ROLE OF SURFACE GAUGING IN EXTENDED PARTICLE INTERACTIONS: THE CASE FOR SPIN

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Abstract. The extended matter should be first characterized by a surface of separation from the empty space. This surface cannot be neatly, i.e. purely geometrically, defined. When it comes to extended particles, which are the fundamental structural units of the matter, the physical evidence is that they are not even stable: they are in a continuous transformation, and so are their limits of separation from space. The present work describes a concept of extended particle with special emphasis on this limit of separation. It turns out that the properties of inertia, as classically understood, are intrinsically related to the spin properties of quantum origin. Thus, the extended particle model cannot be but "holographic" when it comes to imbedding it in a physical structure. The spin properties turn out to be essential, inasmuch as they decide the forces of interaction issuing from the particle.

Keywords: Differential geometry, theory of surfaces, Cartanian approach, affine differential theory, extended particle, limit of separation of particle, extended inertia principle, spin parameters

1. Introduction

The quality of the matter first striking our senses is its space extent: every physical body we notice in our environment occupies a finite space. From a theoretical point of view one cannot talk of the finiteness of a body, without accepting, at least implicitly, a surface delimiting it. Even though, for a long time now, the theoretical physicists have understood the crucial role of this surface, with the present work we push this understanding to its extreme. Specifically, we document here the statement that not only the surface has the essential role of demarcation of a finite body, but when it comes to the description of a constituent particle of a material structure, it plays a fundamental theoretical part, which we have to consider more closely. Mention should be made, however, that the idea of surface here is not to be thought exclusively in plain geometrical terms.

There are indeed, in this respect, two special instances where we can make the best case for physics. First of all we have the separation of matter from the empty space: this is by no means a neatly defined surface, but what we propose to call *a limit of separation*. It is actually a notorious place of wild inhomogeneity, whereby the separation was occasionaly thought of even as a layer of finite thickness, to be defined by a variable density of matter. Poisson was the first to point out the necessity of the theoretical account of material phenomena characterized by the variable density of matter [26]. From the point of view of the existing differential theories of surfaces this situation can be pictured as a structural "evolution" of a geometric surface along a certain transversal direction in space, not necessarily normal, or even along a certain transversal curve in space, over a finite distance. This evolution, however, makes the concept of normal to surface, and even that of general orthonormal frame for that matter, obsolete: the evolution of a surface cannot take place strictly along the Euclidean normal to a point. In other words, from a geometrical point of view, a physical limit of separation presents, in a certain point, different structural properties in different directions in space.

The second instance serving the case for physics in the differential theory of surfaces is that of particle interactions. In any interaction, we have, first and foremost the idea of target particle, like, for example, a target nucleus. This particle sets a local reference frame for the surrounding space, first of all by an ideal fixed point – the origin – with respect to which we reckon the directions in that space. Certainly, then, from a physical point of view, it makes sense to assume that a nucleus has a space extent, and therefore a limit of separation from the empty space, which can be thought of as a "dynamic layer" in the manner described above. A projectile moving towards the target "sees" a succession of geometrical structures in that layer, and the close physical interaction can be described as a "layer interaction" so to speak, inasmuch as the projectile itself is to be conceived as an extended particle, just like the target itself.

We have done some theoretical studies lately (see [21], and the references therein), which allow us to hope for a "holographic" theory that closely parallels the philosophical point of view of the ever-changing structure of the matter. The general idea of such a theory would be that a material structure is made of extended particles. Both the particles themselves and the structure they determine are ephemeral, as they change continuously. Our experience points out that the changes of a particle are reflected in its limit of separation from space. It is therefore only natural to think that the main theoretical tool of description of the matter should be related to this "skin" of the particles. The present work describes the meaning of these statements in their essentials. It is structured along the following lines.

The sections 2, 3 and 4 gradually introduce the idea of the limit of separation of a particle, starting from the Cartanian approach of the classical differential theory of surfaces. Based on this we argue in favor of an affine differential theory, which is then presented in a form convenient for physical interpretation. The classical example of light is brought to bear witness, but mainly in order to illustrate the procedure. Section 5 argues for the dimension five of the space in which a physical surface is imbedded, though only by giving a speculative geometrical argument, and the gist of general current geometrical procedure of imbedding. The physical argument is brought in section 6, whereby the spin parameters associated with the surface curvature and its variations are related to inertia, provided that one accepts an extension of the second law of classical dynamics, which thereby seems quite natural. The classical example of light is again brought to bear witness and to illustrate the case, and a gauging procedure is given. The gauge algebra is a certain realization of the $\mathfrak{sl}(2,R)$. Thereby the parameters of the family of surfaces representing the "skin" are physically decided. Finally, the section 7 "places the particle in an environment", so to speak, even though this time only by constructing a potential in space. This potential is classically introduced, i.e. by a partial differential equation it has to satisfy, and the associated force is calculated as a gradient, in the usual way. Again, the particle limit layer of separation from space turns out to be essential. The nuclear forces are an example of the case.

2. A Natural Imbedding of Surfaces in Space

In classical differential theory of the surface geometry, one usually takes three mutually orthogonal unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ as a reference frame in space. Then, in order to describe a surface locally – to imbed it, as they say – we have to adapt such a reference frame to it, by appropriate rotations and translations. The classical differential geometry of surfaces always uses the notion of metric, as it usually works in the Euclidean space, and thus the definition of an orthonormal frame of reference is always at hand. So, for the moment we undertake a brief review of the local differential geometry of surfaces from such a standpoint, just in order to clarify our own point of view. The best gradual description of the geometrical procedure to constructing such a theory can be found in the extended work of Michael Spivak on differential geometry, especially in the Chapter 2 of the Volume III, pp. 75ff [29].

Assume that ($\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$) is an orthonormal frame on a surface imbedded in the usual threedimensional space, whose unit mormal is $\hat{\mathbf{e}}_3$. Usually, each displacement vector can be written in the form

$$d\mathbf{r} = s^k\hat{\mathbf{e}}_k \equiv s^\alpha\hat{\mathbf{e}}_\alpha + s^3\hat{\mathbf{e}}_3 \tag{1}$$

where $\mathbf{r}$ is the position vector of a generic point of surface. We adopt here the convention that the Latin indices run through values 1, 2, 3, thus referring to the ambient space, while the Greek indices refer to surface, and run only through values 1 and 2. We also use the summation convention over repeated indices all throughout this work. The last identity in equation (1) allows one to define a connection of the position in space with the surface. Specifically, the natural assumption of the Euclidean differential geometry of surfaces is that in order that the position vector remains on the surface, its component normal to surface should always be zero, so that we can write

$$d\mathbf{r} = s^\alpha\hat{\mathbf{e}}_\alpha ; \quad s^3 = 0 \tag{2}$$

The relation of the surface with its environment is then recognized by the normal displacement of a certain point of the surface. This normal displacement is in turn described by the second fundamental form, according to the Cartan's procedure, which can be crudely described as follows.

