Manuscript
Click here to download Manuscript: renamed_3fea6.pdf

The Generalized Hybrid Weighted Average Operator Based on Interval Neutrosophic Hesitant Set and Its Application to Multiple Attribute Decision Making

Peide Liu [a,b,]*, Lanlan Shi [a,]

[a,]*School of Management Science and Engineering, Shandong University of Finance and Economics,Jinan Shandong 250014, China*

[b]*School of Economics and Management, Civil Aviation University of China, Tianjin 300300, China*

Abstract: The Neutrosophic set and Hesitant set are the important tools to describe the fuzzy information, in this paper, we combine the interval neutrosophic sets (INSs) and interval valued hesitant fuzzy sets (IVHFSs), and propose the concept of the interval neutrosophic hesitant fuzzy set (INHFS) in order to use the advantages of them. Then we present the operations and comparison method of INHFS, and develop some new aggregation operators for the interval neutrosophic hesitant fuzzy information, including interval neutrosophic hesitant fuzzy generalized weighted (INHFGWA) operator, interval neutrosophic hesitant fuzzy generalized ordered weighted (INHFGOWA) operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted (INHFGHWA) operator, and discuss some properties. Furthermore, we propose the decision making method for multiple attribute group decision making (MAGDM) problems with interval neutrosophic hesitant fuzzy information, and give the detail decision steps. Finally, we give an illustrate example to show the process of decision making and the effectiveness of the proposed method.

Keywords: interval neutrosophic hesitant fuzzy set; neutrosophic set; generalized weighted aggregation (GWA) operator; multiple attribute decision making

1. Introduction

Decision making has the wide application requirements in the business, service, economic, military and the other aspects. But in real life, the decision information is often incomplete, indeterminate and inconsistent, how to express the decision information is the first task of making decision. The fuzzy set (FS) theory proposed by Zadeh [1] is a good tool to process fuzzy information. Since FS was established, it has been attracted wide attentions, and it is extended from two directions.

One direction, FS only has a membership, and it cannot express some complex fuzzy information. For example, in a voting process, there are 10 persons to vote a matter, three of them give the opinion “agree’’, four of them give the opinion “disagree”, and the others give up the voting. Obviously, it is difficult to express the voting information by FS. Based on FS and real applications, Atanassov [2, 3] presented the concept of the intuitionistic fuzzy set (IFS) by adding a non-membership function on the basis of FS, i.e., IFS is with membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$. The example above can be expressed by membership 0.3 and non-membership 0.4. Because membership and non-membership in IFS are crisp numbers, sometimes, it is difficult to use in real decision making problems. Further, Atanassov and Gargov [4], Atanassov [5] proposed the interval-valued intuitionistic fuzzy set (IVIFS) by extending the membership function and non-membership function to interval numbers. Zhang and Liu [6] proposed the triangular intuitionistic fuzzy number by extending the membership degree and the non-membership degree to triangular fuzzy numbers. Wang [7] proposed the intuitionistic trapezoidal fuzzy number. However, IFSs and IVIFSs can only handle incomplete information not the indeterminate information and inconsistent information. In IFSs, the indeterminacy is $1\text{-}T_A(x)\text{-}F_A(x)$ by default. In some complex decision-making environment, IFS has also some limitations. For example, when we ask an expert for the opinion about a statement, he/she may think the right possibility of the statement is 0.5 and the false possibility of the statement is 0.6 and the degree that he or she is not sure is 0.2 [8]. In this case, IFS doesn't process this type of information. In order to solve this class of decision making problems, based on IFS, Smarandache [9] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership function. Obviously, Ns is a generalization of classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, paraconsistent set, dialetheist set, paradoxist set and so on. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. About the research on NS, some achievements have been made. Wang et al. [10, 11] proposed an interval neutrosophic set (INS) and a single valued neutrosophic set (SVNS), which are an instance of a neutrosophic set. Ye [12, 13] proposed the correlation coefficient and the cross-entropy measure of SVNSs, and then applied them to single valued neutrosophic decision-making problems.

The other direction, FS has only one membership, this is a limitation for some decision making problems. Based on FS, Torra and Narukawa [14] and Torra [15] proposed hesitant fuzzy sets (HFSs). As a generalization of fuzzy sets, HFSs use several possible values of an element to replace the membership degree, which is an important tool to represent indefinite information in multiple attribute decision making. Then, Chen et al. [16] proposed interval valued hesitant fuzzy sets (IVHFSs) in which each value is extended to interval numbers. Zhao et al. [17] proposed hesitant triangular fuzzy set and developed some hesitant triangular fuzzy aggregation operators based on the Einstein operation. Meng et al. [18] proposed linguistic hesitant fuzzy sets (LHFSs), and developed some linguistic hesitant fuzzy hybrid aggregation operators. Farhadinia [19] and Ye [20] proposed the dual hesitant fuzzy Sets and dual interval-valued hesitant fuzzy sets. Peng et al. [21] proposed the hesitant interval-valued intuitionistic fuzzy sets (HIVIFSs), and developed some hesitant interval-valued intuitionistic fuzzy number aggregation operators based on t-conorms and t-norms.

