vanishing moments up to very large orders [10]. However, most of the wavelets mentioned above invariably exhibit increasing moments of higher orders, as a result of which, extensive wavelet based decomposition and reconstruction at multiple levels need to be performed for most real time data before effective burst detections can be carried out [1,5,8]. At a hardware level, this almost invariably and inevitably translates to higher amount of power dissipation.

In the present work, a new kind of wavelet, termed the 'Solitary Wavelet' is proposed based on the hyperbolic secant function, which is known to possess an extremely smooth and compact structure. It is seen that the higher order moments of this wavelet is closer to zero and vanishes at a rate logarithmically faster than existing wavelets mentioned above. The primary application targeted for this wavelet is the effective detection of bursts in signals [5]. In order to understand this, a case study pertaining to the automated detection of the QRS complex in Electrocardiogram Time Series without filtering/reconstruction is presented [11,12].

The extreme compactness and smoothness of the hyperbolic secant based solitary wavelet function, as seen by the rapidly vanishing higher moments, yields in very effective detection of bursts and peaks with minimal amount of filtering, and this forms the significant novelty of the present work, as a consequence of which, the solitary wavelet finds extensive application in analyzing a diverse variety of real time data effectively.

Formulation of the Solitary Wavelet:

1. The Father and Mother Wavelets

The primary inspiration for the concept of solitary wavelet arises from the hyperbolic secant (sech) function. This function is popular in the field of photonics as a solution to various nonlinear differential equations, and the waveform of the sech function is known to be compact and extremely smooth [13]. Based on these properties, the solitary wavelet is formulated according to the following procedure:

1. The first step is to define the Scaling Function, also called the 'Father Wavelet' ϕ in continuous time, based on the hyperbolic secant as follows [14]:

$$\phi(t) = \operatorname{sech}(t) \qquad (1)$$

2. The Solitary Father Wavelet ϕ thus defined is used as the basis to form the Solitary 'Mother Wavelet' ψ, such that the following criteria are satisfied [1,2,5,9,10]:
   a. $\psi(t)$ belongs to a subspace of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the space of absolutely and square integrable measurable functions.
   b. $\phi(t)$ and $\psi(t)$ are orthogonal to each other.
   c. $\psi(t)$ has zero mean, i.e. the following holds:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \qquad (2)$$

   d. $\psi(t)$ has unity square norm, as per the following equation:

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1 \qquad (3)$$

   e. It is preferable, but not a mandatory criterion to ensure that $\psi(t)$ possesses a higher number M vanishing moments. In other words, for all m<M,

$$\int_{-\infty}^{\infty} t^m \psi(t)dt = 0 \qquad (4)$$

3. The Solitary Mother Wavelet ψ is used to define the solitary daughter wavelets $\psi_{(a,b)}(t)$ in the following fashion with $a > 0$ denoting the 'scale' and $b \in \mathbb{R}$ denoting the 'shift' :

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \qquad (5)$$

4. The Father, Mother and Daughter Solitary Wavelets are expressed as discrete signals $\phi(n)$, $\psi(n)$ and $\boldsymbol{\psi}_{(\boldsymbol{a},\boldsymbol{b})}(\boldsymbol{n})$ centered around zero.

Based on the above procedure, the solitary father and mother wavelets have been formed using the MATLAB Wavelet Toolbox [10]. The Father and Mother Wavelet Signals are plotted in Fig. 1.

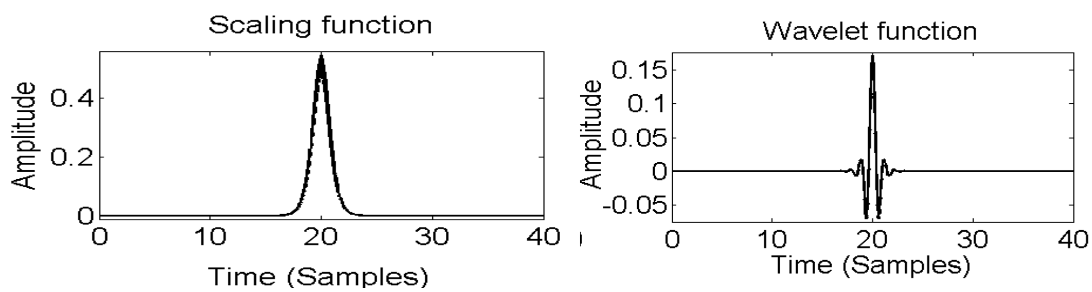

Figure 1 The Father and Mother Wavelet Signals

2. Analysis of Higher Moments

One of the preferable but not mandatory criteria mentioned above in the mother wavelet formulation is the presence of vanishing higher moments, where the 'm'th moment of the mother wavelet ψ is given by Eq. 4. Physically, the existence of vanishing higher moments signifies that the wavelet has a compact, continuous, smooth structure, and that the analysis of bursts in signals with such wavelets can be carried out with minimal filtering [1,2,5,9].