The equation of evolution of this orthonormal frame can be written as

$$|\, d\hat{\mathbf{e}}\rangle = \mathbf{\Omega}\cdot|\,\hat{\mathbf{e}}\rangle ; \quad \mathbf{\Omega} + \mathbf{\Omega}^t = \mathbf{0} \tag{3}$$

where $\mathbf{\Omega}$ is a matrix of differential 1-forms, and the superscript 't' stands for 'transposed'. The last equality here is a direct consequence of the orthonormality of the frame. Now, considering equation (1) as an immersion equation, its conditions of integrability come down to a system of exterior differential equations, which can be written as

$$d \wedge s^{\mu} + \omega^{\mu}_{\nu} \wedge s^{\nu} = 0 \quad \therefore \quad \begin{aligned} d \wedge s^1 + \omega^1_2 \wedge s^2 &= 0 \\ d \wedge s^2 + \omega^2_1 \wedge s^1 &= 0 \\ s^1 \wedge \omega^3_1 + s^2 \wedge \omega^3_2 &= 0 \end{aligned} \tag{4}$$

where ω^{β}_{μ} are the entries of matrix $\mathbf{\Omega}$. Here the last conditions from equations (2) and (3) were used. The bottom equality from (4) leads, via the Cartan's Lemma, to the conclusion that there exists a convenient symmetric matrix **h**, such that

$$\omega^3_i = h_{ik} s^k; \quad h_{ij} = h_{ji} \tag{5}$$

On the other hand, the skew symmetry of the matrix $\mathbf{\Omega}$ shows that this equation practically represents the variation of the local normal vector to surface, i.e. what we would like to call the *curvature vector*. Indeed, from equation (3) we have for the variation of this vector

$$d\hat{\mathbf{e}}_3 = \omega^1_3 \hat{\mathbf{e}}_1 + \omega^2_3 \hat{\mathbf{e}}_2; \quad \omega^{\alpha}_3 + \omega^3_{\alpha} = 0 \tag{6}$$

Then differentiating once more the first equality from equation (2) we get

$$d^2\mathbf{r} = (ds^k + s^{\alpha}\omega^{\beta}_{\alpha})\hat{\mathbf{e}}_{\beta} + (s^{\alpha}\omega^3_{\alpha})\hat{\mathbf{e}}_3 \quad \therefore \quad \hat{\mathbf{e}}_3 \cdot d^2\mathbf{r} = s^{\alpha}\omega^3_{\alpha} = h_{\alpha\beta} s^{\alpha} s^{\beta} \tag{7}$$

The last expression here is the usual second fundamental form. It is the component of the differential of displacement (the second differential of the position vector in space) along the normal to surface.

This Cartanian approach of the differential geometry of surfaces has a tremendous advantage from theoretical physics' point of view. This advantage is offered here by the fact that the matrix **h** can be anything convenient. True, when one comes to evaluating it, one uses purely geometrical methods, but the geometry here is just a means of measurement, so to speak. One can consult [17] for an instructive example of evaluation of the curvature matrix from geometrical point of view. The very determination of the matrix **h** should nevertheless be a matter of physics. Our standpoint is that the three entries of the matrix **h** – here the curvature parameters – are externally defined quantities, allowing us to introduce physics, no matter if we usually evaluate them by geometrical means. As the structure of matter is always in evolution for some physical reasons, such an evolution should be first recognized in changes of its delimiting surface, viz. the surface of the constituent particles of its momentary structure.

Theoretically, one can imagine the evolution of a material structure as having three components. First, there are two components of the evolution proper: the evolution of the matter inside the constituent particles and the evolution of the structure to which they belong as constituent particles. Then, there is a relational component between the two kinds of evolution, in order to accommodate with each other. The theoretical description of this last component involves the limit of the constituent particles, therefore, at least ideally speaking, a surface, and a closed one for that matter, inasmuch as a particle is finite in every direction in space. Which brings us to the conclusion that the theoretical physics of the evolution of matter should be described in terms of a differential geometry of surfaces, more precisely by a geometry of the curvature parameters' space. In view of the brief presentation right above, we would like therefore a geometrical theory where not only the curvature, but also the surface metric is open to physical determination. This means extending the very differential theory of surfaces, a topic that the Cartanian approach of the theory of surfaces has brilliantly accomplished through the affine differential theory.

3. An Affine Differential Generalization

The affine differential theory of surfaces is suggested by the very idea of reference frame, and the fixed point serving as its origin, which can also serve as a center, for instance as a center of force. In an affine theory we miss first of all a metric, which needs to be defined by external means. One can guess that these external means amount, naturally, to a physics of the problem at hand. Once we have a metric, we can define the relative directions, and therefore a normal to surface. In keeping with our manner of introducing the physical considerations into geometrical theory, we shall follow here a purely exterior differential form development – a Cartanian approach, as we call it here – available in quite a few remarkable works descending from a seminal one of S. S. Chern (see [8]). In broad lines, this philosophy can be presented as follows, starting from the idea of volume of a reference frame.

As a reference frame one chooses three vectors in space, with the only requirement of being linearly independent, and delimiting a certain volume theoretically represented as their mixed product. A reference frame is usually further defined by the essential condition that this volume should be constant either for a specific frame, or even for a family of frames. Further on, one maintains the definition of surface as in equation (2) above. However, because the frame is no more orthonormal, when one assumes its evolution like in the equation (3), the matrix **Ω** is no more skew symmetric. The condition of definition of the frame by maintaining its constant volume gives instead the constraint:

$$\mathrm{tr}(\mathbf{\Omega}) \equiv \omega_i^i = 0 \tag{8}$$

The structural equations are, as usual, the integrability conditions for the definition of differential of position and for the evolution of frame. These amount to

$$d \wedge s^j + \omega_k^j \wedge s^k = 0; \quad d \wedge \omega_i^j + \omega_k^j \wedge \omega_i^k = 0 \tag{9}$$

Now, assuming that our surface is convex (for a clear explanation of this concept see [29]), the last condition from equation (2) leads again to equation (5). However, this equation is here taken as defining a metric rather than a second fundamental form as in the regular Euclidean theory of surfaces. This metric is affinely invariant if we write it in the form [34]

$$(ds)^2 \equiv h^{-\frac{1}{4}}\omega_\alpha^3 s^\alpha = h^{-\frac{1}{4}} h_{\alpha\beta} s^\alpha s^\beta \tag{10}$$

where h is the determinant of **h**. One can see indeed, by comparison with equation (7) from the Euclidean differential theory of surfaces sketched above, that this quadratic form would be actually the equivalent of the second fundamental form of a regular surface. This is why, in the affine theory of surfaces, it is usually designated with II among geometers (see [8])

The problem now remains to deal with the other side, namely ω_3^α, of the matrix **Ω**, because now this matrix is no more skew symmetric ($\omega_3^\alpha + \omega_\alpha^3 \neq 0$). From equation (6) above, one can see that these entries of the matrix **Ω** are actually the proper counterparts of the components of the normal from the case of an Euclidean surface. One approach to solving this issue is to choose ω_3^3 an exact differential, which means $d \wedge \omega_3^3 = 0$, in which case, from the corresponding equation (10), we have

$$\omega_3^1 \wedge \omega_1^3 + \omega_3^2 \wedge \omega_2^3 = 0 \quad \therefore \quad \begin{pmatrix} \omega_3^1 \\ \omega_3^2 \end{pmatrix} = \begin{pmatrix} b^{11} & b^{12} \\ b^{12} & b^{22} \end{pmatrix} \begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} \tag{11}$$

Here Cartan's Lemma, guaranteeing the existence of a convenient matrix **b**, was used again. This leads to the definition of a correspondent of the third fundamental form from the regular theory of surfaces, i.e. the square of what we have called earlier the curvature vector [20]. Using here equation (5), we can get from (11) this quadratic form:

$$\omega_3^i = b^{ij} h_{jk} s^k \equiv b_k^i s^k \quad \therefore \quad \mathrm{III} \equiv \omega_3^k \omega_k^3 = b_{ij} s^i s^j \tag{12}$$

where the notation III seems to be, again, geometers' preference.