As mentioned above, HFS and NS can extend the FS from two directions, the HFS can allow the membership function of an element to a set represented by several possible values, which is a good tool to process uncertain information in real decision making process by hesitant manners; however, it cannot handle indeterminate and inconsistent information, while the NS can easily represent uncertainty, incomplete, and inconsistent information. Obviously, each of them has its strengths and weaknesses. So, combined the IVHFS and INS, we further propose the concept of interval neutrosophic hesitant fuzzy sets (INHFSs), which extend truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of an element to a given set to IVHFS, i.e., they may have a few different interval values. So, the INHFS is generalization of fuzzy set, IFS, IVIFS, NS, INS, HS, IVHFS, and so on. In addition, because the aggregation operators are the important tools to process the fuzzy decision making problems and research on aggregation operators has achieved fruitful results [17, 21-23] . This paper's aim is to propose the concept, score function and comparison method of IVHFS, and then to develop some new generalization aggregation operators, including interval neutrisophic hesitant fuzzy generalized weighted average (INHFGWA) operator, an interval neutrisophic hesitant fuzzy generalized ordered weighted average (INHFGOWA) operator and an interval neutrisophic hesitant fuzzy generalized hybrid weighted average (INHFGHWA) operator, further to develop the decision method for multiple attribute decision making problems under interval neutrosophic hesitant fuzzy environment.

To achieve the above purposes, the remainder of this paper is organized as follows. In the next section, we present some concepts of interval fuzzy numbers, HFSs, IVHFSs, generalized weighted average (GWA) operator, generalized ordered weighted average (GOWA) operator and generalized hybrid weighted average (GHWA) operator. In section 3, we propose the concept and operations of INHFSs. In section 4, we present some generalized aggregation operators based on INHFS, including INHFGWA, INHFGOWA and INHFHWA, and introduce some properties and special cases of them. Section 5 establishes the procedure of the decision-making method based on the INHHWA operators. Section 6 gives a numerical example according to our approach. Section 7 summarizes the main conclusion of this paper.

2.Preliminaries

2.1 The interval fuzzy numbers

Definition 1[24]. Let $\tilde{a} = \left[a^L, a^U\right] = \left\{x \middle| a^L \le x \le a^U\right\}$, then $\tilde{a}$ is called an interval fuzzy number.

If $0 \le a^L \le a^R$, then $\tilde{a}$ is called a positive interval fuzzy number.

Definition 2 [24]. Let $\tilde{a} = \left[a^L, a^U\right]$ and $\tilde{b} = \left[b^L, b^U\right]$ be two interval numbers and $l_{\tilde{a}} = a^U - a^L$ and $l_{\tilde{b}} = b^U - b^L$, then the degree of possibility of $\tilde{a} > \tilde{b}$ is formulated by

$$p\ (\tilde{a} > \tilde{b}) = \max\left[1 - \max\left(\frac{b^U - a^L}{l_{\tilde{a}} + l_{\tilde{b}}}, 0\right), 0\right]. \tag{1}$$

We can use the degrees of possibility to compare with the interval numbers.

2.2. The interval neutrosophic set

Definition 3 [10]. Let X be a universe of discourse, with a generic element in X denoted by x .A interval valued neutrosophic set A in X is

$$A = \left\{x(T_A(x), I_A(x), F_A(x)) \middle| x \in X\right\} \tag{2}$$

Where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership, indeterminacy-membership and falsity-membership function, separately. For each point x in X , we have that $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0,1]$, and $0 \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3$.

Definition 4 [10]. If and only if its $\inf T_A(x) = \sup T_A(x) = 0$, $\inf I_A(x) = \sup I_A(x) = 1$, and $\inf F_A(x) = \sup F_A(x) = 0$ for all x in X , we can say the INS A is null.

Definition 5 [10]. A^c denote the complement of an INS A and written as $A^c = \left\{x(T_A^c(x), I_A^c(x), F_A^c(x)) \middle| x \in X\right\}$,i.e. $T_A{}^c(x) = F_A(x)$, $\inf I_A{}^c(x) = 1 - \sup I_A(x)$, $\sup I_A{}^c(x)$

$= 1 - \inf I_A(x)$, $F_A^{\ c}(x) = T_A(x)$ for all x in X.

Definition 6 [10]. If and only if $\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$, $\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$, $\inf F_A(x) \ge \inf F_B(x)$, and $\sup F_A(x) \ge \sup F_B(x)$ for all x in X, we can say an INS A is contained in the INS B and is denoted by A $\subseteq$ B.

Definition 7 [10]. If and only if A $\subseteq$ B and B $\subseteq$ A, we can say two INSs A and B are identical, and written as A = B.

2. 3 Some concepts of HFSs and IVHFSs

Definition 8 [14, 15]. Let X be a no empty finite set, a HFS A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of $[0,1]$, we can express HFSs by:

$$A = \{\langle x, h_A(x)\rangle | x \in X\}, \tag{3}$$

Where $h_A(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to A. we call $h_A(x)$ a hesitant fuzzy element(HFE), denoted by h, which reads $h = \{\gamma | \gamma \in h\}$.

For three hesitant fuzzy elements h, h_1 and h_2, Torra [15] defined three basic operations shown as follows.

