Generation of Frequency Controlled Chaotic Signals using Ramanujan Theta Function

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Abstract

The present work purports to a signal based generation of chaos, thereby offering a radically innovative solution to the issues of tunability caused in conventional system based chaos generation circuits. Specifically, the Ramanujan Theta Function is seen to represent the output signal of a coupled nonlinear system with two driving signals. Using this concept, the iterative map of the Ramanujan Theta Function is developed using the frequency ratio of driving signals as the control parameter, and the ratio dependent dynamics are studied using the bifurcation plot. The proposed system is implemented in hardware using FPGA and the presence of chaos is validated qualitatively using phase portraits and quantitatively using Lyapunov Exponents, whose trend indeed agree with the one observed in the bifurcation plot. The innovative perspective of signal based chaos proposed using the Ramanujan Theta Function enables easy tunability in chaotic generation systems and this forms the novelty of the present work.

Keywords: Chaos Generation, Ramanujan Theta Function, Frequency Controlled Chaos, Bifurcation Analysis

1. Introduction

Chaos theory, the hallmark of nonlinear science, pertains to systems that are essentially deterministic and exhibit an extreme sensitivity to initial conditions, and a distinct signature of chaotic systems are their ornamental and rich patterns exhibited on long term evolution [1, 2]. Owing to these properties, chaos theory has found extensive applications in sciences, where most of the patterns found in nature such as Electroencephalograms and Solar illumination flux exhibiting chaotic dynamics [3, 4, 5, 6, 7, 8, 9, 10], as well as in engineering such as in designing secure communication and encryption systems [11, 12, 13, 14, 15, 16]. The physical realization of chaotic systems using electronic circuits, traditionally builds on the Chua diode, the basic nonlinear element, connected to various amplifiers and passive components such as resistors and capacitors [17, 18, 19, 20, 21]. These circuits are typically op-amp based implementations of coupled nonlinear differential equations such as the Lorenz and Rossler systems [17]. While such methods unmistakably generate chaotic signals, they suffer from an issue of concern: the controllability of chaos is typically achieved using variable resistors or capacitors, which require manual tuning, and this becomes difficult when implemented as Integrated Circuits (IC) level at high frequencies [22, 23].

The present work attempts to address this issue through a radical approach. The key idea is to use signal based tuning of driven chaotic systems, rather than conventional system based tuning. Thus it is necessary to generate chaotic output signals which are sensitive to some parameter (amplitude, frequency, phase or polarity) of the input driving signals. For this purpose, we turn our attention towards q-analog theory, specifically the Ramanujan Theta Function (RTF) [24, 25]. Introduced by the mathematician Ramanujan in his last days as a generalization of Jacobi Theta Functions, the RTF is used typically to determine the critical dimensions in Bosonic String Theory and M-Theory [26, 27, 28, 29, 30, 31, 32]. In the present work, the RTF, a function of two variables a and b is viewed as the output signal from a nonlinearly coupled system. This is done by setting a and b to two sinusoidal signals, representing the driving signals of the nonlinearly coupled system. The behavior and dynamics of the system is studied by forming a differential equation and hence an iterative map using the RTF, and plotting the bifurcation diagram, depicting the output signal as a function of the control parameter r, which in turn is the frequency ratio between the two driving signals. Following this, the phase portrait, examining the first and second derivatives of the output signal X as a function of X are plotted. The presence of chaos, and the dependence of the nature of chaos on the control parameter r is ascertained by computing the largest Lyapunov Exponent.

The concept of a signal based chaos generation, using Ramanujan Theta Functions, provide a radically fresh perspective on the nonlinear dynamics and chaos generation, with the key advantage being easy tunability in physical realizations, and this forms the novelty of the present work.

$$T1(i) = [\pi f_1 n(n+1) sin(2\pi f_1 i)^{(n(n+1)/2)-1} cos(2\pi f_1 i) sin(2\pi r f_1 i)^{n(n-1)/2}] \tag{3}$$

$$T2(i) = [\pi r f_1 n(n-1) sin(2\pi r f_1 i)^{(n(n-1)/2)-1} cos(2\pi r f_1 i) sin(2\pi f_1 i)^{n(n+1)/2}] \tag{4}$$

$$X(i+1) = X(i) + \sum_{n=-\infty}^{\infty} [T1(i) + T2(i)] \tag{5}$$

2. The Ramanujan Theta Function Iterative Map

We start with the general form of the Ramanujan Theta Function, given as follows [28]:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{1}$$

where a and b are variables with the specific condition that $|ab| \leq 1$. Since the present work pertains to a signal based chaos, the variables a and b are represented as sinusoidal signals with frequencies f_1 and f_2=rf_1 respectively, where r represents the ratio of the frequencies r=f_2/f_1. Consequently, the output $f(a,b)$ is represented as a time varying signal $X(t)$. Thus, the output signal is given by

$$X(t) = \sum_{n=-\infty}^{\infty} sin(2\pi f_1 t)^{n(n+1)/2} sin(2\pi r f_1 t)^{n(n-1)/2} \tag{2}$$

