Multiple attribute group decision making method based on neutrosophic number generalized hybrid weighted averaging operator

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Abstract: Neutrosophic number (NN) is an important tool which is used to express indeterminate evaluation information. The purpose of the paper is to propose some aggregation operators based on neutrosophic number, which are used to handle multiple attribute group decision making problems. Firstly, we introduce the definition, the properties and the operational laws of the neutrosophic numbers, and the possibility degree function is briefly introduced. Then, some neutrosophic number operators are proposed, such as the neutrosophic number weighted arithmetic averaging (NNWAA) operator, the neutrosophic number ordered weighted arithmetic averaging (NNOWAA) operator, the neutrosophic number hybrid weighted arithmetic averaging (NNHWAA) operator, the neutrosophic number weighted geometric averaging (NNWGA) operator, the neutrosophic number ordered weighted geometric averaging (NNOWGA) operator, the neutrosophic number hybrid weighted geometric averaging (NNHWGA) operator, the neutrosophic number generalized weighted averaging (NNGWA) operator, the neutrosophic number generalized ordered weighted averaging (NNGOWA) operator, the neutrosophic number generalized hybrid weighted averaging (NNGHWA) operator. Furthermore, some properties of these operators are discussed. Moreover, a multiple attribute group decision making method based on the NNGHWA operator is proposed. Finally, an illustrative example is proposed to demonstrate the practicality and effectiveness of the method.

Keywords: Multiple attribute group decision making; neutrosophic numbers; neutrosophic number generalized aggregation operator.

1. Introduction

Multiple attribute group decision making (MAGDM) is an important branch of decision theory which has been widely applied in many fields. Because of the fuzziness of human thinking and the complexity of objective things, the attribute values expressed by the crisp numbers have difficulty in conveying people's thinking about objective things. Zadeh [1] firstly proposed the fuzzy set (FS) to deal with the fuzzy information. Because the fuzzy set only considered the membership degree and did not take the non-membership degree into account, Atanassov [2] further proposed the intuitionistic fuzzy set (IFS) which was used to overcome the shortcoming of the FS. In other words, the intuitionistic fuzzy set (IFS) consisted of membership degree and non-membership degree. Similar to the FS, IFS paid more attention to the membership degree and non-membership degree and did not consider the indeterminacy-membership degree. On the basis of the intuitionistic fuzzy set, Smarandache [3] further proposed the neutrosophic numbers (NNs), which can be divided into two parts: determinate part and indeterminate part. So the neutrosophic number (NN) was more practical to handle indeterminate information in real situations. Therefore, the neutrosophic number (NN) can be represented as the function $N = a + bI$ in which a is the determinate part and bI is the indeterminate part. Obviously, the indeterminate part related to the neutrosophic number (NN) is fewer, the information conveyed by NN is better. So, the worst scenario is $N = bI$, where the indeterminate part reach the maximum. Conversely, the best case is $N = a$ where there is not indeterminacy related the neutrosophic number. Thus, it is more suitable to handle the indeterminate information in decision making problems. To this day, using neutrosophic numbers to handle indeterminate problems has made little progress in the fields of scientific and engineering techniques. Therefore, it is necessary to propose a new method based on the neutrosophic numbers to handle group decision making problems.

The information aggregation operators have attracted more and more attentions, and they have become a hot research topic. A variety of operators have been proposed to aggregate evaluation information in various environments [4−7,9-13] such as the arithmetic aggregation operator, the geometric aggregation operator and the generalized aggregation operator. Yager [8] firstly proposed the ordered weighted averaging (OWA) operator which was widely used in decision field. The OWA operator can weight the inputs according to the ranking position of all inputs. Many extension of the OWA operator have been proposed, Such as uncertain aggregation operators [12,14,15], the induced aggregation operators [16,17], the linguistic aggregation operators [18,19], the uncertain linguistic aggregation operators [7], the fuzzy aggregation operators [5,20], the fuzzy linguistic aggregation operators [21], the induced linguistic aggregation operators [22], the induced uncertain linguistic aggregation operators [23,24], the fuzzy induced aggregation operators [25] and the intuitionistic fuzzy aggregation operators [26]. Based on the operators mentioned above, Xu and Chen [27] proposed some interval-valued intuitionistic fuzzy arithmetic aggregation (IVIFAA) operators, such as the interval-valued intuitionistic fuzzy weighted aggregation(IVIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted aggregation (IVIFOWA) operator, and the interval-valued intuitionistic fuzzy hybrid aggregation (IVIFHA) operator. Zhao [28] proposed the generalized intuitionistic fuzzy weighted (GIFWA) operator, the generalized intuitionistic fuzzy ordered weighted (GIFOWA) operator, and the generalized intuitionistic fuzzy hybrid (GIFHA) operator.

To this day, there are not the researches on the combination between neutrosophic numbers and generalized aggregation operator. Thus, it is essential to do the research based on neutrosophic numbers aggregation operators. In this paper, we propose a new method, the generalized hybrid weighted averaging operator based on neutrosophic numbers, to handle multiple attribute group decision making problems. The new method not only can handle the indeterminacy of evaluation information but also can consider the relationship between the attributes.

The remainder of this paper is shown as follows. In section 2, we briefly introduce the basic concepts and the operational rules and the characteristics of NNs. In section 3, some aggregation operators based on neutrosophic numbers and these properties are proposed, such as the neutrosophic number weighted arithmetic averaging (NNWAA) operator, the neutrosophic number ordered weighted averaging (NNOWA) operator, the neutrosophic number hybrid weighted averaging (NNHWA) operator, the neutrosophic number weighted geometric averaging (NNWGA) operator, the neutrosophic number ordered weighted geometric averaging (NNOWGA) operator, the neutrosophic number hybrid weighted geometric averaging (NNHWGA) operator, the neutrosophic number generalized weighted averaging (NNGWA)operator, the neutrosophic number generalized ordered weighted averaging (NNGOWA) operator, the neutrosophic number generalized hybrid weighted averaging (NNGHWA) operator. In section 4, we briefly introduce the procedure of multiple attribute group decision making method based on neutrosophic number generalized hybrid weighted averaging (NNGHWA) operator. In section 5, we give a numerical example to demonstrate the effective of the new proposed method.

2. Preliminaries

Definition 1[29-31]. Let $I \in [\beta^-, \beta^+]$ be an indeterminate part, a neutrosophic number N is given by

$$N = a + bI \tag{1}$$

where a and b are real numbers, and I is indeterminacy, such that $I^2 = I$, $0 \cdot I = 0$ and $I / I = undefined$.

Definition 2[30-31]. Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two neutrosophic numbers, then the operational laws are defined as follows.