(1) $$h^c = \bigcup_{\gamma \in h} \{1-\gamma\}, \tag{4}$$

(2) $$h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}, \tag{5}$$

(3) $$h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}. \tag{6}$$

After that, Xia and Xu [25] defined four operations on the HFEs h, h_1, h_2 with a positive scale n:

(1) $$h^n = \bigcup_{\gamma \in h} \{\gamma^n\}, \tag{7}$$

(2) $$nh = \bigcup_{\gamma \in h} \{1-(1-\gamma)^n\}, \tag{8}$$

(3) $$h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \tag{9}$$

(4) $$h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}. \tag{10}$$

Definition 9 [16, 24] Let X be a no empty finite set, an interval valued hesitant fuzzy set (IVHFS) on X is represented by:

$$E = \{\langle x, \tilde{h}_E(x)\rangle | x \in X\} \tag{11}$$

where $\tilde{h}_E(x)$ is a set of some different interval values in $[0,1]$, which denotes the possible membership degrees of the element $x \in X$ to the set E, $\tilde{h}_E(x)$ can be represented by an IVHFE $\tilde{h}$, which reads $\tilde{h} = \{\tilde{\gamma} | \tilde{\gamma} \in \tilde{h}\}$, where $\tilde{\gamma} = [\gamma^L, \gamma^U]$ is an interval number.

Chen et al. [24] gave the operations for three IVHFEs $\tilde{h}, \tilde{h}_1, \tilde{h}_2$ with a positive scale n shown as follows:

(1) $\tilde{h}^n = \bigcup_{\tilde{\gamma} \in \tilde{h}} \{[(\gamma^L)^n, (\gamma^U)^n]\}$; (12)

(2) $n\tilde{h} = \bigcup_{\tilde{\gamma} \in \tilde{h}} \{[1-(1-\gamma^L)^n, 1-(1-\gamma^U)^n]\}$; (13)

(3) $\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \{[\gamma_1^L + \gamma_2^L - \gamma_1^L \gamma_2^L, \gamma_1^U + \gamma_2^U - \gamma_1^U \gamma_2^U]\}$; (14)

(4) $\tilde{h}_1 \otimes \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2} \{[\gamma_1^L \gamma_2^L, \gamma_1^U \gamma_2^U]\}$. (15)

2.4 Some aggregation operators

The generalized weighted average (GWA) operator is a generalization of the weighted average operator, which is defined as follows:

Definition 10 [22]. Let $\mathrm{GWA}: R^n \to R$, if

$$\mathrm{GWA}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} w_j a_j^{\lambda} \right)^{1/\lambda} \tag{16}$$

where $w = (w_1, w_2, \cdots w_n)^T$ is the weighting vector of the $(a_1, a_2, \cdots, a_n)$ such that $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \bigcup (0, +\infty)$, then GWA is called the generalized weighted average (GWA) operator.

Definition 11 [22].Let $GOWA: R^n \to R$, if

$$GOWA(a_1, a_2, \cdots a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda} \right)^{1/\lambda} \tag{17}$$

where b_j is the jth largest of the $(a_1, a_2, \cdots, a_n)$ and $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$ is the associated weighting vector such that $\omega_j \in [0,1]$, $j = 1,2,\cdots,n$ $\sum_{j=1}^{n} \omega_j = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \bigcup (0, +\infty)$, then $GOWA$ is called the generalized ordered weighted average (GOWA) operator.

Definition 12 [23]. Let $GHWA: R^n \to R$, if

$$GHWA\left(a_1, a_2, \cdots a_n\right) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda}\right)^{1/\lambda} \tag{18}$$

where b_j is the jth largest of the weighted arguments $nw_i a_i\left(i = 1,2,\cdots,n\right)$, $w = (w_1, w_2, \cdots w_n)^T$ is the weighting vector of the a_i $(i = 1,2,\cdots,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$, λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$ and $\omega = \left(\omega_1, \ \omega_2, \cdots \omega_n\right)^T$ is the aggregation-associated vector such that $\omega_j \in [0,1]$, $j = 1,2,\cdots,n$, $\sum_{j=1}^{n} \omega_j = 1$. then $GHWA$ is called the generalized hybrid weighted average (GHWA) operator.

3.The interval neutrosophic hesitant fuzzy set

In this section, we will present the concept of interval neutrosophic hesitant fuzzy set based on the combination of interval neutrosophic set and interval valued hesitant fuzzy set.

Definition 13. Let X be a no empty finite set, an interval neutrosophic hesitant fuzzy set (INHFS) on X is represented by:

$$N = \left\{\left\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\right\rangle \middle| x \in X\right\} \tag{19}$$

Where $\tilde{t}(x) = \left\{\tilde{\gamma} \middle| \tilde{\gamma} \in \tilde{t}(x)\right\}$, $\tilde{i}(x) = \left\{\tilde{\delta} \middle| \tilde{\delta} \in \tilde{i}(x)\right\}$, and $\tilde{f}(x) = \left\{\tilde{\eta} \middle| \tilde{\eta} \in \tilde{f}(x)\right\}$ are three sets of some interval values in real unit interval $[0,1]$, which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x \in X$ to the set N, and satisfies these limits : $\tilde{\gamma} = \left[\gamma^L, \gamma^U\right] \subseteq [0,1]$, $\tilde{\delta} = \left[\delta^L, \delta^U\right] \subseteq [0,1]$, $\tilde{\eta} = \left[\eta^L, \eta^U\right] \subseteq [0,1]$ and $0 \le \sup\tilde{\gamma}^+ + \sup\tilde{\delta}^+ + \sup\tilde{\eta}^+ \le 3$, where $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in \tilde{t}(x)} \max\left\{\tilde{\gamma}\right\}$, $\tilde{\delta}^+ = \bigcup_{\tilde{\delta} \in \tilde{i}(x)} \max\left\{\tilde{\delta}\right\}$, and $\tilde{\eta}^+ = \bigcup_{\tilde{\eta} \in \tilde{f}(x)} \max\left\{\tilde{\eta}\right\}$ for $x \in X$.

