

# Smarandache $\mathcal{N}$ -subalgebras (resp. filters) of $CI$ -algebras

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**Abstract.** In this paper, we introduce the notions of  $\mathcal{N}$ -subalgebras and  $\mathcal{N}$ -filters based on Smarandache  $CI$ -algebra and give a number of their properties. The relationship between  $\mathcal{N}(Q, f)$ -subalgebras(filters) and  $\mathcal{N}$ -subalgebras(filters) are also investigated.

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## 1 Introduction

Some recent researchers led to generalizations of the notion of fuzzy set that introduced by Zadeh in 1965 [15]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point  $\{1\}$  into the interval  $[0, 1]$ . In order to provide a mathematical tool to deal with negative information, Jun et. al. introduced  $\mathcal{N}$ -structures, based on negative-valued functions [6]. In 1966, Y. Imai and K. Iseki [3] introduced two classes of abstract algebras:  $BCK$ -algebras and  $BCI$ -algebras. It is known that the class of  $BCK$ -algebras is a proper subclass of the class of  $BCI$ -algebras. H. S. Kim and Y. H. Kim defined a  $BE$ -algebra [5]. Biao Long Meng, defined notion of  $CI$ -algebra as a generalization of a  $BE$ -algebra [9]. It is known that any  $BE$ -algebra is a  $CI$ -algebra. Hence, every  $BE$ -algebra is

a weaker structure than  $CI$ -algebra, thus we can consider in any  $CI$ -algebra a weaker structure as  $BE$ -algebra. Jun et. al. discussed the notion of  $\mathcal{N}$ -structures in  $BCH/BCK/BCI$ -algebras and investigated their properties in [6, 7]. They introduced the notions of  $\mathcal{N}$ -ideals of subtraction algebras and  $\mathcal{N}$ -closed ideals in  $BCK/BCI$ -algebras. We introduce the notions of  $\mathcal{N}$ -subalgebras and  $\mathcal{N}$ -filters in  $CI$ -algebras and give a number of their properties and The relationship between  $\mathcal{N}$ -subalgebras and  $\mathcal{N}$ -filters was discussed in [14]. Also, we discuss on Smarandache  $CI$ -algebra and investigated some of their useful properties in [2]. Beside, we introduced the notion of anti fuzzy set and stated the relationship with the  $\mathcal{N}$ -function of  $CI$ -algebra  $X$ . We showed that every anti fuzzy filter is an anti fuzzy subalgebra in [1]. K. J. Lee and Y. B. Jun introduced the notion of  $\mathcal{N}$ -subalgebras and  $\mathcal{N}$ -ideals based on a sub- $BCK$ -algebra of a  $BCI$ -algebras and their relations/properties are investigated in [8].

In the present paper, we continue study of  $CI$ -algebras and apply the  $\mathcal{N}$ -structures to the filter theory in  $CI$ -algebras and Smarandache  $CI$ -algebras, also investigate the relationship between  $\mathcal{N}$ -subalgebra and  $\mathcal{N}$ -filters based on Smarandache  $CI$ -algebras. We show that any  $\mathcal{N}(Q, f)$ -closed filter is an  $\mathcal{N}(Q, f)$ -subalgebra. We give some conditions for  $\mathcal{N}$ -subalgebras(filters) to be  $\mathcal{N}(Q, \varrho)$ -subalgebras(resp. filters).

## 2 Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

**Definition 2.1.** [9] *An algebra  $(X; *, 1)$  of type  $(2, 0)$  is called a  $CI$ -algebra if it satisfying the following axioms:*

$$(CI1) \quad x * x = 1,$$

$$(CI2) \quad 1 * x = x,$$

$$(CI3) \quad x * (y * z) = y * (x * z), \text{ for all } x, y, z \in X.$$

A  $CI$ -algebra  $X$  satisfying the condition  $x * 1 = 1$  is called a  $BE$ -algebra. In any  $CI$ -algebra  $X$  one can define a binary relation “ $\leq$ ” by  $x \leq y$  if and only if  $x * y = 1$ .