Now, having a metric, we can define an orthonormal frame in the tangent plane of the surface, $(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)$ say, and attach a space vector **n** transversal to surface in a general direction, such that the volume of $(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\mathbf{n})$ is nonzero and remains constant. Then, the following can be proved [34]: there is a unique space vector **n** and some adequate differential 1-forms ϕ^α and ψ^α, satisfying the structural equations

$$\begin{gathered} d\hat{\mathbf{e}}_1 = \omega_1^1 \hat{\mathbf{e}}_1 + \omega_1^2 \hat{\mathbf{e}}_2 + \phi^1 \mathbf{n}; \\ d\mathbf{m} = \phi^1 \hat{\mathbf{e}}_1 + \phi^2 \hat{\mathbf{e}}_2; \quad d\hat{\mathbf{e}}_2 = \omega_2^1 \hat{\mathbf{e}}_1 + \omega_2^2 \hat{\mathbf{e}}_2 + \phi^2 \mathbf{n}; \quad \omega_1^1 + \omega_2^2 = 0 \\ d\mathbf{n} = \psi^1 \hat{\mathbf{e}}_1 + \psi^2 \hat{\mathbf{e}}_2; \end{gathered} \tag{13}$$

One can see from this system that the vector **n** is, formally at least, as close as possible to the normal vector from the regular differential theory of surfaces – the curvature vector d**n** is an intrinsic vector – and it is indeed geometrically known as the *affine normal to surface*. Given any affine frame, $(\mathbf{e}_{10},\mathbf{e}_{20},\mathbf{e}_{30})$ say, the affine normal is uniquely defined by equation

$$\mathbf{n} = a^1 \mathbf{e}_{10} + a^2 \mathbf{e}_{20} + h^{\frac{1}{4}} \mathbf{e}_{30} \tag{14}$$

where the auxiliary vector $|a\rangle$ serving for this definition is determined by the differential equation:

$$\langle a | \mathbf{h} | \phi \rangle + d(h^{\frac{1}{4}}) + h^{\frac{1}{4}} \omega_3^3 = 0 \tag{15}$$

We can see here that, given the metric, one can either define $|a\rangle$ by choosing ω_3^3, or ω_3^3 by choosing $|a\rangle$. The geometers' preference seems to be the first procedure [7]. Indeed, equation (15) can be rewritten in the form

$$\langle a | \mathbf{h} | \phi \rangle + h^{\frac{1}{4}} \{ \tfrac{1}{4} d \ln(h) + \omega_3^3 \} = 0$$

and if one chooses the exact differential ω_3^3 such that

$$\tfrac{1}{4} d \ln(h) + \omega_3^3 = 0 \tag{16}$$

then an *affine invariant* normal vector can be defined with one further special choice, namely $|a\rangle = |0\rangle$, as:

$$\mathbf{n} = h^{\frac{1}{4}}\mathbf{e}_3 \tag{17}$$

This choice is particularly attractive in geometrical exploits of this mathematical theory, for in cases where h = 1, we can write

$$\omega_3^3 = 0; \quad \mathbf{n} = \hat{\mathbf{e}}_3 \tag{18}$$

Notice, however that, while still under the condition of *affine invariance*, but with h ≠ constant, the equation (18) can offer ω_3^3 as the differential of a function depending exclusively on some external parameters already introduced through the metric tensor. Therefore the choice of ω_3^3 as an exact differential can be subjected to the very same physical considerations to which the matrix **h** is subjected.

Moreover, even if we give up the affine invariance for the definition of the affine normal, we still have other possibilities of defining the vector $|a\rangle$, depending on the location in the tangent plane of the affine surface. For once, these posssibilities can be offered, for instance, by the nontrivial solutions of the equation

$$\langle a\,|\,\mathbf{h}\,|\,\phi\rangle = 0 \quad \text{viz.} \quad \frac{a^1}{h_{12}\phi^1 + h_{22}\phi^2} = \frac{-a^2}{h_{11}\phi^1 + h_{12}\phi^2} \tag{19}$$

However, there are still other possibilities, to be exploited here, upon which we do not insist momentarily. The bottom line is that the affine theory of surfaces has this attracting feature, excellently serving our physical point of view that, by a Cartanian approach, both the metric and the curvature are totally external things, so that they can both carry a physical content.

4. A Definition of the Metric Tensor: the Fubini-Pick Cubic Differential Form

One important concept that we were set out to exploit in a previous work [20] is that of algebraic apolarity. The affine differential theory of surfaces makes obvious that the apolarity involves a cubic differential invariant – the so-called *Fubini-Pick cubic form*. This differential form can be best introduced in the Cartanian manner that follows [8]. By exterior differentiating equation (8), within the choice from equation (13), and using the corresponding equality from equation (5), we get

$$Dh_{\alpha\beta} \wedge \phi^\beta = 0; \quad Dh_{\alpha\beta} \equiv dh_{\alpha\beta} - h_{\alpha\mu}\omega_\beta^\mu - \omega_\alpha^\nu h_{\nu\beta} \tag{20}$$

Using again Cartan’s Lemma we get, for a convenient three-index tensor **Φ**:

$$Dh_{\alpha\beta} = \Phi_{\alpha\beta\gamma}\phi^\gamma \tag{21}$$

The tensor **Φ** is obviously symmetric in all three indices, as required by its definition. Geometrically, it measures the difference between the affine connection of the surface and the Levi-Civita connection, $\boldsymbol{\varpi}$ say, of the metric **h**, for which D**h** = **0**:

$$\varpi_\alpha^\beta = \omega_\alpha^\beta + \tfrac{1}{2}h^{\beta\nu}\Phi_{\alpha\nu\sigma}\phi^\sigma \tag{22}$$

The cubic differential form

$$\Phi \equiv \Phi_{\alpha\nu\sigma}\phi^\alpha\phi^\nu\phi^\sigma \tag{23}$$

is known as the Fubini-Pick form of surface. Another important property of this form is that if it vanishes identically for a certain affine surface, then that surface is a quadric [7]. We think that it is mostly this property that came in handy in the past physical problems, as we shall see a little later.