(1) $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$ (2)

(2) $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$ (3)

(3) $N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1 + b_1 b_2)I$ (4)

(4) $N_1{}^2 = a_1{}^2 + (2a_1 b_1 + b_1{}^2)I$ (5)

(5) $\lambda N_1 = \lambda a_1 + \lambda b_1 I$ (6)

(6) $N_1{}^\lambda = a_1{}^\lambda + \left((a_1 + b_1)^\lambda - a_1{}^\lambda\right) I \ \lambda > 0$ (7)

(7) $\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2 (a_2 + b_2)} I \ for\ a_2 \neq 0\ and\ a_2 \neq -b_2$ (8)

Theorem 1. Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two neutrosophic numbers, and $\lambda, \lambda_1, \lambda_2 > 0$, then we have

(1) $N_1 \oplus N_2 = N_2 \oplus N_1$ (9)

(2) $N_1 \otimes N_2 = N_2 \otimes N_1$ (10)

(3) $\lambda(N_1 \oplus N_2) = \lambda N_1 \oplus \lambda N_2$ (11)

(4) $\lambda_1 N_1 \oplus \lambda_2 N_1 = (\lambda_1 + \lambda_2) N_1$ (12)

(5) $N_1{}^\lambda \otimes N_2{}^\lambda = (N_1 \otimes N_2)^\lambda$ (13)

(6) $N_1{}^{\lambda_1} \otimes N_1{}^{\lambda_2} = N_1{}^{\lambda_1 + \lambda_2}$ (14)

Proof.

(1) the formula (9) is obviously right according to the operational rule (1) expressed by (2).

(2) the formula (10) is obviously right according to the operational rule (2) expressed by (3).

(3) for the left of the formula (11)

$$\lambda(N_1 \oplus N_2) = \lambda\left((a_1 + b_1 I) \oplus (a_2 + b_2 I)\right) = \lambda\left((a_1 + a_2) + (b_1 + b_2)I\right)$$

for the right of the formula (11)

$$\begin{aligned}\lambda N_1 \oplus \lambda N_2 &= \lambda(a_1 + b_1 I) \oplus \lambda(a_2 + b_2 I) = (\lambda a_1 + \lambda b_1 I) \oplus (\lambda a_2 + \lambda b_2 I) \\ &= (\lambda a_1 + \lambda a_2) + (\lambda b_1 + \lambda b_2)I = \lambda\left((a_1 + a_2) + (b_1 + b_2)I\right)\end{aligned}$$

So, we can get $\lambda(N_1 \oplus N_2) = \lambda N_1 \oplus \lambda N_2$

which completes the proof of the formula (11).

(4) $\lambda_1 N_1 \oplus \lambda_2 N_1 = \lambda_1(a_1 + b_1 I) + \lambda_2(a_1 + b_1 I) = (\lambda_1 a_1 + \lambda_2 a_1) + (\lambda_1 b_1 + \lambda_2 b_1)I$

$$= (\lambda_1 + \lambda_2)a_1 + (\lambda_1 + \lambda_2)b_1 I = (\lambda_1 + \lambda_2)N_1$$

So, we can proof the formula (12) is right.

(5) for the left of the formula (13)

$$N_1^{\lambda} \otimes N_2^{\lambda} = \left(a_1^{\lambda} + \left((a_1 + b_1)^{\lambda} - a_1^{\lambda}\right)I\right) \otimes \left(a_2^{\lambda} + \left((a_2 + b_2)^{\lambda} - a_2^{\lambda}\right)I\right)$$

$$= a_1^{\lambda} a_2^{\lambda} + a_1^{\lambda}\left((a_2 + b_2)^{\lambda} - a_2^{\lambda}\right)I + a_2^{\lambda}\left((a_1 + b_1)^{\lambda} - a_1^{\lambda}\right)I + \left((a_2 + b_2)^{\lambda} - a_2^{\lambda}\right)\left((a_1 + b_1)^{\lambda} - a_1^{\lambda}\right)I$$

$$= a_1^{\lambda} a_2^{\lambda} + \left(a_1^{\lambda}(a_2 + b_2)^{\lambda} - a_1^{\lambda} a_2^{\lambda}\right)I + \left(a_2^{\lambda}(a_1 + b_1)^{\lambda} - a_2^{\lambda} a_1^{\lambda}\right)I$$

$$+ \left((a_2 + b_2)^{\lambda}(a_1 + b_1)^{\lambda} - a_2^{\lambda}(a_1 + b_1)^{\lambda} - a_1^{\lambda}(a_2 + b_2)^{\lambda} + a_1^{\lambda} a_2^{\lambda}\right)I$$

$$= (a_1 a_2)^{\lambda} + \left((a_2 + b_2)^{\lambda}(a_1 + b_1)^{\lambda} - a_1^{\lambda} a_2^{\lambda}\right)I$$

for the right of the formula (13)

$$(N_1 \otimes N_2)^{\lambda} = \left((a_1 + b_1)I \otimes (a_2 + b_2)I\right)^{\lambda} = \left(a_1 a_2 + (a_1 b_2 + a_2 b_1 + b_1 b_2)I\right)^{\lambda}$$

$$= (a_1 a_2)^{\lambda} + \left((a_1 a_2 + a_1 b_2 + a_2 b_1 + b_1 b_2)^{\lambda} - (a_1 a_2)^{\lambda}\right)I$$

$$= (a_1 a_2)^{\lambda} + \left((a_1 + b_1)^{\lambda}(a_2 + b_2)^{\lambda} - a_1^{\lambda} a_2^{\lambda}\right)I$$

So, we can proof the formula (13) is right.

(6) $N_1^{\lambda_1} \otimes N_1^{\lambda_2} = \left(a_1^{\lambda_1} + \left((a_1 + b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I\right) \otimes \left(a_1^{\lambda_2} + \left((a_1 + b_1)^{\lambda_2} - a_1^{\lambda_2}\right)I\right)$

$$= a_1^{\lambda_1} a_1^{\lambda_2} + \left(a_1^{\lambda_1}\left((a_1 + b_1)^{\lambda_2} - a_1^{\lambda_2}\right)I + a_1^{\lambda_2}\left((a_1 + b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I + \left((a_1 + b_1)^{\lambda_2} - a_1^{\lambda_2}\right)\left((a_1 + b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I\right)$$

$$= a_1^{\lambda_1} a_1^{\lambda_2} + \left((a_1 + b_1)^{\lambda_2}(a_1 + b_1)^{\lambda_1} - a_1^{\lambda_2} a_1^{\lambda_1}\right)I$$

$$= a_1^{\lambda_1 + \lambda_2} + \left((a_1 + b_1)^{\lambda_1 + \lambda_2} - a_1^{\lambda_1 + \lambda_2}\right)I$$

$$= N_1^{\lambda_1 + \lambda_2}$$

So, we can proof the formula (14) is right.