A  $CI$ -algebra  $X$  has the following properties:

$$(i) \quad y * ((y * x) * x) = 1,$$

(ii)  $(x * 1) * (y * 1) = (x * y) * 1,$

(iii) if  $1 \leq x$ , then  $x = 1$ , for all  $x, y \in X$ .

A non-empty subset  $S$  of a  $CI$ -algebra  $X$  is called a subalgebra of  $X$  if  $x * y \in S$  whenever  $x, y \in S$ . A mapping  $f : X \rightarrow Y$  of  $CI$ -algebra is called a homomorphism if  $f(x * y) = f(x) * f(y)$ , for all  $x, y \in X$ . A non-empty subset  $F$  of  $CI$ -algebra  $X$  is called a filter of  $X$  if (1)  $1 \in F$ , (2)  $x \in F$  and  $x * y \in F$  implies  $y \in F$ . A filter  $F$  of  $CI$ -algebra  $X$  is said to closed if  $x \in F$  implies  $x * 1 \in F$ . A nonempty subset  $S$  of a  $CI$ -algebra  $X$  is called a subalgebra of  $X$  if  $x * y \in S$ , for all  $x, y \in S$ . For our convenience, the empty set  $\emptyset$  is regarded as a subalgebra of  $X$ . Denote by  $Q(X, [-1, 0])$  the collection of functions from a set  $X$  to  $[-1, 0]$ . We say that an element of  $Q(X, [-1, 0])$  is a negative-valued function from  $X$  to  $[-1, 0]$  (briefly,  $\mathcal{N}$ -function on  $X$ ). By an  $\mathcal{N}$ -structure we mean an ordered pair  $(X, f)$  of  $X$  and an  $\mathcal{N}$ -function  $f$  on  $X$ .

In what follows, let  $X$  denote a  $CI$ -algebra and  $f$  an  $\mathcal{N}$ -function on  $X$  unless otherwise specified.

**Definition 2.2.** [14] *By a subalgebra of  $X$  based on  $\mathcal{N}$ -function  $f$  (briefly,  $\mathcal{N}$ -subalgebra of  $X$ ), we mean an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  satisfies the following assertion:*

$$f(x * y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in X.$$

**Definition 2.3.** [14] *By a filter of  $X$  based on  $\mathcal{N}$ -function  $f$  (briefly,  $\mathcal{N}$ -filter of  $X$ ), we mean an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  satisfies the following conditions:*

(i)  $f(1) \leq f(y),$

(ii)  $f(y) \leq \max\{f(x * y), f(x)\}, \text{ for all } x, y \in X.$

**Definition 2.4.** [2] *A Smarandache  $CI$ -algebra  $X$  is defined to be a  $CI$ -algebra  $X$  in which there exists a proper subset  $Q$  of  $X$  such that satisfies the following conditions:*

(S1)  $1 \in Q$  and  $|Q| \geq 2,$

(S2)  $Q$  is a  $BE$ -algebra under the operation of  $X$ .

**Example 2.1.** [2] Let  $X := \{1, a, b, c, d\}$  be a set with the following table.

$*$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	1	1	$a$	$a$	$d$
$b$	1	1	1	$a$	$d$
$c$	1	1	1	1	$d$
$d$	$d$	$d$	$d$	$d$	1

Then  $X$  is a  $CI$ -algebra and  $Q = \{1, a, b, c\}$  is a  $BE$ -algebra.

**Definition 2.5.** [2] A nonempty subset  $F$  of  $CI$ -algebra  $X$  is called a Smarandache filter of  $X$  related to  $Q$  (or briefly,  $Q$ -Smarandache filter of  $X$ ) if it satisfies:

- (SF1)  $1 \in F$ ,
- (SF2)  $(\forall y \in Q)(\forall x \in F)(x * y \in F \Rightarrow y \in F)$ .