However, by far the most important property of the Fubini-Pick form is, as we have already said, the property of apolarity, which leads directly to a physical way of defining the metric tensor. The procedure can be described as follows: if we take $\omega_3^3 = 0$, then by equation (16) we have dln(h) = 0, which means tr($\mathbf{h}^{-1}d\mathbf{h}$) = 0. This is to say that the metric of an affine surface is apolar to the quadratic form having as coefficients the variations of the entries of the metric tensor. Now, using from equation (13) the fact that the affine connection of surface is trace-free, and the definition (21) of the Fubini-Pick tensor, the apolarity condition comes down to

$$h^{\mu\nu}\Phi_{\mu\nu\lambda} = 0 \tag{24}$$

This matrix equation can be considered an algebraic homogeneous system for the entries of the metric tensor:

$$h_{22}a_0 - 2h_{12}a_1 + h_{11}a_2 = 0; \quad h_{22}a_1 - 2h_{12}a_2 + h_{11}a_3 = 0 \tag{25}$$

Here we have used the definition of the inverse of the metric tensor, its property of symmetry, and the symmetries of the Fubini-Pick tensor, with the following notations:

$$\Phi_{111} \equiv a_0; \quad \Phi_{112} = \Phi_{121} = \Phi_{211} \equiv a_1; \quad \Phi_{221} = \Phi_{122} = \Phi_{212} \equiv a_2; \quad \Phi_{222} \equiv a_3 \tag{26}$$

With these notations the Fubini-Pick cubic can be written in the form

$$\Phi \equiv a_0(\phi^1)^3 + 3a_1(\phi^1)^2\phi^2 + 3a_2\phi^1(\phi^2)^2 + a_3(\phi^2)^3 \tag{27}$$

If we assume that this cubic is known, which means that its coefficients are known, then from equation (25) we can calculate the affine metric up to an arbitrary factor:

$$\frac{h_{11}}{a_0a_2 - a_1^2} = \frac{2h_{12}}{a_0a_3 - a_1a_2} = \frac{h_{22}}{a_1a_3 - a_2^2} \tag{28}$$

This purely algebraic result gives the entries of the affine metric tensor as coefficients of the Hessian of Fubini-Pick cubic [5], and shows that the affine metric of surface is actually the Hessian of Fubini-Pick cubic form. Summarizing, the apolarity in the case of affine surfaces comes down to the following statement: *for the case of null 33 component of the matrix **Ω**, and constant surface element (h = constant), the affine metric form is the Hessian of Fubini-Pick cubic*.

Obviously, this result can be used in the theory, in order to determine the metric of surface. Indeed, if by a physical reason we have at our disposal a cubic equation for the description of the matter, then this equation can be taken as a Fubini-Pick form in developing a metric theory of the surface delimiting the structure of that matter. The physics has a host of cases, mostly in the mechanics of continua, where the cubic occurs as the eigenvalue equation of the 3×3 matrices, representing either tensions or deformations. The best known classical case illustrating this statement, and the case which started the whole theory of continua, is the Fresnel theory of light [11], first developed based on the classical Euclidean theory of surfaces. Thereby the light is considered a material structure in a continuous medium (the ether), and Fresnel has shown that its surface – the wave surface – can be synthetized from bits and pieces offered by the diffraction experiments. The physical explanation of Fresnel, which was the basis of the future theory of elasticity, is the existence of some elasticities of ether, as a continuous medium supporting the light (for a pertinent and penetrating review of the case for theory of elasticity in physical optics, the reader ca consult the work of Barré de Saint-Venant [36]). Later on, these elasticities were incorporated in a matrix, whose eigenvalue equation is a cubic.

Now, the Fresnel theory was obviously not enough for describing the light. From a global point of view, it had to be completed up to an affine-type theory by Kummer [14]. For a clear and comprehensive account of the Kummer's theory with significant improvements and additions, one can consult Meibauer's work [22]. The necessity of such a theory is mathematically quite natural, and is nowadays obvious via the affine theory of surfaces: if the ether is an elastic matter, then it can be characterized by a cubic representing its tensions, which then can be taken as the Fubini-Pick cubic of the local affine geometry of the wave surface. The calculation of principal elasticities requires the cubic equation obtained when the Fubini-Pick cubic vanish. This is in turn a characteristic of the quadric surfaces, thereby mathematically justifying, on one hand, the necessity of Fresnel's hypothesis of the "ellipsoid of elasticities" of ether and, on the other hand, the necessity of considering an affine differential theory of surfaces in physics. Of course, the theory cannot be as direct as presented in these broad lines but, historically speaking at least, the core of the issue is indeed here.

This historical case can serve as a lesson: in the case of a material structure in general, only the constituent particles of the structure can be assumed to be a material continuum, delimited by a surface. The Fubini-Pick cubic form of this surface is then related to the internal characteristics of this continuum, like stresses and strains, of which one cannot talk for the material structure itself without involving the idea of force and of flux of forces.

5. The Threedimensional Affine Transversal Space

One further point remains to be secured for an affine differential theory of surfaces, in order to make it properly serve the physical purposes: the dimension of the surrounding space. Starting from the 9th decade of the last century, the immersion of 2-dimensional manifolds in spaces of different dimensions higher than 2 – not necessarily 2+1 as in the classical case above – started attracting the attention of geometers from different perspectives. For the use of exterior differential forms in order to treat such a problem, mention should be made of the pioneering work of Wilkinson [32]. However, we have found a particularly inspirational work strictly referring to the immersion of an affine surface in a five dimensional affine space [9]. This work stimulated an approach of immersion of surfaces in five dimensional space in case of a null Fubini-Pick tensor [19], as well as a generalization, showing a property of minimality for the case of immersion of n-dimensional real manifolds into n+n(n+1)/2–dimensional real spaces [28]. The common approach of the problem of immersion here is by considering a general affine frame in the host space, and adapting it to the (hyper)surface in order to induce an affine structure upon it, in connection with that of the space. Considering such a minimality property, the ambient space of a surface should definitely have the dimension five. In order to offer the gist of the matter we shall proceed momentarily more directly though, extending the analogy with the classical theory of surfaces, and only then go a little further, to the physical argument which involves the idea of classical inertia.

First of all, we will consider the classical definition of the displacement as in equation (1) but with three "normal" components of the displacements, corresponding to three arbitrary directions in space

$$d\mathbf{m} = s^k \hat{\mathbf{e}}_k \equiv s^\alpha \hat{\mathbf{e}}_\alpha + s^{31}\mathbf{e}_{31} + s^{32}\mathbf{e}_{32} + s^{33}\mathbf{e}_{33} \tag{29}$$

This corresponds to the fact that the normal direction to the surface "splits" into three different "directions". The notation is explained by the fact that on the surface one can always find means to define a metric tensor and thereby define further the orthonormality. However, it is not at all a clear thing how the orthonormality on a "normal" direction works. Here one has to rely upon some abstract properties of the environmental space. There are thus quadratic representations of these directions, whereby one can think of representation (29) as of a sort of Veronese representation of the affine surface [19]. Anyway, the reference frame should be defined by classical-like equations, written nevertheless in such a way, as to account for the "split" of the normal direction:

$$\begin{aligned} d\hat{\mathbf{e}}_1 &= \omega_1^1 \hat{\mathbf{e}}_1 + \omega_1^2 \hat{\mathbf{e}}_2 + \omega_1^{31}\mathbf{e}_{31} + \omega_1^{32}\mathbf{e}_{32} + \omega_1^{33}\mathbf{e}_{33} \\ d\hat{\mathbf{e}}_2 &= \omega_2^1 \hat{\mathbf{e}}_1 + \omega_2^2 \hat{\mathbf{e}}_2 + \omega_2^{31}\mathbf{e}_{31} + \omega_2^{32}\mathbf{e}_{32} + \omega_2^{33}\mathbf{e}_{33} \end{aligned} \tag{30}$$

This represents the fact that variation of the frame on the surface has both a general in-surface contribution, and a contribution due to the variation of the normal. As to the normal subframe, we maintain the idea that it has only a surface variation, with no normal contribution, for all components:

$$d\mathbf{e}_{31} = \omega_{31}^1 \hat{\mathbf{e}}_1 + \omega_{31}^2 \hat{\mathbf{e}}_2; \quad d\mathbf{e}_{32} = \omega_{32}^1 \hat{\mathbf{e}}_1 + \omega_{32}^2 \hat{\mathbf{e}}_2; \quad d\mathbf{e}_{33} = \omega_{33}^1 \hat{\mathbf{e}}_1 + \omega_{33}^2 \hat{\mathbf{e}}_2 \tag{31}$$

These are just three different copies of the equation (6) above.