The $\tilde{n} = \left\{\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\right\}$ is called an interval neutrosophic hesitant fuzzy element (INHFE) which is the basic unit of the INHFS and be represented by the symbol $\tilde{n} = \left\{\tilde{t}, \tilde{i}, \tilde{f}\right\}$.

Thus, we can regard fuzzy sets, IFSs, IVIFSs, SVNSs, INFs, HFSs, DHFSs, and IVHFSs, as special cases of INHFSs from the definition 13.

Then, we can define the basic operations of INHFEs as follows:

Definition 14. Let $\tilde{n}_1 = \left\{\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\right\}$ and $\tilde{n}_2 = \left\{\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\right\}$ be two INHFEs in a no empty finite set X, then

(1) $\tilde{n}_1 \cup \tilde{n}_2 = \left\{\tilde{t}_1 \cup \tilde{t}_2, \tilde{i}_1 \cap \tilde{i}_2, \tilde{f}_1 \cap \tilde{f}_2\right\}$; (20)

(2) $\tilde{n}_1 \cap \tilde{n}_2 = \left\{\tilde{t}_1 \cap \tilde{t}_2, \tilde{i}_1 \cup \tilde{i}_2, \tilde{f}_1 \cap \tilde{f}_2\right\}$ (21)

Therefore, for two INHFEs $\tilde{n}_1, \tilde{n}_2$ and a positive scale $k > 0$, these operations can be denoted as follows:

(1) $\tilde{n}_1 \oplus \tilde{n}_2 = \left\{\tilde{t}_1 \oplus \tilde{t}_2, \tilde{i}_1 \otimes \tilde{i}_2, \tilde{f}_1 \otimes \tilde{f}_2\right\}$; (22)

$$= \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{i}_2, \tilde{\eta}_2 \in \tilde{f}_2} \left\{\left[\gamma_1^L + \gamma_2^L - \gamma_1^L \gamma_2^L, \gamma_1^U + \gamma_2^U - \gamma_1^U \gamma_2^U\right], \left[\delta_1^L \delta_2^L, \delta_1^U \delta_2^U\right], \left[\eta_1^L \eta_2^L, \eta_1^U \eta_2^U\right]\right\}$$

(2) $\tilde{n}_1 \otimes \tilde{n}_2 = \left\{\tilde{t}_1 \otimes \tilde{t}_2, \tilde{i}_1 \oplus \tilde{i}_2, \tilde{f}_1 \oplus \tilde{f}_2\right\}; = \bigcup_{\tilde{\gamma}_2 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{i}_2, \tilde{\eta}_2 \in \tilde{f}_2}$

$$\left\{\left[\gamma_1^L \gamma_2^L, \gamma_1^U \gamma_2^U\right], \left[\delta_1^L + \delta_2^L - \delta_1^L \delta_2^L, \delta_1^U + \delta_2^U - \delta_1^U \delta_2^U\right], \left[\eta_1^L + \eta_2^L - \eta_1^L \eta_2^L, \eta_1^U + \eta_2^U - \eta_1^U \eta_2^U\right]\right\} \quad (23)$$

(3) $k\tilde{n}_1 = \bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1} \left\{\left[1-(1-\gamma_1^L)^k, 1-(1-\gamma_1^U)^k\right], \left[(\delta_1^L)^k, (\delta_1^U)^k\right], \left[(\eta_1^L)^k, (\eta_1^U)^k\right]\right\}$ (24)

(4) $\tilde{n}_1^k =$

$$\bigcup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{i}_1, \tilde{\eta}_1 \in \tilde{f}_1} \left\{\left[(\gamma_1^L)^k, (\gamma_1^U)^k\right]\left[1-(1-\delta_1^L)^k, 1-(1-\delta_1^U)^k\right], \left[1-(1-\eta_1^L)^k, 1-(1-\eta_1^U)^k\right]\right\} \quad (25)$$

Definition 15. For an INHFE $\tilde{n}$,

$$s(\tilde{n}) = \left[\frac{1}{l}\sum_{i=1}^{l} \tilde{\gamma}_i + \left(\frac{\sum_{i=1}^{p}\left(1-\tilde{\delta}_i\right)}{p}\right) + \left(\frac{\sum_{i=1}^{q}\left(1-\tilde{\eta}_i\right)}{q}\right)\right] \Big/ 3 \quad (26)$$

is called the score function of $\tilde{n}$. where l, p, q being the number of the interval values in $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{\eta}$, respectively. Obviously, $s(\tilde{n})$ is an interval value belong to $[0,1]$. For two INHFE $\tilde{n}_1$ and $\tilde{n}_2$, because $s(\tilde{n}_1)$ and $s(\tilde{n}_2)$ are interval numbers, they can be compared by the degrees of possibility defined in definition 1. If $s(\tilde{n}_1) > s(\tilde{n}_2)$, then $\tilde{n}_1 > \tilde{n}_2$.