**Definition 2.6.** [11] A fuzzy set  $\mu : X \rightarrow [0, 1]$  is called an anti fuzzy subalgebra of  $X$  if it satisfy:

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}, \text{ for all } x, y \in X.$$

**Definition 2.7.** [1] A fuzzy set  $\mu : X \rightarrow [0, 1]$  is called an anti fuzzy filter of  $X$  if it satisfies:

- (AFF1)  $\mu(1) \leq \mu(x)$ ,
- (AFF2)  $\mu(y) \leq \max\{\mu(x * y), \mu(x)\}$ , for all  $x, y \in X$ .

### 3 Smarandache $\mathcal{N}$ -subalgebras

**Definition 3.1.** Let  $X$  be a  $Q$ -Smarandache  $CI$ -algebra and  $\varrho \in [-1, 0]$ . An  $\mathcal{N}$ -structure  $(X, f)$  is called an  $\mathcal{N}$ -subalgebra of  $X$  based on  $Q$  and  $\varrho$  (briefly,  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ ) if it is an  $\mathcal{N}$ -subalgebra of  $X$  such that satisfies the following condition:

- (type 1)  $(\forall x \in Q) (\forall y \in X \setminus Q) (f(x) \leq \varrho \leq f(y))$ ,
- (type 2)  $(\forall x \in Q) (\exists y \in X \setminus Q) (f(x) \leq \varrho \leq f(y))$ ,
- (type 3)  $(\exists x \in Q) (\forall y \in X \setminus Q) (f(x) \leq \varrho \leq f(y))$ ,

- (type 4)  $(\exists x \in Q) (\exists y \in X \setminus Q) (f(x) \leq \varrho \leq f(y))$ .

**Note.** If  $\varrho := 0$ , then  $f(y) = 0$ , for all  $y \in X \setminus Q$ . So,  $(Q, f)$  is an  $\mathcal{N}$ -subalgebra. If  $\varrho := -1$ , then  $f(x) = -1$ , for all  $x \in Q$ . And so  $(X, f) = \mathcal{N}(Q, \varrho)$ .

**Example 3.1.** a) In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = f(a) = -0.7, f(b) = -0.4, f(c) = -0.6$  and  $f(d) = -0.3$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of all types on  $X$ , for  $\varrho \in [-0.4, -0.3]$  and  $Q = \{1, a, b, c\}$ .

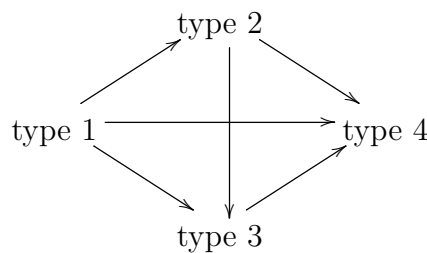
b) In Example 2.1, an  $\mathcal{N}$ -structure  $(X, g)$  in which  $g$  is defined by  $g(1) = g(a) = -0.7, g(b) = -0.2, g(c) = -0.6$  and  $g(d) = -0.3$  is not an  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$  because  $g(d) = -0.3 \not\leq g(b) = -0.2$ .

c) In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = f(a) = -0.7, f(b) = -0.4, f(c) = -0.5$  and  $f(d) = -0.3$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of type 2, type 3 and type 4 on  $X$ , for  $\varrho \in [-0.4, -0.3]$  and  $Q = \{1, a, b\}$ , but it is not of type 1, because  $f(c) \not\leq \varrho$ .

d) In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = f(a) = -0.7, f(b) = -0.2, f(c) = -0.3$  and  $f(d) = -0.1$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of type 3 and type 4 on  $X$ , for  $\varrho \in [-0.7, -0.3]$  and  $Q = \{1, a, b\}$ , but it is not of type 1 and type 2 on  $X$ , because  $f(b) \not\leq \varrho$ .

e) In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = f(a) = -0.7, f(b) = -0.2, f(c) = -0.5$  and  $f(d) = -0.3$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of type 4 on  $X$ , for  $\varrho \in [-0.7, -0.3]$  and  $Q = \{1, a, b\}$ , but it is not of type 1, type 2, type 3 on  $X$ .

Now, in the following diagram we summarize the results of this definition. The mark  $A \rightarrow B$ , means that  $A$  implies  $B$ .