6. A Physical Argument Involving the Idea of Spin

Theoretically, one can safely argue indeed that the dimension of the environmental space cannot be three, as classically suggested, but the issue is a little more involved, for it cannot be purely geometrical. Of course, one cannot bring arguments against the intuitive feeling that the residence space of the matter, as it presents itself to our senses, is threedimensional. The simplest argument for this statement is provided by the fact that in order to evaluate quantitatively the shape of a body, no matter of the space scale, we need at least three measurements of its dimension along three different directions. However, from the very foundation of the affine theory of surfaces delimiting the matter – the linear independence of the vectors – the things cannot remain at the intuitive level, because the physics itself points out to a environmental space of geometrical dimension five. Indeed, our line of introducing the physics here is based on the fact that both the metric form and the second fundamental form in a point of an affine surface are *binary quadratic forms*. The binary variable is provided, of course, by the components of differential displacements along the tangent plane to surface in the chosen point. It is from this perspective that the affine normal direction is, in a sense, an affine vector in a three dimensional real space, even though it has to be considered as a direction in space, as we usually know it. The mathematical basis of this statement can be presented as follows.

Any binary quadratic form can be uniquely expressed as a linear combination of three nonsingular, mutually apolar, binary quadratic forms. The proof here is due to Dan Barbilian [3], but for the general theory of apolarity of the binary quadratic forms, one can consult the treatise of Burnside & Panton [5]. The chapter XVI of the second volume, particularly the example 6, pp. 136–137, is particularly illuminating for what we have to say. Start with designating a generic binary quadratic form by

$$Q(x,y) \equiv ax^2 + 2bxy + cy^2$$

and write it as a linear combination

$$Q(x,y) \equiv \lambda_1 Q_1(x,y) + \lambda_2 Q_2(x,y) + \lambda_3 Q_3(x,y)$$

where Q_k represent three quadratic forms whose coefficients are a, b, c with different indices k. This identity is equivalent to the linear algebraical system:

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = a; \quad b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = b; \quad c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 = c \tag{32}$$

Considered as a linear system for the unknowns $\lambda_{1,2,3}$, this system can be compatible, and if the three basic quadratics are *nonsingular and mutually apolar*, it is always compatible and has unique solution. Indeed, the square of its third order principal determinant can be expressed as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = -\begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{vmatrix}; \quad 2I_{kl} \equiv a_k c_l + c_k a_l - 2b_k b_l \tag{33}$$

Now, if the three basic quadratic forms are apolar $I_{kl} = 0$ for $k \neq l$ (see also [20], for the definition of the notion of apolarity of two quadratics), so that only the diagonal entries survive here, and these are the discriminants of the quadratic forms. Because they are assumed nonsingular, the equation (33) shows that the determinant of system (32) is always nonzero, therefore the system has unique solution for the unknowns $\lambda_{1,2,3}$. If the system is homogeneous, it has only the trivial null solution. Therefore the set of all quadratic binary forms, defined by some external reasons in a point of an affine surface, is a linear threedimensional set.

Likewise, inasmuch as the second fundamental form is to be taken into consideration, the affine normal is not simply a normal direction to a surface, but *a transversal linear threedimensional space*. If it is to think of an immersion of the affine surface here, the ambient affine space should be therefore five dimensional. This corresponds to the natural fact that in order to completely characterize the profile of a surface in a point, one needs three different perspectives of the surface in three different, but otherwise arbitrary, directions through that point. The curvature of a surface in a point simply cannot be described by a single quadratic form.

Strange as it may seem, the dimension three for the transversal space to a physical surface has indeed strong physical reasons from the very classical mechanics' point of view. That point of view is given by the Newtonian relation between force and acceleration – the second law – within a proper perspective though. In a classical differential geometry of surfaces, as presented above, this perspective amounts to the following. Assume that **v** is the local speed of a motion along the geodesics of a surface in a certain point:

$$\mathbf{v} = \dot{s}^1 \hat{\mathbf{e}}_1 + \dot{s}^2 \hat{\mathbf{e}}_2$$

Here the overdot denotes a time rate as usual. This is an intrinsic vector, just like the elementary displacement on the surface itself, from which it is derived. The rate of displacements is taken with respect to the time of geodesics of surface, which is to assume that they exist and the motion along them is physical. The variation of vector **v** along geodesics has only a component, normal to surface:

$$d\mathbf{v} = (\dot{s}^\alpha \omega_\alpha^3)\hat{\mathbf{e}}_3$$

Using the definition of the curvature vector, one can see that the quadratic generated by the second fundamental form in association with the time of geodesics on surface, i.e. the rate

$$\frac{dv}{dt} = \dot{s}^\alpha h_{\alpha\beta} \dot{s}^\beta \tag{34}$$

is the magnitude of a *normal acceleration to surface*. By the same token, the intrinsic vector normal to geodesics

$$\mathbf{p} = \dot{s}^2 \hat{\mathbf{e}}_1 + (-\dot{s}^1)\hat{\mathbf{e}}_2$$

when transported by paralelism along geodesics, yields a normal vector

$$d\mathbf{p} = (\dot{s}^2 \omega_1^3 - \dot{s}^1 \omega_2^3)\hat{\mathbf{e}}_3 \tag{35}$$

having the magnitude

$$\frac{dp}{dt} = \dot{s}^\alpha h^*_{\alpha\beta} \dot{s}^\beta; \quad \mathbf{h}^* \equiv \mathbf{h} \cdot \mathbf{i}; \quad \mathbf{i} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{36}$$

This is an acceleration due to the geodesic torsion: the vector (35) represents the torsion of any curve touching the geodesic in the given point. The two quadratic forms from equations (34) and (36), represent two different acceleration, both vectors oriented along the normal to surface. Therefore in a regular geometry they are just two collinear vectors. Nevertheless, as quadratic forms, they are insufficient for characterizing the whole magnitude of the normal acceleration, inasmuch as, considered as a quadratic polynomial, this one is a point in a threedimensional linear space, as shown above. This is the right place to use the property of apolarity in order to properly make up the magnitude of acceleration.

Indeed, as one can see directly, the two quadratics (34) and (36) are reciprocally apolar. Then we can naturally construct a third quadratic, apolar with each one of them, the so-called resultant:

$$\dot{s}^\alpha h^{**}_{\alpha\beta} \dot{s}^\beta; \quad \mathbf{h}^{**} \equiv \tfrac{1}{2}(h\mathbf{1} - \mathbf{h}^2); \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{37}$$

where h is the determinant of **h**, i.e. the Gaussian curvature of surface. If the support function of the surface (see [29] for a clear explanation of the concept) can be represented by a quadratic form, then the proper representation of the acceleration when referred to the time of the geodesics on the surface, is by a linear combination of the quadratics (34), (36) and (37).