From this relation it is seen that the output signal results from a mixing (multiplication) operation of two nonlinearly waveshaped (exponents of sinusoids) inputs, both the mathematical terms giving rise to new frequencies, other than f_1 and rf_1 [22]. In order to understand the evolution and dynamics of the system, an iterative map has to be formed. In order to do this, the derivative of $X(t)$ is found as $d(X(t))/dt$. By discretizing $X(t)$ as well as its derivative, the latter is expressed as the difference equation between successive samples of $X(n)$ as $X(i+1) - X(i)$. By rearranging, $X(i+1)$ is obtained as a function of its previous sample $X(i)$ and the derivative of $X(i)$ as follows, depicting the dependence of the current sample on previous samples, and for this reason termed the 'Iterative Map' [1, 2].

As seen from the iterative map, the evolution of X depends intricately on r. Thus, a mapping of X as a function of r, termed the 'Bifurcation Plot' is the ideal tool to study the evolution and dynamics of the system represented by the RTF, and this is plotted in Fig. (1) for values of r from 0 to 4. [1, 2]

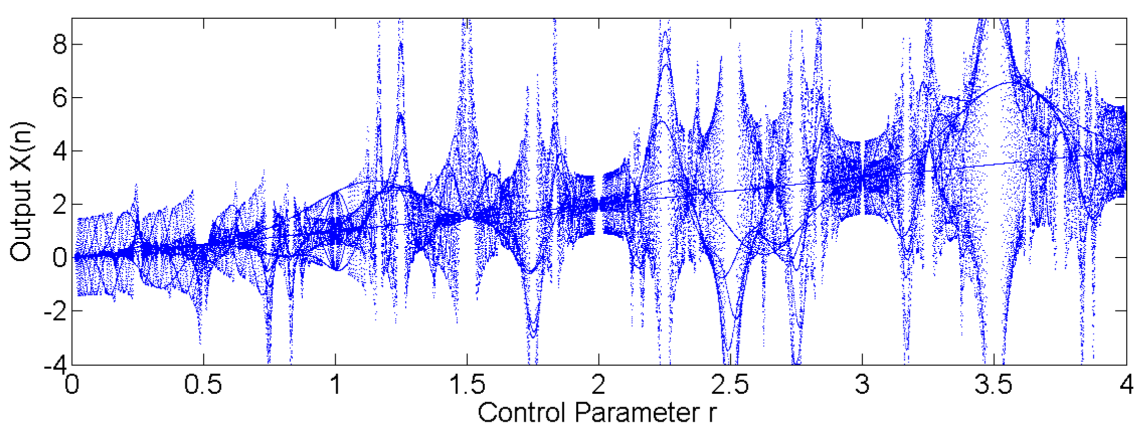

Figure 1: Bifurcation Plot of the Ramanujan Theta Function

It is clear from the bifurcation plot that the system represented by the RTF indeed shows an evolutionary behavior dependent on r, with certain regions, viewed as 'sparse' and 'dense' corresponding to non-chaotic and chaotic regimes of operation respectively. It must be noted that this evolutionary behavior is for a nonlinearly coupled frequency controlled system, and as with other similar systems in literature, such as the standard circle map, the proposed system follows a quasiperiodic route to chaos [33, 34, 35, 36, 37, 38].

3. Characterization of the RTF Chaos

The chaotic system specified by Fig. (1) is physically realized at the hardware level by implementing the using Equation 2 in Altera Cyclone II 2C20 FPGA, with the clock frequency set to 25MHz and a large value (10^6) used to approximate the summation limit. The obtained output waveform X is plotted in Fig. (2). It is noteworthy that from a superficial glance, the output waveform gives the appearance of a quasiperiodic signal, while the specific features of the waveform, its crests and troughs clearly exhibit chaotic patterns, testifying to the quasiperiodic route to chaos [37, 38].