4. Some aggregation operators based on interval neutrisophic hesitant fuzzy numbers

Since the GWA, GOWA and GHWA operators can only aggregate the crisp numbers, and cannot aggregate the interval neutrisophic hesitant fuzzy information. In this section, we will extend the GWA, GOWA and GHWA operators to aggregate interval neutrisophic hesitant fuzzy information, and propose an interval neutrisophic hesitant fuzzy generalized weighted average (INHFGWA) operator, an interval neutrisophic hesitant fuzzy generalized ordered weighted average (INHFGOWA) operator and an interval neutrisophic hesitant fuzzy generalized hybrid weighted average (INHFGHWA) operator described as follows.

Definition 16. Let $\lambda > 0$ and $\tilde{n}_j = \left(\tilde{t}_j, \tilde{i}_j, \tilde{f}_j\right)\left(j = 1, 2, \cdots, n\right)$ be a collection of interval neutrisophic hesitant fuzzy numbers with the weight vector $w = \left(w_1, w_2, \cdots, w_n\right)^T$ such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, then an interval neutrisophic hesitant fuzzy generalized weighted average (INHFGWA) operator of dimension n is a mapping INHFGWA: $\Omega^n \rightarrow \Omega$, and has

$$\text{INHFGWA}\left(\tilde{n}_1, \tilde{n}_2 \cdots, \tilde{n}_n\right) = \left(\sum_{j=1}^{n} w_j \tilde{n}_j^{\lambda}\right)^{1/\lambda} \tag{27}$$

where Ω is the set of all the interval neutrisophic hesitant fuzzy numbers.

Based on the operational rules of the interval neutrisophic hesitant fuzzy numbers, we have the following theorems.

Theorem 1. Let $\lambda > 0$ and $\tilde{n}_i = \left(\left[\gamma_i^L, \gamma_i^U\right], \left[\delta_i^L, \delta_i^U\right], \left[\eta_i^L, \eta_i^U\right]\right)\left(i = 1, 2, \cdots, n\right)$ be the collection of INHEs, then the result aggregated from Definition 16 is still an INHEs, and even

$$\begin{aligned}\text{INHFGWA}\left(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n\right) = &\left\{\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right],\right.\\
&\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right],\\
&\left.\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right]\right\}\end{aligned} \tag{28}$$

Proof.

(1) We firstly prove that

$$\begin{aligned}\sum_{j=1}^{n} w_j \tilde{n}_j^{\lambda} = &\left\{\left[1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right],\right.\\
&\left[\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j}, \prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right],\\
&\left.\left[\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j}, \prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right]\right\}\end{aligned} \tag{29}$$

(29) can be proved by Mathematical induction on n as follows

(i) When $n=1$,

According to the operational rules of INHFs, we have

$$\tilde{n}_1^{\lambda}=\bigcup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{i}_1,\tilde{\eta}_1\in\tilde{f}_1}\left\{\left[(\gamma_1^L)^{\lambda},(\gamma_1^U)^{\lambda}\right],\left[1-(1-\delta_1^L)^{\lambda},1-(1-\delta_1^U)^{\lambda}\right],\left[1-(1-\eta_1^L)^{\lambda},1-(1-\eta_1^U)^{\lambda}\right]\right\}$$

$$w_1\tilde{n}_1^{\lambda}=\bigcup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{i}_1,\tilde{\eta}_1\in\tilde{f}_1}\left\{\begin{array}{l}\left[1-\left(1-\left(\gamma_1^L\right)^{\lambda}\right)^{w_1},1-\left(1-\left(\gamma_1^U\right)^{\lambda}\right)^{w_1}\right],\left[\left(1-\left(1\text{-}\delta_1^L\right)^{\lambda}\right)^{w_1},\left(1-\left(1\text{-}\delta_1^U\right)^{\lambda}\right)^{w_1}\right],\\ \left[\left(1-\left(1\text{-}\eta_1^L\right)^{\lambda}\right)^{w_1},\left(1-\left(1\text{-}\eta_1^U\right)^{\lambda}\right)^{w_1}\right]\end{array}\right\}$$

So, when n=1, (29) is right.

(ii) Suppose when $n=k$, (29) is right, i.e.,

$$\sum_{j=1}^{k}w_j\tilde{n}_j^{\lambda}=\left\{\left[1-\prod_{j=1}^{k}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j},1-\prod_{j=1}^{k}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right],\right.$$

$$\left.\left[\prod_{j=1}^{k}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right],\left[\prod_{j=1}^{k}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right]\right\}$$