As a first conclusion therefore, assuming the classical Newtonian relation between acceleration and force, the forces with which a particle surface responds in accomodating with each other the exterior and interior changes of the matter, are linear combinations of the three accelerations. One can say that the coefficients entering such an expression of the forces represent *three types of inertia*, which only accidentally reduce to one, in those cases where the force can be sufficiently characterized as vector, for instance when the constituent particles of the matter structure can be represented as a material points. This was actually the case that allowed introducing forces in physics.

Indeed, this was the manner of conceiving forces responsible for the celestial harmony in the first place: by the ratio of their magnitudes (Newton, *Principia*, Book I, Proposition VII, Corollary 3). Only, in the original case of Newton, the forces were considered as acting in different directions in space, for instance from a planet toward the focus of its orbit, and from the same planet toward the center of the orbit. However, a more explicit example of forces having as magnitude a linear combination of magnitudes of forces, is provided by the case of revolving orbits, whereby the magnitude of the force responsible for the motion along a revolving Kepler orbit is a linear combination of the magnitude of a force inversely proportional to the square of distance and the magnitude of a force inversely proportional to the cube of distance (*Principia*, Book I, Proposition XLIV). For a treatment of the problem in modern theoretical terms see [6, 18, 31].

There is, however, a very instructive classical case illustrating the issue even from the very standpoint of the very theory of surfaces. Like anticipating the case of Fresnel two centuries later, Newton explained the phenomena of reflection and refraction of light by a force acting at the surface of matter, along the normal to that surface (see [25], pp. 79ff). Nowadays, we should be able to say that such a force can be explained within differential geometry, by the extension of the principle of inertia, as long as the light itself is considered as a material structure, as it always was actually. From this perspective, one can say that the diffraction phenomena studied by Fresnel, only added a local structure to the Newtonian force, which was actually based upon the experience regarding only reflection and refraction of light. The existence of such force was indeed confirmed as a physical fact, on one hand through the electromagnetic structure of the light and, on the orther hand, through the experiments of P. N. Lebedev [15,16].

It would seem therefore that, by an extended principle of inertia, the *magnitude of the force* itself is a linear space with the dimension decided by the dimension of the curvature algebra, which is obviously a $\mathfrak{sl}(2,R)$ algebra. Indeed, the three quadratics above, representing accelerations, can be considered as connected with the homographic action of the three matrices which belong to a basis of $\mathfrak{sl}(2,R)$ algebra:

$$\mathbf{e}_1 = \begin{pmatrix} -\beta & -\gamma \\ \alpha & \beta \end{pmatrix}; \quad \mathbf{e}_2 = \begin{pmatrix} (\alpha-\gamma)/2 & \beta \\ \beta & (\gamma-\alpha)/2 \end{pmatrix}; \quad \mathbf{e}_3 = \begin{pmatrix} -\beta(\alpha+\gamma) & -\beta^2-\gamma^2+\Delta \\ \alpha^2+\beta^2-\Delta & \beta(\alpha+\gamma) \end{pmatrix}$$

The quadratics in question are the symplectic forms corresponding to the linear action of these matrices. As long as we have to deal with a single surface in describing the limit of separation of a particle – in which case α, β and γ are constants – one can argue that this basis is just enough in order to write the physics in a geometric language. However, a physical limit of an extended particle cannot be simply a geometrically fixed surface, but should also contain its local deformations due to physical causes. This situation can only be modeled as a succession of geometrical surfaces, resembling to the known geometrical image of a 1-form for instance (see [23]). This means that the curvature parameters vary in a certain way along a direction crossing the evolving surface, in any of its momentary instances.

In a previous work [21], we have defined such a variable geometry in terms of the second fundamental form of an initial surface in a point, and its different deformations. These deformations are represented by three mutually apolar quadratics which can be interpreted as symplectic forms corresponding to the following three involutive anticommuting matrices

$$\mathbf{e}_1 \equiv \frac{1}{\sqrt{\Delta}}\begin{pmatrix} -\beta & -\gamma \\ \alpha & \beta \end{pmatrix}; \mathbf{e}_2 \equiv \frac{1}{\sqrt{\Delta'}}\begin{pmatrix} -\omega_2/2 & -\omega_3 \\ \omega_1 & \omega_2/2 \end{pmatrix}; \mathbf{e}_3 = \frac{1}{\sqrt{\Delta\Delta'}}\begin{pmatrix} -\Omega_2/2 & -\Omega_3 \\ \Omega_1 & \Omega_2/2 \end{pmatrix} \tag{38}$$

where the entries of the vectors $\mathbf{e}_2$ and $\mathbf{e}_3 \equiv \mathbf{e}_1\,\mathbf{e}_2$ are exterior 1-forms in the curvature parameters,

$$\omega_1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}; \quad \omega_2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}; \quad \omega_3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta}; \quad \Delta \equiv \alpha\gamma - \beta^2 \tag{39}$$

and

$$\Delta' \equiv (\omega_2/2)^2 - \omega_1\omega_3$$

The multiplication table of these three involutive matrices is given by the relations

$$\begin{matrix} \mathbf{e}_1^2 = -\mathbf{1}; & \mathbf{e}_2^2 = \mathbf{1}; & \mathbf{e}_3^2 = \mathbf{1}; \\ \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_3; & \mathbf{e}_3 \cdot \mathbf{e}_1 = \mathbf{e}_2; & \mathbf{e}_2 \cdot \mathbf{e}_3 = -\mathbf{e}_1 \end{matrix} \quad \therefore \quad \begin{matrix} [\mathbf{e}_1, \mathbf{e}_2] = 2\mathbf{e}_3; \\ [\mathbf{e}_3, \mathbf{e}_1] = 2\mathbf{e}_2; \\ [\mathbf{e}_2, \mathbf{e}_3] = -2\mathbf{e}_1 \end{matrix}$$

As vectors, they can represent indeed an orthonormal frame, but with respect to the product

$$2(\mathbf{e}_k, \mathbf{e}_j) = \mathrm{tr}(\mathbf{e}_k \cdot \mathbf{e}_j)$$

where the dot means matrix multiplication. For further details regarding such a matrix frame and the inspiring literature, the reader should consult the work cited right above, for our present treatment will turn in some other direction, in order to reveal an inedit classical connection, between inertia and spin.

It is quite clear that the three quadratic forms corresponding to the vectors from equation (38) provide us with a basis of three binary quadratics, a, b, c say, in the components of displacement of an initial surface with curvature parameters given by α, β and γ. These quadratics are nonsingular, reciprocally apolar and, if suitably normalized (see especially [3] for a thorough treatment of this issue), they satisfy the quadratic equation [5]

$$a^2 + b^2 + c^2 = 0 \tag{40}$$

In other words, these binary quadratics represent in physics what we would like to call the *spin parameters*. This connection was first outlined by the middle of the last century [33] under "complex space of null radius". In view of our classical inspiration, these parameters should be directly related to inertia properties of the matter. An image then emerges as the basis of description of the free extended particle concept.

An extended particle is thereby aptly represented by its limit of separation from space, which can be organized as a family of surfaces deriving from one another by a process of deformation. One can say that they form a bundle coordinated by the transversal threedimensional space of the curvature parameters. This structure is analogous with a bundle of planes from the classical case of the Yang-Mills gauge fields [10]. In that case a specific choice of frames in the planes would amount to a gauge choice, and the rotation of the frames would amount to a gauge transformation. Here the construction of the gauge can be explicitly made, and it involves not one, but three parameters. The procedure can be described as follows.