In order to investigate and characterize the performance of the solitary wavelet, the moments upto the tenth order of the solitary mother wavelet (SOL) are computed and compared with the corresponding moments of six established wavelets, namely Daubechies 4 (DB4), Biorthogonal 4.4 (BIOR4.4), Reverse Biorthogonal 4.4 (RBIO4.4), Symlet 4 (SYM4), Coiflet 4 (COIF4) and the Discrete Meyer Wavelet (DMEY) [10]. The moments are tabulated in Table 1.

Table 1 Moments of Various Wavelets upto the Tenth Order

Moments	DB4	BIOR4.4	RBIO4.4	SYM4	COIF4	DMEY	SOL
First	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Second	1.33E-01	1.09E-01	1.16E-01	1.33E-01	4.26E-02	9.90E-03	1.00E-02
Third	2.05E-02	5.41E-02	8.01E-02	6.96E-02	1.96E-02	3.30E-03	1.50E-03
Fourth	1.13E-01	9.95E-02	1.27E-01	1.30E-01	3.74E-02	7.60E-03	2.80E-03
Fifth	3.25E-02	8.78E-02	1.37E-01	1.14E-01	3.19E-02	5.40E-03	9.00E-04
Sixth	1.05E-01	1.12E-01	1.73E-01	1.52E-01	4.10E-02	7.30E-03	9.38E-04
Seventh	4.19E-02	1.16E-01	2.05E-01	1.58E-01	4.19E-02	6.60E-03	4.67E-04
Eighth	9.95E-02	3.45E-01	2.49E-01	1.90E-01	4.85E-02	7.60E-03	3.50E-04
Ninth	4.96E-02	1.47E-01	2.99E-01	2.08E-01	5.24E-02	7.60E-03	2.00E-04
Tenth	9.56E-02	1.66E-01	3.62E-01	2.42E-01	5.87E-02	8.30E-03	1.36E-04

From Table 1, it is seen that the higher moments of the solitary wavelet tend toward zero. From this trend, it is seen that even the Meyer wavelet moments increase after a certain order (sixth). In order to better capture the trends of the higher moments, the moments of the various wavelets from the third order onwards are plotted on a logarithmic scale in Fig. 2. It is clearly seen that while all the other wavelet moments including those of the Daubechies and Meyer show an increasing trend, the solitary wavelet moments show a decreasing trend with a negative logarithmic slope. This indicates that the moments of the solitary wavelet rapidly decay and vanish toward zero. This gives the solitary wavelet the exclusive advantages of smoothness, compactness and effective detection of bursts as explained earlier.

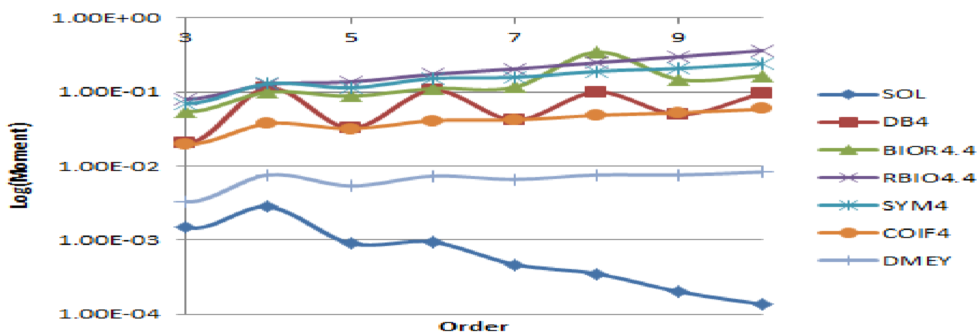

Figure 2 Higher order Moments of various wavelets plotted with logarithmic scale.

Applications of the Solitary Wavelet

One of the most popular applications of wavelet based signal analysis is in the detection of various features in the electrocardiogram (ECG) signal [11]. One cycle of a healthy ECG signal typically consists of three bursts, named the P-wave, the QRS-complex and the T-wave [11]. These are caused due to the contraction of the atria, contraction of the ventricles and the repolarization of the atria respectively. It can be seen that the three bursts have different scales (pulse widths) [11,12]. The detection of the onsets and widths of the three waves prove to be of immense value in diagnosing various ailments such as hypertrophy, arrhythmia, blocks, infarctions and ischemia [11,12,15].