Then when $n=k+1$, we have

$$w_{k+1}\tilde{n}_{k+1}^{\lambda}=$$

$$\bigcup_{\tilde{\gamma}_{k+1}\in\tilde{t}_{k+1},\tilde{\delta}_{k+1}\in\tilde{i}_{k+1},\tilde{\eta}_{k+1}\in\tilde{f}_{k+1}}\left\{\begin{array}{l}\left[1-\left(1-\left(\gamma_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},1-\left(1-\left(\gamma_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right],\left[\left(1-\left(1\text{-}\delta_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},\left(1-\left(1\text{-}\delta_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right],\\ \left[\left(1-\left(1\text{-}\eta_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},\left(1-\left(1\text{-}\eta_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right]\end{array}\right\}$$

and

$$\sum_{j=1}^{k+1}w_j\tilde{n}_j^{\lambda}=\sum_{j=1}^{k}w_j\tilde{n}_j^{\lambda}\oplus w_{k+1}\tilde{n}_{k+1}^{\lambda}$$

$$=\left\{\left[1-\prod_{j=1}^{k}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j},1-\prod_{j=1}^{k}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right],\right.$$

$$\left.\left[\prod_{j=1}^{k}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right],\left[\prod_{j=1}^{k}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right]\right\}$$

$$\oplus$$

$$\bigcup_{\tilde{\gamma}_{k+1}\in\tilde{t}_{k+1},\tilde{\delta}_{k+1}\in\tilde{i}_{k+1},\tilde{\eta}_{k+1}\in\tilde{f}_{k+1}}\left\{\begin{array}{l}\left[1-\left(1-\left(\gamma_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},1-\left(1-\left(\gamma_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right],\left[\left(1-\left(1\text{-}\delta_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},\left(1-\left(1\text{-}\delta_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right],\\ \left[\left(1-\left(1\text{-}\eta_{k+1}^L\right)^{\lambda}\right)^{w_{k+1}},\left(1-\left(1\text{-}\eta_{k+1}^U\right)^{\lambda}\right)^{w_{k+1}}\right]\end{array}\right\}$$

$$=\left\{\left[1-\prod_{j=1}^{k+1}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j},1-\prod_{j=1}^{k+1}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right],\right.$$

$$\left.\left[\prod_{j=1}^{k+1}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k+1}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right],\left[\prod_{j=1}^{k+1}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j},\prod_{j=1}^{k+1}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right]\right\}$$

So, when $n=k+1$, (29) is also right.

According to (i) and (ii), we can get when (29) is right for all n.

(2) According step (1), we have

$$\mathrm{INHFGWA}\left(\tilde{n}_1,\tilde{n}_2\cdots,\tilde{n}_n\right)=\left(\sum_{j=1}^{n}w_j\tilde{n}_j^{\lambda}\right)^{1/\lambda}$$

$$=\left\{\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right],\right.$$

$$\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right],$$

$$\left.\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right]\right\}.$$

which completes the proof Theorem 1..

Moreover, the INHFGWA operator has the following properties.

(1) Theorem 2 (Idempotency).

Let $\tilde{n}_j=\tilde{n}=\left\{\tilde{t},\tilde{i},\tilde{f}\right\}$,where $\tilde{t}=\left[\gamma^L,\gamma^U\right]$, $\tilde{i}=\left[\delta^L,\delta^U\right]$ and $\tilde{f}=\left[\eta^L,\eta^U\right]$,

then

$$\mathrm{INHFGWA}\left(\tilde{n}_1,\tilde{n}_2\cdots,\tilde{n}_n\right)=\tilde{n}\,. \tag{30}$$

Proof. Since $\tilde{n}_j=\tilde{n}=\left\{\left[\gamma^L,\gamma^U\right],\left[\delta^L,\delta^U\right],\left[\eta^L,\eta^U\right]\right\}$, for all j, we have

$$\mathrm{INHFGWA}(\tilde{n}_1,\tilde{n}_2,...\tilde{n}_n)=\left(\sum_{j=1}^{n}w_j\tilde{n}_j^{\lambda}\right)^{1/\lambda}=\left(\sum_{j=1}^{n}w_j\tilde{n}^{\lambda}\right)^{1/\lambda}$$

$$=\left\{\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\delta^{L}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\delta^{U}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right]\right.$$

$$\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{L}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{U}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right],$$

$$\left.\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{L}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{U}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right]\right\}$$

$$=\left\{\left[\left(1-\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma^{L}\right)^{\lambda}\right)^{\sum_{j=1}^{n}w_j}\right)^{1/\lambda},\left(1-\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma^{U}\right)^{\lambda}\right)^{\sum_{j=1}^{n}w_j}\right)^{1/\lambda}\right],\right.$$

$$\left[1-\left(1-\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{L}\right)^{\lambda}\right)^{\sum_{j=1}^{N}w_j}\right)^{1/\lambda},1-\left(1-\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{U}\right)^{\lambda}\right)^{\sum_{j=1}^{N}w_j}\right)^{1/\lambda}\right],$$

$$\left.\left[1-\left(1-\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{L}\right)^{\lambda}\right)^{\sum_{j=1}^{N}w_j}\right)^{1/\lambda},1-\left(1-\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{U}\right)^{\lambda}\right)^{\sum_{j=1}^{N}w_j}\right)^{1/\lambda}\right]\right\}$$

$$=\left\{\left[\left(1-\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma^{L}\right)^{\lambda}\right)\right)^{1/\lambda},\left(1-\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma^{U}\right)^{\lambda}\right)\right)^{1/\lambda}\right],\right.$$

$$\left[1-\left(1-\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{L}\right)^{\lambda}\right)\right)^{1/\lambda},1-\left(1-\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta^{U}\right)^{\lambda}\right)\right)^{1/\lambda}\right],$$

$$\left.\left[1-\left(1-\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{L}\right)^{\lambda}\right)\right)^{1/\lambda},1-\left(1-\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta^{U}\right)^{\lambda}\right)\right)^{1/\lambda}\right]\right\}$$

$=\tilde{n}$.