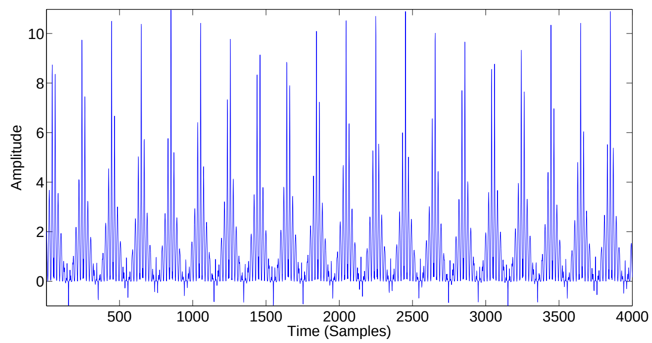

Figure 2: Experimentally obtained Output RTF Chaotic Waveform

In order to examine the system dynamics for specific values of r, the phase portrait, a plot of the time derivative of a signal dX/dt as a function of the signal X illustrating the phase space dynamics, qualitatively serving as a tool to assess various chaotic parameters such as sensitivity and ergodicity, is used [1, 2]. From Fig. (1), it is seen that a non-integral r value close to 2, such as 2.0322 corresponds to a dense patch indicative of chaos, and to validate this inference, three phase portraits of X are plotted in Fig. (3) - (5), using the first and second derivatives of X.

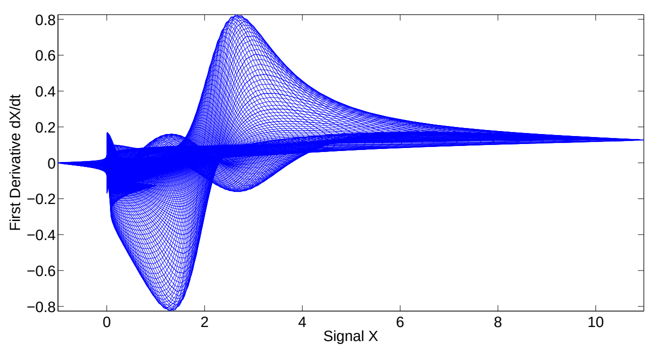

Figure 3: Signal Phase Portrait of the RTF Output with $r = 2.0322$

The phase portraits exhibit dense and ornamental patterns of X, qualitatively asserting the presence of chaos. Moreover, values of r corresponding to two other regions of Fig. (1) are considered - a non-chaotic $r = 2$, and a quasiperiodic $r = 1.159$, and the corresponding phase portraits are plotted in Fig. (6) and Fig. (7) respectively.

The phase portraits indeed confirm the non-chaotic nature of X at $r = 2$ and the quasiperiodic patterned nature at $r = 1.159$ verifying the predictions in accordance with the bifurcation plot.

The nature of chaos in X is assertively established by calculating the largest Lyapunov Exponent (LLE), quantifying the systems sensitive dependence on initial conditions [39]. The Rosensteins algorithm is used to compute the Lyapunov Exponents λ_i from the time series of Fig. (2), where the sensitive dependence is characterized by the divergence samples $d_j(i)$ between nearest trajectories represented by j given as $d_j(i) = C_j e^{\lambda_i(i\delta t)}$, C_j being a normalization constant [40].

It is seen that the LLE corresponding to the chaotic r value of 2.0322 is obtained as 3.36, whereas the LLE values of $r = 2$ and $r = 1.159$ are obtained as -0.03 and 1.12 respectively, thereby ascertaining the presence of chaos and validating the ratio dependent trends observed in the bifurcation plot and phase portraits.