Definition 3[32-33]. Let $N_i = a_i + b_i I$ be a neutrosophic number in which $I \in [\beta^-, \beta^+](i = 1,2,...,n)$, $a_i, b_i, \beta^-, \beta^+ \in R$, where R is all real numbers, the neutrosophic number N_i is equivalent to $N_i \in [a_i + b_i\beta^-, a_i + b_i\beta^+]$, then the possibility degree is

$$P_{ij} = P(N_i \geq N_j) = \max\left\{1 - \max\left(\frac{(a_j + b_j\beta^+) - (a_i + b_i\beta^-)}{(a_i + b_i\beta^+) - (a_i + b_i\beta^-) + (a_j + b_j\beta^+) - (a_j + b_j\beta^-)}, 0\right), 0\right\} \quad (15)$$

Thus, the matrix of possibility degrees can be simplified as $P = (P_{ij})_{n \times n}$, where $P_{ij} \geq 0$,

$P_{ij} + P_{ji} = 1$, and $P_{ii} = 0.5$. Then, the value of N_i $(i = 1,2,...,n)$ for ranking order is given as follows:

$$q_i = \frac{\left(\sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1\right)}{n(n-1)} \quad (16)$$

Hence, the bigger values of q_i $(i = 1,2,...,n)$ is, the more precise information of neutrosophic numbers conveyed can be acquired, so, the neutrosophic numbers of N_i $(i = 1,2,...,n)$ can be ranked in an ascending order according to the values of q_i $(i = 1,2,...,n)$.

3. Neutrosophic Number Aggregation Operators

A neutrosophic number includes two parts, determinate part a and indeterminate part bI. Therefore, the neutrosophic number has an advantage in expressing indeterminate and incomplete information in real decision making. On the basis of neutrosophic numbers, it is necessary to propose some aggregation operators and apply them to the MAGDM problems in which the attribute values take the form of NNs. Here, some neutrosophic number aggregation operators are proposed firstly.

3.1 The neutrosophic number hybrid weight arithmetic averaging operator

Definition 4. Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of neutrosophic numbers (NNs), and NNWAA : $NNS^n \rightarrow NNS$. If

$$NNWAA(N_1, N_2,..., N_n) = \sum_{i=1}^{n} \omega_i N_i \quad (17)$$

where $\omega = (\omega_1, \omega_2,..., \omega_n)$ is the weight vector of $N_i (i = 1,2,...,n)$ satisfying $\omega_i \in [0,1]$ $(i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then NNWAA is called neutrosophic number weighted arithmetic averaging operator. Specially, when $\omega = \left(\frac{1}{n}, \frac{1}{n},..., \frac{1}{n}\right)$, the NNWAA operator will degenerate into neutrosophic number arithmetic averaging (NNAA) operator:

$$NNAA(N_1, N_2,..., N_n) = \frac{1}{n} \sum_{i=1}^{n} N_i \quad (18)$$

Theorem 2. Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of NNs, and $\omega = (\omega_1, \omega_2,..., \omega_n)$ be the weight vector of $N_i (i = 1,2,...,n)$ satisfying $\omega_i \in [0,1]$ $(i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the result obtained by Eq. (17) is still an NN and

$$NNWAA(N_1, N_2,..., N_n) = \sum_{i=1}^{n} \omega_i a_i + \sum_{i=1}^{n} \omega_i b_i I \quad (19)$$

The Eq.(19) can be proved by Mathematical induction on n as follows:

Proof.

(i) when n=1, the Eq. (19) is right obviously.

(ii) Suppose when $n = k$, the Eq.(19) is right, i.e.,

$$NNWAA(N_1, N_2,..., N_k) = \sum_{i=1}^{k} \omega_i a_i + \sum_{i=1}^{k} \omega_i b_i I$$

Then when $n = k+1$, we have

$$NNWAA(N_1, N_2, ..., N_{k+1}) = NNWAA(N_1, N_2, ..., N_k) \oplus \omega_{k+1} N_{k+1}$$

$$= \left(\sum_{i=1}^{k} \omega_i a_i + \sum_{i=1}^{k} \omega_i b_i I \right) + \left(\omega_{k+1} a_{k+1} + \omega_{k+1} b_{k+1} I \right) = \sum_{i=1}^{k+1} \omega_i a_i + \sum_{i=1}^{k+1} \omega_i b_i I$$

So, when $n = k+1$, the Eq.(19) is also right.

According to (i) and (ii), we can get when the Eq.(19) is right for all n.

Theorem 3. (Idempotency).

Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI$ $(i = 1,2,...,n)$, then

$$NNWAA(N_1, N_2, ..., N_n) = N_0 .$$

Proof.

Since $N_i = N_0$, for all A_i, we have

$$NNWAA(A_1, A_2, ..., A_n) = NNWAA(A_0, A_0, ..., A_0) = \sum_{i=1}^{k} \omega_i a + \sum_{i=1}^{k} \omega_i bI = a + bI = N_0$$

which completes the proof of theorem 3.

Theorem 4. (Monotonicity).

Let $N_i = a_i + b_i I$ and $N^*{}_i = a^*{}_i + b^*{}_i I$ be two sets of NNs satisfying $a_i \le a_i^*$, $b_i^* \le b_i$,for all i,i=1,2,...,n ,then

$$NNWAA(N_1, N_2, ..., N_n) \le NNWAA(N_1^*, N_2^*, ..., N_n^*) .$$

Proof.

Since $a_i \le a_i^*$, $b_i^* \le b_i$, for all i, we can get $\sum_{i=1}^{n} \omega_i a_i \le \sum_{i=1}^{n} \omega_i a^*{}_i$, $\sum_{i=1}^{n} \omega_i b^*{}_i I \le \sum_{i=1}^{n} \omega_i b_i I$

So, we can get $NNWAA(N_1, N_2, ..., N_n) \le NNWAA(N_1^*, N_2^*, ..., N_n^*)$.

which complete the proof of theorem 4.

Theorem 5. (Boundedness).

Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of NNs. If $N_{\max} = \max(N_1, N_2, ..., N_n) = a_{\max} + b_{\min} I$ and $N_{\min} = \min(N_1, N_2, ..., N_n) = a_{\min} + b_{\max} I$,

then

$$N_{\min} \le NNWAA(N_1, N_2, ..., N_n) \le N_{\max}$$

Proof.