(2) Theorem 3 (monotonicity)

Let $\tilde{n}_j\left(j=1,2,\cdots,n\right)$ and $\tilde{n}'_j\left(j=1,2,\cdots,n\right)$ be two sets of INHFS, and $w=\left(w_1,w_2,\cdots w_n,\right)^T$ be the weighting vector of $\tilde{n}_j,\tilde{n}'_j\left(j=1,2,\cdots,n\right)$ which satisfying $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$,if $\tilde{n}_j\geq\tilde{n}'_j$ for all j ,then

$$\text{INHFGWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right)\geq\text{INHFGWA}\left(\tilde{n}'_1,\tilde{n}'_2,\cdots,\tilde{n}'_n\right)\tag{31}$$

Proof.

Because $\tilde{n}_j \geq \tilde{n}'_j$ for all j, we may suppose $\gamma_j^L \geq \gamma'^L_j$, $\gamma_j^U \geq \gamma'^U_j$, $\delta_j^L \leq \delta'^L_j$, $\delta_j^U \leq \delta'^U_j$, $\eta_j^L \leq \eta'^L_j$ and $\eta_j^U \leq \eta'^U_j$ for all j.

As the aggregated result of INHFGWA$\left(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n\right)$ operator is still an INHEs, and it has three parts, i.e., truth-membership, indeterminacy-membership and falsity-membership. We can prove them separately. The proof is shown as follows.

(i) We firstly prove the truth-membership part.

As $\gamma_j^L \geq \gamma'^L_j$ and $\lambda > 0, w_j \geq 0$ for all j, then

$$\left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j} \leq \left(1-\left(\gamma'^L_j\right)^{\lambda}\right)^{w_j}$$

And

$$\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j} \leq \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma'^L_j\right)^{\lambda}\right)^{w_j}$$

So we have

$$\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda} \geq \left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma'^L_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

Similarly, we have

$$\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda} \geq \left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1-\left(\gamma'^U_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

(ii) We prove the indeterminacy-membership part.

As $\delta_j^L \leq \delta'^L_j$ and $\lambda > 0, w_j \geq 0$ for all j, then

$$\left(1\text{-}\delta_j^L\right)^{\lambda} \geq \left(1\text{-}\delta'^L_j\right)^{\lambda}$$

$$\left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j} \leq \left(1-\left(1\text{-}\delta'^L_j\right)^{\lambda}\right)^{w_j}$$

$$\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda} \geq \left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta'^L_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

So,

$$1-\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda} \leq 1-\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta'^L_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

Similarly, we have

$$1-\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda} \leq 1-\left(1-\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1-\left(1\text{-}\delta'^U_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

(iii) We prove the falsity-membership part.

Similar to the indeterminacy-membership part, we have

$$1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\eta_j^L\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\leq 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\eta_j'^{L}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

Similarly, we have

$$1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\eta_j^U\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\leq 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1\text{-}\eta_j'^{U}\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}$$

According to (i) and (ii), we can get

$$\text{INHFGWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right)\geq \text{INHFGWA}\left(\tilde{n}_1',\tilde{n}_2',\cdots,\tilde{n}_n'\right)$$

(3) Theorem 4 (Boundedness)

The INHFGWA operator lies between the max and min operators:

$$\min(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n)\leq INHFGWA\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)\leq \max\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right) \tag{32}$$

Proof.

Let $m=\min(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n)$, $M=\max\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)$,

Since $m\leq\tilde{n}_j\leq M$,we can get the result as follows according to theorem 3.

$$\left(\sum_{j=1}^{n}w_j m^{\lambda}\right)^{1/\lambda}\leq\left(\sum_{j=1}^{n}w_j\tilde{n}_j^{\ \lambda}\right)^{1/\lambda}\leq\left(\sum_{j=1}^{n}w_j M^{\lambda}\right)^{1/\lambda}$$

that is

$$m\leq\left(\sum_{j=1}^{n}w_j\tilde{n}_j^{\ \lambda}\right)^{1/\lambda}\leq M$$

i.e. $\min(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n)\leq INHFGWA\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)\leq \max\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)$.

Now we can consider some special cases of INHFGWA operator.

(1) If $\lambda=1$, then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted averaging (INHFWA) operator

$$\text{INHFWA}(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n)=\sum_{j=1}^{n}w_j\tilde{n}_j$$

$$=\left\{\begin{array}{l}\left[1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\gamma_j^L\right)^{w_j},1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\gamma_j^L\right)^{w_j}\right],\left[\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{t}_j}\left(\delta_j^L\right)^{w_j},\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{t}_j}\left(\delta_j^U\right)^{w_j}\right],\\ \left[\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(\eta_j^L\right)^{w_j},\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(\eta_j^U\right)^{w_j}\right]\end{array}\right\} \tag{33}$$