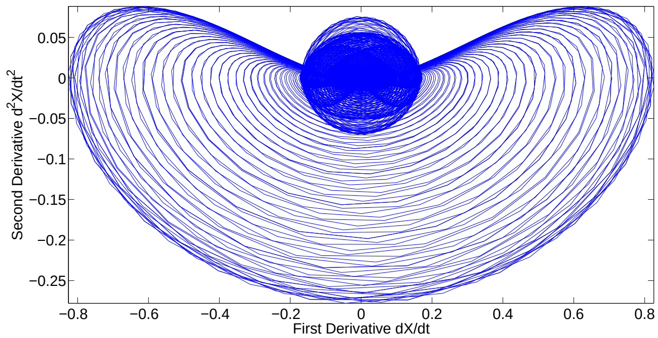

Figure 4: Derivative Phase Portrait of the RTF Output with $r = 2.0322$

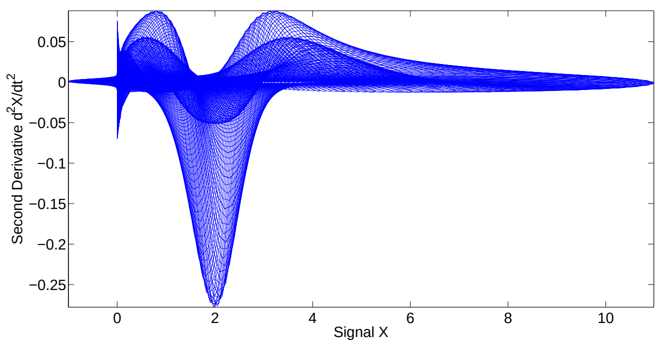

Figure 5: Transderivative Phase Portrait of the RTF Output with $r = 2.0322$

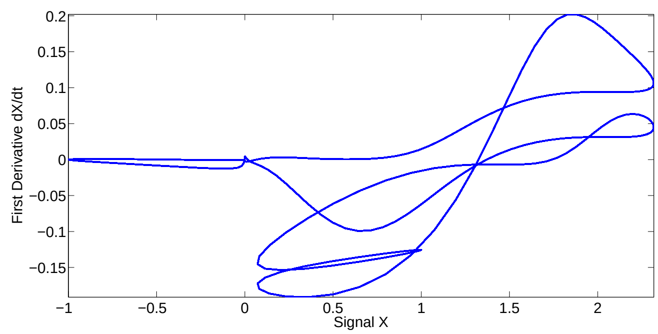

Figure 6: Signal Phase Portrait of the RTF Output with $r = 2$

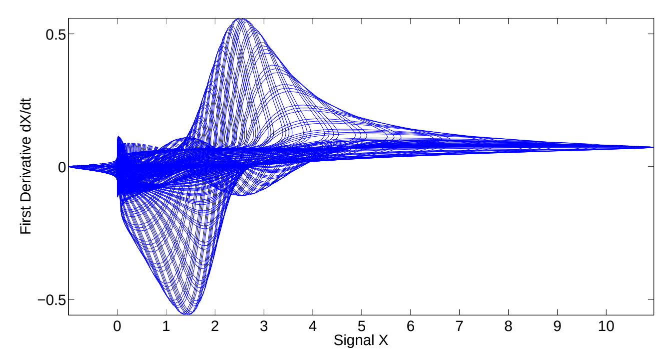

Figure 7: Signal Phase Portrait of the RTF Output with $r = 1.159$

The worthiness of the generated signal as a potential telecommunication carrier can be established by ascertaining the amount of information that can be carried by the signal. This is precisely quantified by the Kolmogorov Entropy, a statistical measure of the uncertainty of the signal [39]. By assigning each of the R quantifiable states of the amplitude of the output signal as an event i, the Kolmogorov Entropy K obtained depends on their probabilities p_i according to the relation as $K = -\sum_{i=1}^{N} p_i \log p_i$. Based on this relation, the values of K are computed and tabulated for selected values of r, along with their corresponding LLE Values in Table 1.

Table 1: Effect of r on the nature of output chaos

Ratio r	LLE	K (nats/sym)
1.0	-0.45	0.67
1.1	2.85	2.92
1.2	1.08	2.23
1.3	2.76	2.54
1.4	1.57	2.18
1.5	2.66	2.77
1.6	2.91	3.01
1.7	-0.02	0.93
1.8	0.87	2.28
1.9	2.95	3.23
1.8	9.66	6.82

4. Conclusion

Considering the tunability issues in system based chaos generation circuits such as the Chua Diode, the present work proposes a radically fresh perspective using signal based chaos, and to achieve this, the Ramanujan Theta Function is considered. It is see that this function depicts nonlinear coupling of two variables, and by assigning these variables as sinusoidal signals, the RTF is adapted as an output signal. Deriving the Iterative map and plotting the bifurcation diagram, it is seen that the dynamics of the RTF based chaotic system is indeed dependent on the the ratio of the driving signal frequencies, and for certain values of the ratio, chaotic behavior is observed. The presence of chaos is validate qualitatively using the phase portrait and quantitatively using the Lyapunov Exponent.

Finally, it is noteworthy that since the behavior of the output signal depends on frequency ratio r, this ratio serves as a secure 'key', enabling the use of the Ramanujan Theta Function based Frequency Controlled Chaos in secure communication and encryption systems. The signal oriented approach to generating chaos from mathematical functions, coupled with the easy tunability hence obtained form the novelty of the present work.

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