Since $a_{\min} \le a_i \le a_{\max}$, $b_{\max} \le b_i \le b_{\min}$, for all i ,we can get

$$\sum_{i=1}^{n} \omega_i a_{\min} \le \sum_{i=1}^{n} \omega_i a_i \le \sum_{i=1}^{n} \omega_i a_{\max}, \quad \sum_{i=1}^{n} \omega_i b_{\max} \le \sum_{i=1}^{n} \omega_i b_i \le \sum_{i=1}^{n} \omega_i b_{\min}$$

So, we can get

$$NNWAA(N_{\min}, N_{\min}, ..., N_{\min}) \le NNWAA(N_1, N_2, ..., N_n) \le NNWAA(N_{\max}, N_{\max}, ..., N_{\max}),$$

According to theorem 3, we can know

$$NNWAA(N_{\min}, N_{\min}, ..., N_{\min}) = N_{\min}$$

$$NNWAA(N_{\max}, N_{\max}, ..., N_{\max}) = N_{\max}$$

So, we can get $N_{\min} \le NNWAA(N_1, N_2, ..., N_n) \le N_{\max}$, which complete the proof of the theorem 5.

Definition 5. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, and NNOWAA : NNS^n→ NNS. If

$$NNOWAA(N_1,N_2,...,N_n) = \sum_{i=1}^{n} \omega_i \widetilde{N}_i \tag{20}$$

Where $\omega = (\omega_1, \omega_2,..., \omega_n)$ is the weight vector correlative with the NNOWAA operator satisfying $\omega_i \in [0,1] \ (i = 1,2,...,n)$, and $\sum_{i=1}^{n} \omega_i = 1$. $\widetilde{N}_i$ is the ith largest of the $N_i (i = 1,2,...,n)$. Then NNOWAA operator is called neutrosophic number ordered weighted arithmetic averaging operator.

Theorem 6. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, $\omega = (\omega_1, \omega_2,..., \omega_n)$ is the weight vector correlative with the NNOWAA operator satisfying $\omega_i \in [0,1] \ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$, $\widetilde{N}_i = a'_i + b'_i I$ be the value of the ith largest $N_i \ (i = 1,2,...,n)$. Then the result obtained using Eq. (20) is still an NN and

$$NNOWAA(N_1, N_2,..., N_n) = \sum_{i=1}^{n} \omega_i a'_i + \sum_{i=1}^{n} \omega_i b'_i I \tag{21}$$

The proof is similar with theorem 2, it is omitted here.

Similar to Theorems 3-5, it is easy to prove the NNOWAA operator has the following 7-9 properties.

Theorem 7 (Idempotency).

Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI$, then

$$NNOWAA(N_1, N_2,..., N_n) = N_0.$$

Theorem 8(Monotonicity)**.**

Let $N_i = a_i + b_i I$ and $N_i^* = a_i^* + b_i^* I$ be two sets of NNs satisfying $a_i \le a_i^*$, $b_i^* \le b_i$, for all i,i=1,2,...,n ,then

$$NNOWAA(N_1, N_2,..., N_n) \le NNOWAA(N_1^*, N_2^*,..., N_n^*).$$

Theorem 9. (Boundedness).

Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, If $N_{\max} = a_{\max} + b_{\min} I$ and $N_{\min} = a_{\min} + b_{\max} I$, then

$$N_{\min} \le NNOWAA(N_1, N_2,..., N_n) \le N_{\max}$$

Theorem 10. (Commutativity).

Let $(N'_1, N'_2,..., N'_n)$ is any permutation of $(N_1, N_2,..., N_n)$, then

$$NNOWAA(N'_1,N'_2,...,N'_n) = NNOWAA(N_1,N_2,...,N_n)$$

Proof.

Suppose the weight of $(N_1', N_2',..., N_n')$ is $(\omega'_1, \omega'_2,..., \omega'_n)$, then since $(N_1', N_2',..., N_n')$ is any permutation of $(N_1, N_2,..., N_n)$,we have

$$\sum_{i=1}^{n} \omega_i a_i = \sum_{i=1}^{n} \omega'_i a'_i, \quad \sum_{i=1}^{n} \omega_i b_i = \sum_{i=1}^{n} \omega'_i b'_i$$

So, we can get $\sum_{i=1}^{n}\omega_i N_i = \sum_{i=1}^{n}\omega_i^{'} N_i^{'}$, then

$$NNOWAA(N^{'}_1,N^{'}_2,...,N^{'}_n) = NNOWAA(N_1,N_2,...,N_n)$$

Definition 6. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, and NNHWAA : NNS^n→ NNS. If

$$NNHWAA(N_1,N_2,...,N_n) = \sum_{i=1}^{n}\omega_i \tilde{N}_{\sigma(i)} \quad (22)$$

Where $\omega = (\omega_1, \omega_2,...,\omega_n)$ is the weight vector correlative with the NNHWAA operator satisfying $\omega_i \in [0,1] \, (i = 1,2,...,n)$ and $\sum_{i=1}^{n}\omega_i = 1$; $\tilde{N}_{\sigma(i)}$ is the ith largest of the $nw_i N_i (i = 1,2,...,n)$,such that $\tilde{N}_{\sigma(i-1)} \geq \tilde{N}_{\sigma(i)}$ and $w = (w_1, w_2,..., w_n)^T$ is the weighting vector of $N_i \ (i = 1,2,...,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$,Then, NNHWAA is called neutrosophic number hybrid weighted arithmetic averaging operator.

Theorem 11. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, then the result obtained using Eq. (22) can be expressed as

$$NNHWAA(N_1, N_2,..., N_n) = \sum_{i=1}^{n}\omega_i a^{'}_{\sigma(i)} + \sum_{i=1}^{n}\omega_i b^{'}_{\sigma(i)} I \quad (23)$$

The proof is similar with theorem 2, it is omitted here.

3.2 The neutrosophic number hybrid weighted geometric averaging operator

Definition 7. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, and NNWGA : NNS^n→ NNS, if

$$NNWGA(N_1, N_2,..., N_n) = \prod_{i=1}^{n} N_i^{\omega_i} \quad (24)$$

where $\omega = (\omega_1, \omega_2,...,\omega_n)$ is the weight vector of $N_i (i = 1,2,...,n)$ satisfying $\omega_i \in [0,1] \, (i = 1,2,...,n)$ and $\sum_{i=1}^{n}\omega_i = 1$. Then, NNWGA is called neutrosophic number weighted geometric averaging operator. Especially, when $\omega = \left(\frac{1}{n}, \frac{1}{n},..., \frac{1}{n}\right)$, the NNWGA operator will degenerate into neutrosophic number geometric averaging (NNGA) operator.

$$NNWGA(N_1, N_2,..., N_n) = \prod_{i=1}^{n} N_i^{\frac{1}{n}} \quad (25)$$

Theorem 12. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, and $\omega = (\omega_1, \omega_2,...,\omega_n)$ be the weight vector of $N_i (i = 1,2,...,n)$ satisfying $\omega_i \in [0,1] \, (i = 1,2,...,n)$ and $\sum_{i=1}^{n}\omega_i = 1$. Then the result obtained using Eq. (25) is still an NN and

$$NNWGA(N_1, N_2,..., N_n) = \prod_{i=1}^{n} a_i^{\omega_i} + \left(\prod_{i=1}^{n}(a_i + b_i)^{\omega_i} - \prod_{i=1}^{n} a_i^{\omega_i}\right) I \quad (26)$$

The proof of this theorem is similar with theorem 2, it's omitted here.