Conventionally, to perform ECG analysis, the Daubechies Wavelet is chosen, since for a given support N, the Daubechies wavelet DbA yields A=N/2 vanishing moments. Typically, Db4 or Db6 wavelets are used [16]. The typical procedure is that the raw ECG signal are decomposed upto a certain level X and then shifted by the approximation of X-1 to get a baseline drift free signal. Following this, the signal is again decomposed to a different level and the higher order details are used to construct the signal. Finally, by performing wavelet analysis at the desired scale, the peaks are identified [17-24].

The primary objective of the solitary wavelet introduced in the present work is to avoid the tedious multiple filtering process. Since the solitary wavelet has a high number of vanishing moments, it is able to approximate bursts effectively.

The solitary wavelet analysis of a function f(t) is formally defined as follows [9]:

$$\boldsymbol{F(a,b)} = \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{a}}}\int_{-\infty}^{\infty} \boldsymbol{f(t)\psi}\left(\frac{\boldsymbol{t-b}}{\boldsymbol{a}}\right)\boldsymbol{dt} \quad (6)$$

Here, a and b denote the scales and shifts respectively. In order to effectively identify the peaks corresponding to an ECG waveform, it is important to know the typical scales at which the P, QRS and T waves occur. The ECG typically has a period of 1s, resulting in a frequency of 1Hz [15]. In the present work, a healthy ECG sample is obtained from the "Common Standard for Electrocardiography" library's ECG databank [25]. The samples obtained are of a sampling frequency of 900Hz, and each ECG cycle typically consists of 800-1000 samples [15,25]. The Shift-Scale contour plotted for six cycles of this ECG data are shown in Fig. 3.