The reciprocal position of two vectors over a surface from the bundle, say $\mathbf{e}_1$ and $\mathbf{e}_2$ is in general not a standard Euclidean one, due to deformation. Once we have a metric at our disposal, it is described by equations of the form

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \lambda\mu\cos\theta; \quad \mathbf{e}_1^2 = \lambda^2; \quad \mathbf{e}_2^2 = \mu^2 \tag{41}$$

where λ and μ are the measured quantities. On the other hand, the surface can be characterized by the local orthonormal frame, taken as a standard:

$$\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 = 0; \quad u_1^2 = 1; \quad u_2^2 = 1 \tag{42}$$

Here $\hat{\mathbf{u}}_{1,2}$ are the unit vectors of an orthonormal frame adapted to surface. A transformation between the two bases of vectors – properly a gauging – is given by a nonsingular matrix that reduces the general situation to the standard one:

$$|\mathbf{e}\rangle = \mathbf{m}\cdot|\hat{\mathbf{u}}\rangle \tag{43}$$

Now, if a, b, c, d are the entries of the matrix **m**, the equations (41), (42) and (43) imply

$$ac + bd = \lambda\mu\cos\theta; \quad a^2 + b^2 = \lambda^2; \quad c^2 + d^2 = \mu^2 \tag{44}$$

These relations determine the matrix a up to an arbitrary phase. Indeed, we can choose

$$a = \lambda\cos\phi; \quad b = \lambda\sin\phi; \quad c = \mu\cos\varphi; \quad d = \mu\sin\varphi \tag{45}$$

so that the area determined by the two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ is

$$|\mathbf{e}_1 \times \mathbf{e}_2| = \lambda\mu\sin\theta \quad \leftrightarrow \quad ad - bc = \lambda\mu\sin(\varphi - \phi) \tag{46}$$

So, if we choose $\varphi = \theta + \phi$, we get

$$|\mathbf{m} \equiv \begin{pmatrix} \lambda\cos\phi & \lambda\sin\phi \\ \mu\cos(\phi+\theta) & \mu\sin(\phi+\theta) \end{pmatrix} \tag{47}$$

This is an Iwasawa representation of the matrix **m**. Indeed, the matrix is of the form

$$\mathbf{m} \equiv \mathbf{A}\cdot\mathbf{N}\cdot\mathbf{K} \tag{48}$$

with the three factors given by

$$\mathbf{A} \equiv \begin{pmatrix} \lambda/(\mu\sin\theta) & 0 \\ 0 & (\mu\sin\theta)/\lambda \end{pmatrix}; \quad \mathbf{N} \equiv \begin{pmatrix} 1 & 0 \\ (\lambda\cos\theta)/(\mu\sin^2\theta) & 1 \end{pmatrix}; \quad \mathbf{K} \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Therefore a frame deformation, or even a statistics, which actually comes to the same from a mathematical point of view, is represented by a Iwasawa matrix, which can sustain the theory of a stochastic process for the “skin” of the extended particle, to be described in the manner offered, for instance, by Albeverio & Gordina [1].

The most important point to be noticed here is the mathematical nature of this stochastic process: it is a process with variance given by a metric [35]. This metric is what we would like to call a *Barbilian metric* for the Riemannian space of families of binary cubics [2]. This can be easiest shown as follows: use the notations

$$\xi \equiv \lambda\cos\phi; \quad \eta \equiv \lambda\sin\phi; \quad u \equiv \frac{\mu}{\lambda}\cos\theta; \quad v \equiv \frac{\mu}{\lambda}\sin\theta \tag{49}$$

in which case the matrix **m** can be written in the form

$$\mathbf{m} \equiv \begin{pmatrix} \xi & \eta \\ \xi u - \eta v & \xi v + \eta u \end{pmatrix} \tag{50}$$

Now, the Killing-Cartan metric of this variable matrix is given by the quadratic form:

$$\left(\frac{dv}{v}\right)^2 - (2d\phi)^2 - 2(2d\phi)\left(\frac{du}{v}\right) \equiv \frac{(du)^2 + (dv)^2}{v^2} - \left(2d\phi + \frac{du}{v}\right)^2 \tag{51}$$

which is indeed the Barbilian metric up to a sign.

It is therefore also important to construct a coframe on a surface of the family. In the parametrization from equation (50) the components of this coframe are:

$$\begin{aligned} \omega_1 &= \cos^2\phi\frac{du}{v} - \sin\phi\cos\phi\frac{dv}{v} + d\phi; \\ \omega_2 &= \sin 2\phi\frac{du}{v} + \cos 2\phi\frac{dv}{v}; \\ \omega_3 &= \sin^2\phi\frac{du}{v} + \sin\phi\cos\phi\frac{dv}{v} + d\phi \end{aligned} \tag{52}$$

Thus for a *Sasaki parametrization* [27] of the local geometry of a surface from the family representing the limit of separation of an extended particle, we have

$$\begin{aligned} \omega_1 - \omega_3 &= \cos 2\phi\frac{du}{v} - \sin 2\phi\frac{dv}{v}; \\ \omega_1 + \omega_3 &== \frac{du}{v} + 2d\phi \end{aligned} \tag{53}$$

One can see that (ω_2, ω_1–ω_3) represents a Bäcklund transformation of the fundamental forms of the Lobachevsky plane, while the form ω_1+ω_3 is the corresponding connection form. This explains the form of the gauge metric from the right hand side of the equation (51).

Bringing into consideration the standard orthonormal frame on the surface seems to add a subjective note to the procedure, which brings the question: is this gauging natural? As long as the internal structure of a particle can be taken as a continuum, the answer is definitely affirmative. Indeed, the evolution of the structure of particle is then sufficiently represented by a family of cubics which should vanish on the particle surface, in order to give the corresponding “eigenvalues”. The geometry just presented is actually a Riemannian geometry of such a family of cubics, as it was first presented [2]. The problem is only the reflection of this geometry in the external forces with which the particle is manifested in a matter structure. As a matter of fact, the light is again “illuminating” here: the above gauging is historically related to the electromagnetic structure of light. This issue is presented in detail in one of our previous works (see [37], pp. 148 – 173)

7. Extended Particle Forces and Spin Parameters

Assuming that we have an extended particle model, built along the lines sketched above, the problem is to incorporate such a particle within a physical structure of the matter. A first issue in solving the problem, at least in a classical view, is the construction of a potential, and we are able to show here that this potential is strictly determined by the limit of separation of the extended particle. The expression of the force obtained from this potential depends solely on the curvature and its variation. This can be shown as follows.

By the beginning of the last century, Edmund Whittaker was able to produce solutions of the nonhomogeneous Laplace equation introducing arbitrary functions in an integral over the unit sphere [30]. Specifically, if we parameterize the sphere by the polar angles θ, ϕ with respect to its center, then we find that the potential written in the form

$$U(\mathbf{r}) = \oiint_{\text{Sphere}} e^{x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta} f(\theta,\phi)\, d\theta d\phi$$

with f(θ, ϕ) an arbitrary function, is a solution of the equation $\Delta U(\mathbf{r}) = U(\mathbf{r})$.