In this paper, we focus on  $\mathcal{N}(Q, \varrho)$ -subalgebra of type 1 and from now on  $X$  is a  $Q$ -Smarandache  $CI$ -algebra.

The following example shows that there exists an  $\mathcal{N}$ -structure  $(X, f)$  in  $X$  such that it satisfies the condition (type 1), but it is not an  $\mathcal{N}$ -subalgebra of  $X$ .

**Example 3.2.** In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = -0.7, f(a) = -0.2, f(b) = -0.4, f(c) = -0.6$  and  $f(d) = -0.3$ .

Then  $(X, f)$  satisfies the condition (2.1) for  $\varrho \in [-0.2, -0.1]$ , but it is not an  $\mathcal{N}$ -subalgebra. Because

$$f(b * c) = f(a) = -0.2 \not\leq -0.4 = \max\{f(b), f(c)\}.$$

**Proposition 3.1.** *If an  $\mathcal{N}$ -structure  $(X, f)$  satisfies the following condition:*

$$(\forall x \in Q)(\forall y \in X \setminus Q)(f(x) \leq f(y)),$$

*then  $(X, f)$  is an  $(Q, \varrho)$ -subalgebra of  $X$ , for every  $\varrho \in [\bigvee_{x \in Q} f(x), \bigwedge_{y \in X \setminus Q} f(y)]$ .*

**Theorem 3.2.** *Let  $\varrho \in [-1, 0]$ . If  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ , then*

- (i)  $Q \subseteq C(f; \varrho)$ ,
- (ii)  $(\forall \beta \in [-1, 0]) (\beta < \varrho \Rightarrow C(f; \beta) \text{ is a subalgebra of } Q)$ .

*Proof.* Let  $(X, f)$  be a  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ . Obviously,  $Q \subseteq C(f; \varrho)$ . If  $\beta \in [-1, 0]$  be such that  $\beta < \varrho$ , then  $C(f; \beta) \subseteq Q$ . Let  $x, y \in C(f; \beta)$ . Then  $f(x) \leq \beta$  and  $f(y) \leq \beta$ . Thus  $f(x * y) \leq \max\{f(x), f(y)\} \leq \beta$ , and so  $x * y \in C(f; \beta)$ . Thus  $C(f; \beta)$  is a subalgebra of  $Q$ .  $\square$

In the following theorem we give some conditions for an  $\mathcal{N}$ -subalgebra to be an  $\mathcal{N}(Q, \varrho)$ -subalgebra.

**Theorem 3.3.** *Let  $\varrho \in [-1, 0]$ . If  $(X, f)$  is an  $\mathcal{N}$ -subalgebra of  $X$  satisfies the conditions (i) and (ii) in Theorem 3.2, then  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ .*

*Proof.* Let  $x \in Q$  and  $y \in X \setminus Q$ . Then by Theorem 3.2(i),  $x \in C(f; \varrho)$ , and so  $f(x) \leq \varrho$ . Let  $f(y) = \beta$ . If  $\beta < \varrho$ , then by Theorem 3.2(ii),  $y \in C(f; \beta) \subseteq Q$ , which is a contradiction. Hence  $f(x) \leq \varrho \leq \beta = f(y)$ . Thus  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ .  $\square$

## 4 Smarandache $\mathcal{N}$ -filters

**Definition 4.1.** *Let  $X$  be a  $Q$ -Smarandache CI-algebra and  $\varrho \in [-1, 0]$ . An  $\mathcal{N}$ -structure  $(X, f)$  is called an  $\mathcal{N}$ -filter of  $X$  based on  $Q$  and  $\varrho$  (briefly,  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ ) if it satisfies the following conditions:*

- (i)  $(\forall x \in Q) (\forall y \in X \setminus Q) (f(1) \leq f(x) \leq \varrho \leq f(y))$ .

$$(ii) \quad (\forall x, y \in Q) \quad (f(y) \leq \max\{f(x * y), f(x)\}).$$

**Example 4.1.** In Example 2.1, an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = -0.6$ ,  $f(a) = -0.4$ ,  $f(b) = -0.5$ ,  $f(c) = -0.4$  and  $f(d) = -0.3$  is an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$  for  $\varrho \in [-0.4, -0.3]$ .