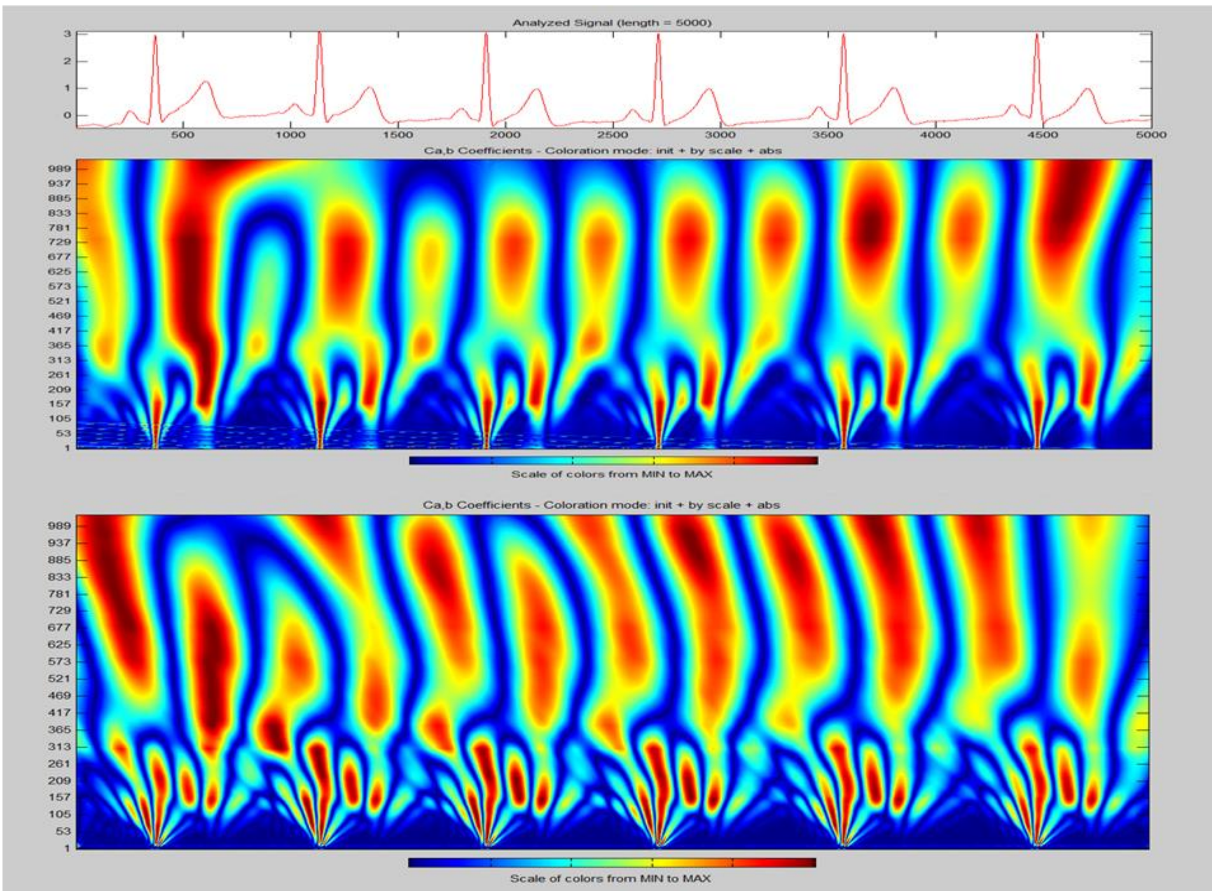

Figure 3 Analysis of ECG Data using Solitary (top) and Daubechies Db4 Wavelet (bottom)

As can be observed from Fig. 3, in the case of Solitary wavelet, in the scale range of 1-150, six well defined peaks are present, and these peaks directly coincide with the QRS complex of each cycle. On the other hand, the same scale range of 1-150 in the Daubechies 4 analysis shows multiple peaks corresponding to each QRS complex. These multiple peaks give the appearance of a 'dog's paw', and for each QRS complex, all other peaks except for the longest are spurious peaks. The same conclusion can be derived for the T waveforms of each cycle (scale value 150-300) and the P waveforms of each cycle (scale value 300-450). The presence of spurious peaks in the Daubechies analysis is due to the oscillatory nature of the wavelet [9]. It is to avoid these spurious peaks that multiple levels of filtering are required in the Daubechies wavelet case, whereas such filtering is unnecessary for the Solitary wavelet [1,5].

As an example illustrating this concept, the QRS complex of the ECG signal are detected using solitary and db4 wavelets and are then compared. It can be seen that the QRS peaks in each cycle of ECG are about 25 samples wide and thus a scale value a of 25 is chosen as he basis. The solitary wavelet analysis of the ECG signal given by Eq. 6 is thus carried out for a scale of 25 and the analysis is overlaid on the original ECG data in Fig. 4. A similar analysis using the Db4 wavelet at the same scale value of 25 is also plotted. As can be seen, while the solitary wavelet analysis shows clear unambiguous peaks corresponding to the QRS complexes of each cycle, the Daubechies wavelet oscillates heavily and leads to ambiguity in detection of QRS complexes, necessitating the need for multiple levels of filtering and reconstruction in the latter.

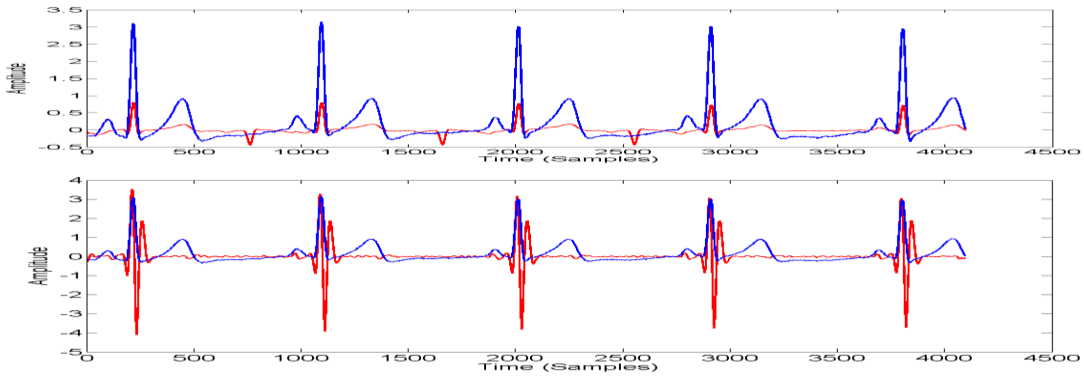

Figure 4 Solitary (top) and Db4 (bottom) Analysis (Red) of ECG Signal (Blue) for a typical QRS scale of 25

As an extension of this inference, a practical application of the solitary wavelet is in detecting specific cycles in a given ECG data where the heartbeat is slower or faster than the normal, thus pertaining to cases of tachycardia or bradycardia [11,12,15]. In order to demonstrate this, data from an ECG simulator operating at 200 samples/second is taken at two different rates, 75 beats per second (bps) and 140 beats per second. The scale for QRS detection is set to 40, corresponding to the slower rate (75bps). The analysis is plotted in Fig. 5.