Theorem 13. (Idempotency).

Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI \ (i = 1, 2,..., n)$, then

$$NNWGA(N_1, N_2,..., N_n) = N_0 .$$

Definition 8. Let $N_i = a_i + b_i I \ (i = 1,2,...,n)$ be a set of NNs, and NNOWGA : NNS^n→ NNS. If

$$NNOWGA(N_1,N_2,...,N_n) = \prod_{i=1}^{n} \tilde{N}_i^{\omega_i} \tag{27}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNOWGA operator satisfying $\omega_i \in [0,1]\ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$; $\tilde{N}_i$ is the ith largest of the $N_i\ (i = 1,2,...,n)$.Then NNOWGA operator is called neutrosophic number ordered weighted geometric averaging operator.

Theorem 14. Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a set of NNs, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNOWGA operator satisfying $\omega_i \in [0,1]\ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$, $\tilde{N}_i = a'_i + b'_i I$ be the ith largest of $N_i\ (i = 1,2,...,n)$.Then, the result obtained using Eq. (27) is still an NN and

$$NNOWGA(N_1, N_2, ..., N_n) = \prod_{i=1}^{n} a_i'^{\omega_i} + \left(\prod_{i=1}^{n} \left(a_i'^{\omega_i} + b_i'^{\omega_i} \right) - \prod_{i=1}^{n} a_i'^{\omega_i} \right) I \tag{28}$$

The proof of this theorem is similar with theorem 2, it's omitted here.

Theorem 15. (Idempotency).

Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI$, then

$$NNOWGA(N_1, N_2, ..., N_n) = N_0 .$$

Theorem 16. (Commutativity).

Let $(N'_1, N'_2, ..., N'_n)$ is any permutation of $(N_1, N_2, ..., N_n)$, then

$$NNOWGA(N'_1,N'_2,...,N'_n) = NNOWGA(N_1,N_2,...,N_n)$$

Definition 9. Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a set of NNs, and NNHWGA : NNS^n→ NNS. If

$$NNHWGA(N_1,N_2,...,N_n) = \prod_{i=1}^{n} \tilde{N}_{\sigma(i)}^{\omega_i} \tag{29}$$

Where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNGHWA operator satisfying $\omega_i \in [0,1]\ (i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$; $\tilde{N}_{\sigma(i)}$ is the ith largest of the $nw_i N_i (i = 1,2,...,n)$,such that $\tilde{N}_{\sigma(i-1)} \geq \tilde{N}_{\sigma(i)}$; $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector of the $N_i\,(i = 1,2,...,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$,

Then, NNHWGA is called neutrosophic number hybrid weighted geometric averaging operator.

Theorem 17. Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a set of NNs, then the result obtained using Eq. (29) can be expressed as

$$NNHWGA(N_1, N_2, ..., N_n) = \prod_{i=1}^{n} a'^{\omega_i}_{\sigma(i)} + \left(\prod_{i=1}^{n} \left(a'^{\omega_i}_{\sigma(i)} + b'^{\omega_i}_{\sigma(i)} \right) - \prod_{i=1}^{n} a'^{\omega_i}_{\sigma(i)} \right) I \tag{30}$$

The proof is similar with the theorem 2, it is omitted here.

It is easy to prove that when $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)$, the NNHWGA operator will reduce to NNOWGA operator, and when $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)$, the NNHWGA operator will reduce to NNWGA operator.

3.3 The neutrosophic number generalized hybrid weighted averaging operator

Definition 10. Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a set of NNs, and NNGWA : NNS^n→ NNS, If

$$NNGWA(N_1, N_2, ..., N_n) = (\sum_{i=1}^{n} \omega_i N_i^{\lambda})^{1/\lambda} \quad (31)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector of $N_i (i = 1,2,...,n)$ satisfying $\omega_i \in [0,1]$ $(i = 1,2,...,n)$ and $\sum_{i=1}^{n} \omega_i = 1$, and $\lambda \in (0, +\infty)$. Then NNGWA is called neutrosophic number generalized weighted averaging operator. Specially, when $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, the NNGWA operator will degenerate into neutrosophic number generalized averaging (NNGA) operator.

$$NNGA(N_1, N_2, ..., N_n) = (\sum_{i=1}^{n} \frac{1}{n} N_i^{\lambda})^{1/\lambda} \quad (32)$$

Theorem 18. Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a collection of NNs, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNGWA operator satisfying $\omega_i \in [0,1]$ $(i = 1,2,...,n)$, $\sum_{i=1}^{n} \omega_i = 1$, and $\lambda \in (0, +\infty)$. Then the result obtained using Eq. (32) is still an NN and

$$NNGWA\ (N_1, N_2, ..., N_n) = (\sum_{i=1}^{n} \omega_i a_i^{\lambda})^{1/\lambda} + \left(\left(\sum_{i=1}^{n} \omega_i (a_i + b_i)^{\lambda}\right)^{1/\lambda} - (\sum_{i=1}^{n} \omega_i a_i^{\lambda})^{1/\lambda}\right) I$$

The proof is similar with the theorem 2, it is omitted here.

Obviously, there are some properties for the NNGWA operator as follows.

(1) When $\lambda \to 0$,

$$NNGWA(N_1, N_2, ..., N_n) = \left(\sum_{i=1}^{n} \omega_i N_i^{\lambda}\right)^{1/\lambda} = \prod_{i=1}^{n} a_i^{\omega_i} + \left(\prod_{i=1}^{n} (a_i + b_i)^{\omega_i} - \prod_{i=1}^{n} a_i^{\omega_i}\right) I = \prod_{i=1}^{n} N_i^{\omega_i},$$

So, the NNGWA operator is reduced to the NNWGA operator.

(2) When $\lambda = 1$,

$$NNGWA(N_1, N_2, ..., N_n) = \left(\sum_{i=1}^{n} \omega_i N_i^{\lambda}\right)^{1/\lambda} = \sum_{i=1}^{n} \omega_i a_i + \sum_{i=1}^{n} \omega_i b_i I = \sum_{i=1}^{n} \omega_i N_i$$

So, the NNGWA operator is reduced to the NNWAA operator.

Therefore, the NNWGA operator and NNWAA operator are two particular cases of the NNGWA operator, and the NNGWA operator is the generalized form of the NNWGA operator and NNWAA operator.

Theorem19. (Idempotency).

Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI$ $(i = 1,2,...,n)$, then

$$NNGWA(N_1, N_2, ..., N_n) = N_0.$$

Definition 11. Let $N_i = a_i + b_i I$ $(i = 1,2,...,n)$ be a set of NNs, and NNGOWA : $NNS^n \to NNS$. If

$$NNGOWA(N_1, N_2, ..., N_n) = (\sum_{i=1}^{n} \omega_i \tilde{N}_i^{\lambda})^{1/\lambda} \quad (33)$$

Where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNGOWA operator satisfying $\omega_i \in [0,1]$ $(i = 1,2,...,n)$, $\sum_{i=1}^{n} \omega_i = 1$ and $\lambda \in (0, +\infty)$; $\tilde{N}_i$ is the ith largest of the

N_i $(i=1,2,...,n)$ .Then NNGOWA is called neutrosophic number generalized ordered weighted averaging operator.

Theorem 20. Let $N_i = a_i + b_i I$ $(i=1,2,...,n)$ be a set of NNs, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNGOWA operator satisfying $\omega_i \in [0,1]$ $(i=1,2,...,n)$, $\sum_{i=1}^{n} \omega_i = 1$ and $\lambda \in (0,+\infty)$, $\tilde{N}_i = a'_i + b'_i I$ be the ith largest N_i $(i=1,2,...,n)$ .Then the result obtained using Eq. (33) is still an NN and

$$NNGOWA\ (N_1, N_2, ..., N_n) = (\sum_{i=1}^{n} \omega_i a_i'^{\lambda})^{1/\lambda} + \left(\left(\sum_{i=1}^{n} \omega_i (a'_i + b'_i)^{\lambda} \right)^{1/\lambda} - (\sum_{i=1}^{n} \omega_i a_i'^{\lambda})^{1/\lambda} \right) I \tag{34}$$

The proof is similar with the theorem 2, it is omitted here.

Obviously, there are some properties for the NNGOWA operator as follows.

(1)When $\lambda \to 0$,

$$NNGOWA(N_1, N_2, ..., N_n) = \left(\sum_{i=1}^{n} \omega_i \tilde{N}_i^{\lambda} \right)^{1/\lambda} = \prod_{i=1}^{n} a_i'^{\omega_i} + \left(\prod_{i=1}^{n} \left(a'_i + b'_i \right)^{\omega_i} - \prod_{i=1}^{n} a_i'^{\omega_i} \right) I = \prod_{i=1}^{n} \tilde{N}_i^{\omega_i} ,$$

So, the NNGOWA operator is reduced to the NNOWGA operator.

(2)When $\lambda = 1$,

$$NNGOWA(N_1, N_2, ..., N_n) = \left(\sum_{i=1}^{n} \omega_i \tilde{N}_i^{\lambda} \right)^{1/\lambda} = \sum_{i=1}^{n} \omega_i a'_i + \sum_{i=1}^{n} \omega_i b'_i I = \sum_{i=1}^{n} \omega_i \tilde{N}_i$$

So, the NNGOWA operator is reduced to the NNOWAA operator.

Therefore, the NNOWGA operator and NNOWAA operator are two particular cases of the NNGOWA operator, and the NNGOWA operator is the generalized form of the NNOWGA operator and NNOWAA operator.

Theorem 21. (Idempotency).

Let $N_i = a_i + b_i I$ $(i=1,2,...,n)$ be a set of NNs, if $N_i = N_0 = a + bI$ $(i=1,2,...,n)$, then

$$NNGOWA(N_1, N_2, ..., N_n) = N_0 .$$

Theorem 22. (Commutativity).

Let $(N'_1, N'_2, ...N'_n)$ is any permutation of $(N_1, N_2, ..., N_n)$, then

$$NNGOWA(N'_1, N'_2, ..., N'_n) = NNGOWA(N_1, N_2, ..., N_n)$$

Definition 12. Let $N_i = a_i + b_i I$ $(i=1,2,...,n)$ be a collection of NNs, and NNGHWA : $NNS^n \rightarrow$ NNS. If

$$NNGHWA(N_1, N_2, ..., N_n) = \left(\sum_{i=1}^{n} \omega_i \tilde{N}_{\sigma(i)}^{\lambda} \right)^{1/\lambda} \tag{35}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector correlative with the NNGHWA operator satisfying $\omega_i \in [0,1]$ $(i=1,2,...,n)$, $\sum_{i=1}^{n} \omega_i = 1$ and $\lambda \in (0,+\infty)$; $\tilde{N}_{\sigma(i)}$ is the ith largest of the $nw_i N_i (i=1,2,...,n)$,such that $\tilde{N}_{\sigma(i-1)} \geq \tilde{N}_{\sigma(i)}$ and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector of the $N_i\ (i=1,2,...,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Then NNGHWA is called neutrosophic number generalized hybrid weighted averaging operator.

Theorem 23. Let $N_i = a_i + b_i I\ (i = 1,2,...,n)$ be a collection of NNs, then the result obtained using Eq. (35) can be expressed as

$$NNGHWA(N_1,N_2,...,N_n) = (\sum_{i=1}^{n}\omega_i a'^{\lambda}_{\sigma(i)})^{1/\lambda} + + \left(\left(\sum_{i=1}^{n}\omega_i (a'_{\sigma(i)} + b'_{\sigma(i)})^{\lambda}\right)^{1/\lambda} - (\sum_{i=1}^{n}\omega_i a'^{\lambda}_{\sigma(i)})^{1/\lambda}\right) I \quad (36)$$

The proof is similar with the theorem 2, it is omitted here.

It is easy to prove that when $w = \left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)$, the NNGHWA operator reduce to the NNGOWA operator, and when $\omega = \left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)$, the NNGHWA operator reduce to the NNGWA operator.

Obviously, there are some properties for the NNGHWA operator as follows.

(1)When $\lambda \to 0$,

$$NNGHWA(N_1,N_2,...,N_n) = \left(\sum_{i=1}^{n}\omega_i \tilde{N}_{\sigma(i)}^{\ \lambda}\right)^{1/\lambda} = \prod_{i=1}^{n} a'^{\omega_i}_{\sigma(i)} + \left(\prod_{i=1}^{n}\left(a'_{\sigma(i)} + b'_{\sigma(i)}\right)^{\omega_i} - \prod_{i=1}^{n} a'^{\omega_i}_{\sigma(i)}\right) I = \prod_{i=1}^{n} \tilde{N}_{\sigma(i)}^{\ \omega_i},$$

So, the NNGHWA operator is reduced to the NNHWGA operator.

(2)When $\lambda = 1$,

$$NNGHWA(N_1,N_2,...,N_n) = \left(\sum_{i=1}^{n}\omega_i \tilde{N}_{\sigma(i)}^{\ \lambda}\right)^{1/\lambda} = \sum_{i=1}^{n}\omega_i a'_{\sigma(i)} + \sum_{i=1}^{n}\omega_i b'_{\sigma(i)} I = \sum_{i=1}^{n}\omega_i \tilde{N}_{\sigma(i)}$$

So, the NNGHWA operator is reduced to the NNHWAA operator.