**Theorem 4.1.** Let  $\{\mathcal{N}(Q_i, \varrho_i) : i \in \Delta\}$  be a family of  $\mathcal{N}(Q_i, \varrho_i)$ -subalgebras (filters) of  $X$  where  $\Delta \neq \emptyset$  and  $\varrho_i \in [-1, 0]$ , for all  $i \in \Delta$ . Then  $\mathcal{N}(\cap Q_i, \min\{\varrho_i\}_{i \in \Delta})$  is a subalgebra (filter) of  $X$ , too.

**Theorem 4.2.** Let  $\varrho \in [-1, 0]$ . If  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ , then

$$(i) \quad Q \subseteq C(f; \varrho),$$

$$(ii) \quad (\forall \beta \in [-1, 0]) \quad (\beta < \varrho \Rightarrow C(f; \beta) \text{ is a filter of } Q).$$

*Proof.* Let  $(X, f)$  be an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ . Obviously,  $Q \subseteq C(f; \varrho)$ . Let  $\beta \in [-1, 0]$  be such that  $\beta < \varrho$ . If  $x \in C(f; \beta)$ , then  $f(x) \leq \beta < \varrho$ , and so  $x \in Q$ . Hence  $C(f; \beta) \subseteq Q$ . by Definition 4.1(i),  $f(1) \leq f(x)$ , for all  $x \in X$ . Hence  $f(1) \leq f(x) \leq \beta$  for all  $x \in C(f; \beta)$ , and so  $1 \in C(f; \beta)$ . Let  $x, y \in Q$  be such that  $x * y \in C(f; \beta)$  and  $x \in C(f; \beta)$ . Then  $f(x * y) \leq \beta$  and  $f(x) \leq \beta$ . If  $x, y \in C(f; \beta)$ , then  $f(x) \leq \beta$ . Now by Definition 4.1(ii),  $f(y) \leq \max\{f(x * y), f(x)\} \leq \beta$ . Thus  $y \in C(f; \beta)$ . Therefore,  $C(f; \beta)$  is a filter of  $Q$ .  $\square$

For a  $Q$ -Smarandache  $CI$ -algebra  $X$  and  $\varrho \in [-1, 0]$ , the following example shows that an  $\mathcal{N}$ -filter  $(X, f)$  of  $X$  may not be an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ .

**Example 4.2.** Let  $X := \{1, a, b, c\}$  be a set with the following table.

$*$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$
$a$	1	1	$b$	$c$
$b$	1	$a$	1	$c$
$c$	$c$	$c$	$c$	1

Then  $X$  is a  $CI$ -algebra and  $Q := \{1, a, b\}$  is a  $BE$ -algebra [13]. Define an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = -0.7$ ,  $f(a) = -0.2$ ,  $f(b) = -0.4$ ,  $f(c) = -0.2$ . Then  $(X, f)$  is an  $\mathcal{N}$ -filter of  $X$ . But it is not an  $\mathcal{N}(Q, \varrho)$  of  $X$  for  $\varrho \in [-0.7, -0.3]$ . Because  $f(a) = -0.2 > \varrho$ .

In the following theorem we give conditions for an  $\mathcal{N}$ -filter to be an  $\mathcal{N}(Q, \varrho)$ -filter.

**Theorem 4.3.** Let  $\varrho \in [-1, 0]$  and  $(X, f)$  be an  $\mathcal{N}$ -filter of  $X$  satisfies the conditions (i) and (ii) of Theorem 4.2. Then  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ .

*Proof.* Let  $x \in Q$  and  $y \in X \setminus Q$ . Then by Theorem 4.2(i),  $x \in C(f; \varrho)$ , and so  $f(x) \leq \varrho$ . Let  $f(y) = \beta$ . If  $\beta < \varrho$ , then by Theorem 4.2(ii),  $y \in C(f; \beta) \subseteq Q$ , which is a contradiction. Hence  $\varrho \leq \beta = f(y)$ . Since  $f(1) \leq f(x)$  for all  $x \in X$ , it follows that  $f(1) \leq f(x) \leq \varrho \leq \beta = f(y)$  so that condition (i) of Definition 4.1 is valid. Since  $f$  is an  $\mathcal{N}$ -filter of  $X$ , the condition (ii) of Definition 4.1 is obvious. Therefore,  $(X, f)$  is an  $\mathcal{N}(Q, \varrho)$ -filter of  $X$ .  $\square$

The following example shows that an  $\mathcal{N}(Q, \varrho)$ -subalgebra may not be an  $\mathcal{N}(Q, \varrho)$ -filter.

**Example 4.3.** Let  $X := \{1, a, b, c, d\}$  be a set with the following table.

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	a	d
b	1	1	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then  $X$  is a  $CI$ -algebra and  $Q = \{1, a, b, c\}$  is a  $BE$ -algebra. Define an  $\mathcal{N}$ -structure  $(X, f)$  in which  $f$  is defined by  $f(1) = -0.4$ ,  $f(a) = -0.4$ ,  $f(b) = -0.3$ ,  $f(c) = -0.2$  and  $f(d) = -0.1$ . Then  $(X, f)$  is an  $\mathcal{N}$ -subalgebra, for  $\varrho \in [-0.2, 0]$ , but it is not an  $\mathcal{N}$ -filter because

$$f(c) = -0.2 \not\leq -0.3 = \max\{f(b * c), f(b)\}.$$

**Definition 4.2.** An  $\mathcal{N}$ -function on  $X$  is called closed  $\mathcal{N}$ -filter if  $f$  satisfies:

$$f(x * 1) \leq f(x) \leq \max\{f(y * x), f(y)\}, \text{ for all } x, y \in X.$$

**Example 4.4.** Let  $X := \{1, a, b\}$  be a set with the following table:

*	1	a	b
1	1	a	b
a	a	1	1
b	a	1	1

Then  $X$  is a  $CI$ -algebra [10]. Define an  $\mathcal{N}$ -function  $f : X \rightarrow [0, 1]$  by  $f(1) = -0.7$ ,  $f(a) = -0.3$  and  $f(b) = -0.4$ . Then  $(X, f)$  is an  $\mathcal{N}$ -filter of  $X$ . But it is not an  $\mathcal{N}$ -closed filter because

$$f(b * 1) = f(a) = -0.3 \not\leq f(b) = -0.4.$$



**Example 4.5.** In Example 4.4, if define  $\mathcal{N}$ -function  $f : X \rightarrow [0, 1]$  by  $f(1) = -0.7$ ,  $f(a) = -0.4$  and  $f(b) = -0.4$ . Then  $(X, f)$  is an  $\mathcal{N}$ -closed filter of  $X$ .

**Proposition 4.4.** *Let  $(X, f)$  be an  $\mathcal{N}$ -closed filter. Then  $f(1) \leq f(x)$ , for all  $x \in X$ .*

*Proof.* Let  $x \in X$ . Now, by Definition 4.2, we have

$$f(1) \leq \max\{f(x * 1), f(x)\} \leq \max\{f(x), f(x)\} = f(x).$$

□

**Theorem 4.5.** *Let  $(X, f)$  be an closed  $\mathcal{N}$ -filter and  $\varrho \in [-1, 0]$ . Then every  $\mathcal{N}(Q, \varrho)$ -filter is  $\mathcal{N}(Q, \varrho)$ -subalgebra of  $X$ .*

*Proof.* Let  $(X, f)$  be  $\mathcal{N}(Q, \varrho)$ -filter and  $x, y \in X$ . Then by (CI3) and Definition 4.2, we have

$$\begin{aligned} f(x * y) &\leq \max\{f(y * (x * y)), f(y)\} \\ &= \max\{f(x * (y * y)), f(y)\} \\ &= \max\{f(x * 1), f(y)\} \\ &\leq \max\{f(x), f(y)\}. \end{aligned}$$