It was Pierre Humbert who noticed the fact that the partial differential equation to be satisfied by the potential of a surface in space is actually dictated by the equation of the surface itself, if it is represented by a known algebraical expression, as in the case of Whittaker’s unit sphere, and that such a determination asks for means depending exclusively on the geometry of surface. This can be shown quite directly, considering $U(\mathbf{r},\mathbf{a})$ the potential between points **r** and **a** in space [13]. Humbert chooses for illustration a Gaussian central potential, but he soon notices that the theory goes with arbitrary forms of potential. If the point **a** describes a surface, S say, the potential of that surface in the space point **r** is naturally given by the integral

$$U(\mathbf{r}) = \iint_S U(\mathbf{r},\mathbf{a})\, d^2\mathbf{a}$$

where $d^2\mathbf{a}$ is the elementary measure on that surface. For a potential of the form

$$U(\mathbf{r},\mathbf{a}) = e^{\mathbf{a}\cdot\mathbf{r}} f(\mathbf{a}) \tag{54}$$

if the surface is algebraic, we have

$$\sum\nolimits_{m,n,p} a^m b^n c^p = 1 \quad \therefore \quad \sum\nolimits_{m,n,p} \partial_x^m \partial_y^n \partial_z^p U(\mathbf{r}) = U(\mathbf{r}) \tag{55}$$

These results generalize the one due to Whittaker, for the representation of the solutions of Laplace equation: in that case it is sufficient to consider that the algebraic equation in (55) is a sphere.

The Humbert's results can yet be seen from another angle, namely as an "eigenvalue problem", involving "level surfaces", so to speak. Indeed, if the equation of the algebraical surface is of the general form

$$\sum\nolimits_{m,n,p} A_{mnp} a^m b^n c^p = \lambda \tag{56}$$

where A_{mnp} and λ are some parameters, then the partial differential equation satisfied by $U(\mathbf{r})$, as given by equation (54), is

$$\sum\nolimits_{m,n,p} A_{mnp} \partial_x^m \partial_y^n \partial_z^p U(\mathbf{r}) = \lambda U(\mathbf{r}) \tag{57}$$

This allows us to include among the representations of Humbert some important homogeneous equations like, for instance, the Laplace equation proper. The solutions of this equation are then represented by an integral of the form

$$U(\mathbf{r}) = \iint_S e^{\mathbf{r}\cdot\mathbf{a}} f(\mathbf{a}) d^2\mathbf{a} \tag{58}$$

where $f(\mathbf{a})$ is an arbitrary function on the surface $\mathbf{a}^2 = 0$ (see also [4], Chapter VIII; [12], pp. 18 – 20).

Thus, the potential of a particle in a point, considered as a solution of the Laplace equation in that point of space, *is decided by the spin parameters*. The force generated by an extended particle in any point in space is offered by the gradient of the potential, as usual:

$$\nabla U(\mathbf{r}) = \iint_S \mathbf{a} e^{\mathbf{r}\cdot\mathbf{a}} f(\mathbf{a}) d^2\mathbf{a} \tag{59}$$

a conclusion in concordance with the extended inertia principle as restated above. The arbitrary function $f(\mathbf{a})$ can be given from further invariance considerations, but related to surface only. In a previous work, already cited above [20], we treated such a case by statistical theoretical methods, but the results are in concordance with the group theoretical methods, allowing us to describe the classical theory of forces as a gauge theory, for the elaboration of which we reserve a future work.

Conclusions and a Perspective

Any matter structure is theoretically thought in terms of particles. Our experience shows that such particles are by no means "elementary", but they exhibit a structure in concordance with the matter structure to which they momentarily belong. In a word, neither that matter structure, nor the structure of the constituent particles are fixed, but are both essentially ephemeral. The classical case of molecules, atoms and nuclei is notorious: none of these structural particles proved eventually to be what they were intended to be. Closer to our times, the modern realm of elementary particles exhibits a wild variety, thus making the term "elementary" obsolete, if it is to be taken within the initial understanding.

Neither the classic nor the quantum theory of forces can properly cope with this emerging image of the world, and we think that this is mainly due to the fact that the prevailing theoretical image of the particle is that of a material point. This image can indeed be properly used only in the limit of large distances between particles. However, in close particle encounters it obviously fails: there the material point needs to be replaced with an extended particle. The present work is dedicated to such a task.

We start from the observation that an extended particle means a finite volume of matter in space, and thus *a fortiori* a limit of separation of the matter from the empty space. It seems just natural that, when incorporated in a material structure, an extended particle should need adaptation to that structure. This means that its internal evolution should be in concordance with the evolution of the material structure to which it belongs momentarily, and this fact brings the "skin", i.e. the limit of separation of the particle, to the fore. Our declared task here is thus accomplished by describing this limit of separation from geometrical point of view, with emphasis on the possibilities of introducing the physics into the geometrical theory. We show that the Cartanian approach of the differential geometry of surfaces can properly accomodate the physical needs.

Thus, our main result is that the forces issuing from such a particle depend exclusively on the geometry of its limit of separation from the empty space. It is like these forces are actually determined "holographically" – to use a theoretical term in fashion nowadays. As a matter of fact, the holography acquires in our theory a precise meaning, upon which we resolve to elaborate in a future work. For the moment being our results indicate that the classical principle of inertia needs to be extended in order to include three "mass" coefficients, as for instance in the old theory of electrons. That theory has partially failed only due to assigning an exclusive electromagnetic nature to the mass, which should not be the case: as our historical examples show, the electromagnetic theory is only particular here.

If it is to assign masses to forces by an extended principle of inertia, then these masses cannot be tensors, as in the classical case of the electromagnetic mass, but coefficients connecting the forces to the spin properties of matter which, again, cannot be revealed but within an extended particle model. The masses are simply three coefficients which, if geometrically conceived, cannot be but coordinates in a threedimensional Lorentz geometry. The whole theory of forces is then a gauge theory based on spin properties of matter, of which we presented in this work a representative, based on the following general philosophy.

The structured matter is a collection of extended particles. The structure of a particle can be sufficiently considered as a continuum, for otherwise the collection of extended particles would not be the same. The internal continuum of a particle generates the Fubini-Pick invariant cubic form characterizing the limit of separation of the particles. The forces between particles can then be calculated from the spin structure generated by the limit of separation of particle.

One of the advantages of the gauging theory as presented in this work is its versatility: it can be used in a stochastic approach of the interaction, which is particularly attractive, for instance, in a plasma theory, or in the theory of nanostructures, to name two fields of technological interest today. Furthermore, the explicit connection of the force with the limit of separation of extended particle, limit that can, on occasion, be described via fractals, allows us to entertain the idea that the theory of fractals should be somehow essential here. Indeed, the approach of fractal by the so-called scale theory, is based upon partial differential equations of the kind encountered here in the theory of potential. It then can be shown that a wave theory is instrumental in applying the fractals to physical, or even technological problems, involving matter structures [24].

Appendix: A Few Definitions of Symbols Used

$\mathbf{A}$, $\mathbf{a}$, $\mathbf{\Omega}$, $\mathbf{e}_k$ matrices and vectors, as usual

$\hat{\mathbf{e}}_k$, $\hat{\mathbf{e}}_\alpha$, $\hat{\mathbf{u}}_\alpha$ unit vectors

$|a\rangle$ vectors represented as column matrices; notation used in cases where the basis vectors are immaterial

$|\mathbf{e}\rangle$, $|\hat{\mathbf{e}}\rangle$ collection of matrices or vectors, arranged in a column; for instance a reference frame

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