Therefore, the NNHWGA operator and the NNHWAA operator are two particular cases of the NNGHWA operator, and the NNGHWA operator is the generalized form of the NNHWGA operator and NNHWAA operator.

4. Multiple Attribute Group decision-making method based on Neutrosophic Number Generalized Aggregation Operator

As we all known, the objective things are complex in real decision making, so it is difficult to express people's judgments to some objective things by the crisp numbers. The neutrosophic number is a more suitable and effective tool which is used to express the indeterminate information in decision making problems. The decision makers can evaluate the alternatives with respect to every attribute and give the final evaluation results by the neutrosophic number. Therefore, we show a method for processing group decision making problems with neutrosophic numbers, including a de-neutrosophication process and a possibility degree ranking method for neutrosophic numbers.

In a multiple attribute group decision making problem with neutrosophic numbers, let $A = \{A_1, A_2,..., A_m\}$ be a discrete set of alternatives, $C = \{C_1, C_2,..., C_n\}$ be a set of attributes, and $D = \{D_1, D_2,..., D_s\}$ be a set of decision makers. If the kth $k = (1,2,...,s)$ decision maker provides an evaluation value for the alternative $A_i\ (i = 1,2,...,m)$ under the attribute $C_j\ (j = 1,2,...,n)$ by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I, the evaluation value can be represented by the form of neutrosophic number $N_{ij}^k = a_{ij}^k + b_{ij}^k I$ for $a_{ij}^k, b_{ij}^k \in R$ $(k = 1,2,...,s; j = 1,2,...,n; i = 1,2,...,m)$. Therefore, we can get the kth neutrosophic number decision matrix N^k:

$$N^k = \begin{bmatrix} N_{11}^k & N_{12}^k & \cdots & N_{1n}^k \\ N_{21}^k & N_{22}^k & \cdots & N_{2n}^k \\ \vdots & \vdots & \vdots & \vdots \\ N_{m1}^k & N_{m2}^k & \cdots & N_{mn}^k \end{bmatrix}$$

The weights of attributes symbolize the importance of each attribute C_j $(j = 1,2,...,n)$. The weighting vector of attributes is given by $W = (w_1, w_2,..., w_n)^T$ with $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$. Similar to the attributes, the weights of decision makers symbolize the importance of each decision maker D_k $(k = 1,2,..,s)$. And the weighting vector of decision makers is $V = (v_1, v_2,..., v_s)^T$ with $v_k \geq 0$, $\sum_{k=1}^{s} v_k = 1$.

Then, the steps of the decision making method are described as follows:

Step 1: Utilized the NNGHWA operator

$$N_i^k = a_i^k + b_i^k I = NNGHWA(N_{i1}^k, N_{i2}^k,..., N_{in}^k) \quad (38)$$

to derive the comprehensive values N_i^k $(i = 1,2,..., m; k = 1,2,..., s)$ of each decision maker.

Step 2: Utilized the NNGHWA operator

$$N_i = a_i + b_i I = NNGHWA(N_i^k, N_i^k,..., N_i^k) \quad (39)$$

to derive the collective overall values N_i $(i = 1,2,..., m)$.

Step 3: Calculate the possibility degree $P_{ij} = P(N_i \geq N_j)$ can be given by the Eq.(16)

$$P_{ij} = P(N_i \geq N_j) = \max\left\{1 - \max\left(\frac{(a_j + b_j\beta^+) - (a_i + b_i\beta^-)}{(a_i + b_i\beta^+) - (a_i + b_i\beta^-) + (a_j + b_j\beta^+) - (a_j + b_j\beta^-)}, 0\right), 0\right\}$$

So, the matrix of possibility degrees is structured as $P = (P_{ij})_{m \times m}$.

Step 4: The values of q_i $(i = 1,2,..., m)$ for ranking order are calculated by using Eq.(17)

$$q_i = \frac{\left(\sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1\right)}{n(n-1)}$$

Step 5: The alternatives are ranked according to the values of q_i $(i = 1,2,..., m)$,and then the best one(s) is obtained.

5. A numerical example

In this section, we give a numerical example to demonstrate the multiple attribute group decision making method based on neutrosophic number generalized hybrid weighted averaging operator (which is cited from [34]). An investment company wants to choose a best investment project. There are four possible alternatives : (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company makes a choice according to the following three attributes: (1) C_1 is the risk factor; (2) C_2 is the growth factor; (3) C_3 is the environmental factor. Assume that the weighting vector of the attributes is $W=(0.35,0.25,0.4)^T$. There are three experts $\{D_1, D_2, D_3\}$ who are asked to evaluate the four alternatives in the evaluation process. The weighting vector of three experts is $V=(0.37,0.33,0.3)^T$, the kth (k=1,2,3) expert evaluates the four possible alternatives of A_i $(i=1,2,3,4)$ with respect to the three attributes of $C_j\,(j=1,2,3)$ by the form of neutrosophic number $N_{ij}^k = a_{ij}^k + b_{ij}^k I$ for $a_{ij}^k, b_{ij}^k \in R$, $(k=1,2,...,s; j=1,2,...,n; i=1,2,...,m)$.

Table 1 The evaluation values of four alternatives with respect to the three attributes by the expert D_1

	C1	C2	C3
A1	4+I	5	3+I
A2	6	6	5
A3	3	5+I	6
A4	7	6	4+I

Table 2 The evaluation values of four alternatives with respect to the three attributes by the expert D_2

	C1	C2	C3
A1	5	4	4
A2	5+I	6	6
A3	4	5	5+I
A4	6+I	6	5

Table 3 The evaluation values of four alternatives with respect to the three attributes by the expert D_3

	C1	C2	C3
A1	4	5+I	4
A2	6	7	5+I
A3	4+I	5	6
A4	8	6	4+I

5.1 The evaluation steps of the new MAGDM method based on NNGHWA operator

(1) Calculate the comprehensive evaluation values N_i^k $(i=1,2,3,4; k=1,2,3)$ of each expert D_k by the formula (39) (suppose $\lambda=1$), we can get

$N_1^1 = 3.95 + 0.65I$, $N_2^1 = 5.6$, $N_3^1 = 4.55 + 0.25I$, $N_4^1 = 5.55 + 0.4I$

$N_1^2 = 4.35$, $N_2^2 = 5.6 + 0.4I$, $N_3^2 = 4.6 + 0.35I$, $N_4^2 = 5.6 + 0.35I$

$N_1^3 = 4.35 + 0.35I$, $N_2^3 = 5.95 + 0.4I$, $N_3^3 = 4.95 + 0.4I$, $N_4^3 = 5.9 + 0.4I$

(2) Calculate the collective overall values N_i $(i = 1,2,3,4)$ by the formula (39) (suppose $\lambda = 1$), we can get

$$N_1 = 4.23 + 0.3245I \ , \ N_2 = 5.7295 + 0.28I$$
$$N_3 = 4.7145 + 0.3385I \ , \ N_4 = 5.696 + 0.3835I$$

(3) Calculate the possibility degree $P_{ij} = P(N_i \geq N_j)$ by the formula (17) (suppose $I \in [0,0.5]$).

$$P = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.5000 & 1.0000 & 0.5230 \\ 1.0000 & 0.0000 & 0.5000 & 0.0000 \\ 1.0000 & 0.4770 & 1.0000 & 0.5000 \end{bmatrix}$$

(4) Calculate the values of $q_i \, (i = 1,2,...,m)$ by the formula (18).

$$q_1 = 0.125 \ , \ q_2 = 0.3352 \ , \ q_3 = 0.2083 \ , \ q_4 = 0.3314$$

(5) Rank the four alternatives.