Therefore,  $(X, f)$  is an  $\mathcal{N}$ -subalgebra of  $X$ . □

**Theorem 4.6.** *Let  $(X, f)$  and  $(X, g)$  be  $\mathcal{N}(Q_1, \varrho_1)$  and  $\mathcal{N}(Q_2, \varrho_2)$ -subalgebra (filter) of  $X$  respectively. Then  $(X \times X, f \times g)$  is an  $\mathcal{N}(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\})$ -subalgebra(filter) of  $X \times X$ .*

*Proof.* Let  $(x, y) \in (Q_1 \times Q_2)$  and  $(z, t) \in (X \times X) \setminus (Q_1 \times Q_2)$ . Then we have

$$\begin{aligned} (f \times g)(1, 1) = \max\{f(1), g(1)\} &\leq \max\{f(x), g(y)\} \\ &\leq \max\{\varrho_1, \varrho_2\} \\ &\leq \max\{f(z), f(t)\} = (f \times g)(z, t). \end{aligned}$$

Now, let  $(x_1, x_2), (y_1, y_2) \in (Q_1 \times Q_2)$ . Then

$$\begin{aligned} (f \times g)((x_1, x_2) * (y_1, y_2)) &= (f \times g)((x_1 * y_1), (x_2 * y_2)) \\ &= \max\{f(x_1 * y_1), g(x_2 * y_2)\} \\ &\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} \\ &= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} \\ &= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}. \end{aligned}$$

Hence  $(X \times X, f \times g)$  is an  $\mathcal{N}(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\})$ -subalgebra(resp. filter) of  $X \times X$ . □

**Proposition 4.7.** *Let  $Q_1$  and  $Q_2$  be two  $BE$ -algebras which are properly contained in  $X$ ,  $Q_1 \subseteq Q_2$  and  $\varrho \in [-1, 0]$ . Then every  $\mathcal{N}(Q_2, \varrho)$ -subalgebra(filter) of  $X$  is an  $\mathcal{N}(Q_1, \varrho)$ -subalgebra(filter) of  $X$ .*

**Note.** By the following example we show that the converse of above theorem is not correct in general.

**Example 4.6.** Let  $X := \{1, a, b, c\}$  be a set with the following table.

$*$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$
$a$	1	1	$b$	$c$
$b$	1	$a$	1	$c$
$c$	$c$	$c$	$c$	1

Then  $Q_1 = \{1, a\}$ ,  $Q_2 = \{1, a, b\}$  are  $BE$ -algebras which are properly contained in  $X$  and  $f(1) = -0.7$ ,  $f(a) = -0.4$ ,  $f(b) = -0.2$  and  $f(c) = -0.1$ . Then  $(X, f)$  is an  $\mathcal{N}(Q_1, \varrho)$ -subalgebra, for all  $\varrho \in [-0.4, 0]$ , but it is not an  $\mathcal{N}(Q_2, \varrho)$ -subalgebra, because, if  $\varrho := -0.3$ , then  $f(b) = -0.2 \not\leq -0.3$ .

## 5 Conclusion

A Smarandache structure on a set  $A$  means a weak structure  $W$  on  $A$  such that there exist a proper subset  $B$  of  $A$  which is embedded with a strong structure  $S$ . It is that any  $BE$ -algebra is a  $CI$ -algebra. Hence, every  $BE$ -algebra is a weaker structure than  $CI$ -algebra, thus we can consider in any  $CI$ -algebra a weaker structure as  $BE$ -algebra.

In this paper, we have introduced the concept of  $\mathcal{N}$ -subalgebra (filter) based on Smarandache  $CI$ -algebras and some related properties are investigated. We show that any  $\mathcal{N}(Q, f)$ -closed filter is an  $\mathcal{N}(Q, f)$ -subalgebra. We give some conditions for an  $\mathcal{N}$ -subalgebras (filters) to be  $\mathcal{N}(Q, \varrho)$ -subalgebras (filters).

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