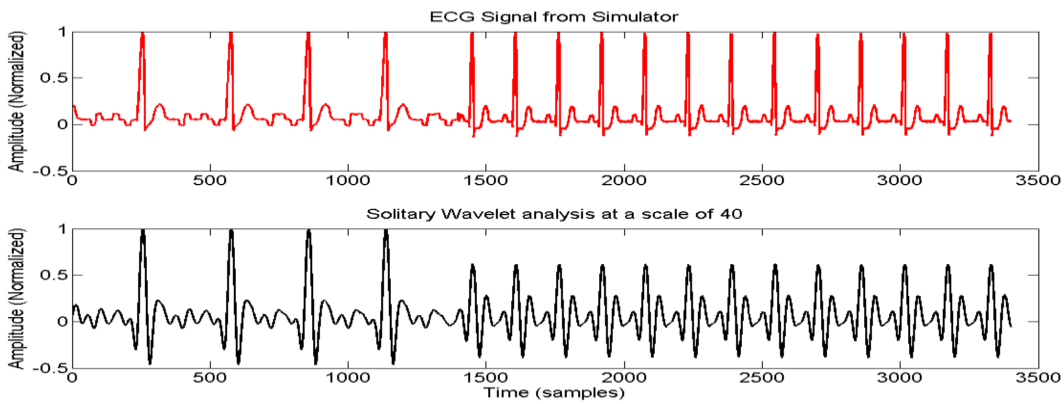

Figure 5 Solitary analysis of ECG signal with two different beat rates

It can be seen from the analysis in Fig. 5 that since the analysis scale is set to 40 which corresponds to 75bps, the peaks of ECG cycles not corresponding to this heart rate (140bps; at times 1500 onwards) exhibit significant attenuation as opposed to the first four cycles corresponding to a 75bps rate. In practical scenarios, such variations in analyzed amplitude corresponding to variations in heart rates can be used as a calibration tool for the solitary analysis, and a sensitivity factor can be established.

An interesting observation holds in data pertaining to catch of American Plaice Fish in Iceland from 1992 to 2012, as shown in Fig. 6 [26]. Here, the data is seasonal, and slowly declining over the years. Though this is evident in the long term time series data, if a person had done the wavelet analysis in the middle instant of the data (around 150), the big red patch at 70 and the big blue patch at 140 in the scale of 50, would have clearly heralded the befall, where the next dataset at around 210 shows the complete absence of any patch at 50 scale. In other words, the decline of fish catch could easily be predicted and appropriate precautions taken with the help of Solitary wavelet.

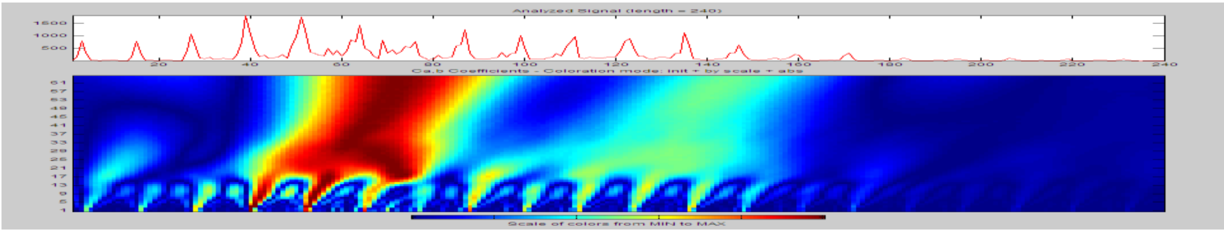

Figure 6 Solitary Wavelet analysis of Fish Catch data in Iceland

Conclusion

Taking inspiration from the hyperbolic secant function, the father and mother wavelets corresponding to the solitary wavelet have been formulated, investigated and various applications have been explored.

One significant property observed in the solitary mother wavelet is that the higher order moments tend towards zero, vanishing with a negative logarithmic slope, a feature not exhibited by most other wavelet functions. A direct physical consequence of this property is that the mother wavelet is compact, smooth, has a symmetrical structure and yields to effective burst type signal detection with minimal filtering. In order to ascertain this inference, a real-time ECG data is taken and a direct scale based analysis is performed using both solitary and Daubechies wavelets. The results clearly indicate the increased effectiveness to which the solitary wavelet is able to detect the QRS complex of the ECG signal. Other pattern and trend detection applications such as fisheries have also been explored.

The simple structure of the solitary wavelet with minimal oscillation, resulting in a high number of vanishing moments and yielding effective detection of bursts and peaks in real time data pertaining to diverse variety of fields form the key highlights of the present work on the solitary wavelet.

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