Since $q_2 > q_4 > q_3 > q_1$, the ranking order of the four alternatives $A_2 > A_4 > A_3 > A_1$.

5.2 The influence of the parameter λ and the indeterminate range for I on the ordering of the alternatives

We use the values of parameter λ to express the mentality of the decision makers. The bigger λ is, the more optimistic decision makers are. In this part, in order to verify the influence of the parameter λ on decision making results, the different values λ are used to compute the ordering results. The final ranking results are shown in Table 4.

Table 4 Ordering of the alternatives by utilizing the different λ in NNGHWA operator

λ	q_i	Ranking
$\lambda = 0.1$	$q_1 = 0.1250, q_2 = 0.3560$ $q_3 = 0.2083, q_4 = 0.3107$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 1.0$	$q_1 = 0.1250, q_2 = 0.3352$ $q_3 = 0.2083, q_4 = 0.3314$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 1.1$	$q_1 = 0.1250, q_2 = 0.3327$ $q_3 = 0.2083, q_4 = 0.3340$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 1.2$	$q_1 = 0.1250, q_2 = 0.3300$ $q_3 = 0.2083, q_4 = 0.3366$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 2.0$	$q_1 = 0.1250, q_2 = 0.3062$ $q_3 = 0.2083, q_4 = 0.3605$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 3.0$	$q_1 = 0.1250, q_2 = 0.2917$ $q_3 = 0.2083, q_4 = 0.3750$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 10$	$q_1 = 0.1250, q_2 = 0.2917$ $q_3 = 0.2083, q_4 = 0.3750$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 15$	$q_1 = 0.1250, q_2 = 0.2917$ $q_3 = 0.2083, q_4 = 0.3750$	$A_4 \succ A_2 \succ A_3 \succ A_1$

As we can see from Table 4, the ordering of the alternatives may be different for the different values λ in NNGHWA operator.

(1) When $0 < \lambda \le 1$, the ordering of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$ and the best alternative is A_2.

(2) When $\lambda > 1$, the ordering of the alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$ and the best alternative is A_4.

Similar to the parameter λ, in order to demonstrate the influence of indeterminate range for I on decision making results of this example, we use the different values I in NNGHWA operator to rank the alternatives. The ranking results are shown in Table 5. (suppose $\lambda = 1$)

Table 5 Ordering of the alternatives by different indeterminate ranges for I in NNGHWA operator

I	q_i	Ranking
$I = 0$	/	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.2]$	$q_1 = 0.1250, q_2 = 0.3479$ $q_3 = 0.2083, q_4 = 0.3188$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.4]$	$q_1 = 0.1250, q_2 = 0.3374$ $q_3 = 0.2083, q_4 = 0.3293$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.6]$	$q_1 = 0.1250, q_2 = 0.3327$ $q_3 = 0.2083, q_4 = 0.3328$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I \in [0, 0.8]$	$q_1 = 0.1250, q_2 = 0.3321$ $q_3 = 0.2083, q_4 = 0.3346$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I \in [0,1]$	$q_1 = 0.1250, q_2 = 0.3310$ $q_3 = 0.2083, q_4 = 0.3356$	$A_4 \succ A_2 \succ A_3 \succ A_1$

As we can see from Table 5, the ordering of the alternatives may be different for the different value I in NNGHWA operator.

(1)When $I = 0, I \in [0, 0.2], I \in [0, 0.4]$, the ordering of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$ and the best alternative is A_2.

(2)When $I \in [0, 0.6], I \in [0, 0.8], I \in [0,1]$, the ordering of the alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$ and the best alternative is A_4.

In order to demonstrate the effective of the new method in this paper, we compare the ordering results of the new method with the ordering results of the method proposed by Ye[34]. From the table 6 and the table 5, we can find that the two methods produce the same ranking results.

Table 6 The ordering results produced by the old method (proposed by Ye[34]).

I	q_i	Ranking
$I = 0$	/	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.2]$	$q_1 = 0.1250, q_2 = 0.3368$ $q_3 = 0.2083, q_4 = 0.3298$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.4]$	$q_1 = 0.1250, q_2 = 0.3301$ $q_3 = 0.2083, q_4 = 0.3366$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I \in [0, 0.6]$	$q_1 = 0.1250, q_2 = 0.3279$ $q_3 = 0.2083, q_4 = 0.3388$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I \in [0, 0.8]$	$q_1 = 0.1250, q_2 = 0.3267$ $q_3 = 0.2083, q_4 = 0.3399$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I \in [0,1]$	$q_1 = 0.1250, q_2 = 0.3261$ $q_3 = 0.2083, q_4 = 0.3406$	$A_4 \succ A_2 \succ A_3 \succ A_1$

The method proposed by Ye [34] is based on de-neutrosophication process, it does not realize the importance of the aggregation information. The new proposed in this paper is based on the neutrosophic number general hybrid weighted averaging operators, and it provides the more general and flexible features as I is assigned different values.

6. Conclusions

In this paper, we propose a new multiple attribute group decision making method based on neutrosophic number generalized hybrid weighted averaging (NNGHWA) operator, which is a widely practical tool used to handle indeterminate evaluation information in decision making problems. Furthermore, it also considers the relationship of the decision arguments and reflects the mentality of the decision makers. So, the method can be more appropriate to handle multiple attribute group decision making problems. The decision makers can properly get the desirable alternative according to their interest and the actual need by changing the values of λ, which make the decision making results of the proposed method more flexible and reliable. In order to choose the best alternative, we give the possibility degree ranking method for neutrosophic numbers from the probability viewpoint as a methodological support for the group decision making problems. Lastly, we give a numerical example to demonstrate the practicability of the proposed method. Especially, we use the different values of λ and different indeterminate ranges for I to analyze the effectiveness. In further study, we should study the applications of the above operators. At the same time, we should continue studying other aggregation operators based on the neutrosophic numbers.

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