(2) If $\lambda \to 0$,then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted geometric (INHFWG) operator:

$$\text{INHFWG}(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \prod_{j=1}^{n} (\tilde{n}_j)^{w_j}$$

$$= \left\{ \left[\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} (\gamma_j^L)^{w_j}, \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} (\gamma_j^U)^{w_j} \right], \left[1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} (1-\delta_j^L)^{w_j}, 1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} (1-\delta_j^U)^{w_j} \right], \right. \\ \left. \left[1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} (1-\eta_j^L)^{w_j}, 1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} (1-\eta_j^U)^{w_j} \right] \right\} \quad (34)$$

(3)If $\lambda = 2$,then INHFGWA operator becomes the interval neutrosophic hesitant fuzzy weighted quadratic averaging (INHFWQA)operator:

$$\text{INHFWQA}(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \left(\sum_{j=1}^{n} w_j \tilde{n}_j^2 \right)^{1/2}$$

$$= \left\{ \left[\left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \left(\gamma_j^L \right)^2 \right)^{w_j} \right)^{1/2}, \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \left(\gamma_j^U \right)^2 \right)^{w_j} \right)^{1/2} \right] \right. \\ \left[1 - \left(1 - \prod_{j-1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - \left(1 - \delta_j^L \right)^2 \right)^{w_j} \right)^{1/2}, 1 - \left(1 - \prod_{j-1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - \left(1 - \delta_j^U \right)^2 \right)^{w_j} \right)^{1/2} \right], \quad (35) \\ \left. \left[1 - \left(1 - \prod_{j-1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - \left(1 - \eta_j^U \right)^2 \right)^{w_j} \right)^{1/2}, 1 - \left(1 - \prod_{j-1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - \left(1 - \eta_j^U \right)^2 \right)^{w_j} \right)^{1/2} \right] \right\}$$

In definition 16, we considered the weight vector of attributes. However, in some cases, we may need to consider the position of aggregated data. For example, in the diving contest of Olympic Games, in general, after removing the most high and low scores, we can take the average value of the remaining. i.e., we can assign the weights of the most high and low scores are 0. So, the positional weights are very important in some real decision making problems. thus, we further define a new aggregation operator to process this case.

Definition 17. An interval neutrosophic hesitant fuzzy generalized ordered weighted averaging (INHFGOWA) operator of dimension n is a mapping I : $\Omega^n \to \Omega$,defined by an associated weighting vector $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$ such that $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$ and parameter $\lambda \in (0, +\infty)$, so,

$$\text{INHFGOWA}\left(\tilde{n}_1,\tilde{n}_2\cdots,\tilde{n}_n\right)=\left(\sum_{j=1}^{n}\omega_j\tilde{n}_{\sigma(j)}^{\lambda}\right)^{1/\lambda}. \tag{36}$$

where, the $\tilde{n}_{\sigma(j)}$ is the jth largest of the $\tilde{n}_i$ $(i=1,2,\cdots,\ n)$, $(\sigma(1),\ \sigma(2),\cdots,\ \sigma(n))$ is a permutation of $(1,2,\cdots,\ n)$,such that $\tilde{n}_{\sigma(j-1)}\geq\tilde{n}_{\sigma(j)}$ for all $j=2,\cdots,n$,and $\tilde{n}_j=\left(\tilde{t}_j,\tilde{i}_j,\tilde{f}_j\right)(j=1,2,\cdots,n)$ are interval neutrosophic hesitant fuzzy elements (INHFE).

Theorem 5. Let $\tilde{n}_j=\left(\left[\gamma_j^L,\gamma_j^U\right],\left[\delta_j^L,\delta_j^U\right],\left[\eta_j^L,\eta_j^U\right]\right)$ $(j=1,2,\cdots,n)$ be the collection of INHEs, $\tilde{n}_{\sigma(j)}$ is the jth largest of the $\tilde{n}_i$ $(i=1,2,\cdots,\ n)$ then the result aggregated from Definition 17 is still an INHEs, and even

$$\begin{aligned}\left(\sum_{j=1}^{n}\omega_j\tilde{n}_{\sigma(j)}^{\lambda}\right)^{1/\lambda}=\Bigg\{&\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_{\sigma(j)}^{L}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda},\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\gamma_{\sigma(j)}^{U}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right]\\&\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta_{\sigma(j)}^{L}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\delta_{\sigma(j)}^{U}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right],\\&\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta_{\sigma(j)}^{U}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\eta_{\sigma(j)}^{U}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right]\Bigg\}\end{aligned} \tag{37}$$

The proof of this theorem is similar with theorem 1, it's omitted here.

Similar to Theorems 2-4, it is easy to prove the INHFGOWA operator has the following properties.

(1) Theorem 6 (Idempotency).

If $\tilde{n}_j=\tilde{n}=\left\{\tilde{t},\tilde{i},\tilde{f}\right\}$ for all j, then INHFGOWA ($\tilde{n}_1\ \tilde{n}_2,..\tilde{n}_n$)= $\tilde{n}$.

(2) Theorem 7 (monotonicity)

Let $\tilde{n}_j(j=1,2,\cdots,n)$ and $\tilde{n}'_j(j=1,2,\cdots,n)$ be two sets of INHFS,if $\tilde{n}_j\geq\tilde{n}'_j$,then

$$\text{INHFGOWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right)\geq\text{INHFGOWA}\left(\tilde{n}'_1,\tilde{n}'_2,\cdots,\tilde{n}'_n\right)$$

(3) Theorem 8 (Boundedness)

The INHFGOWA operator lies between the max and min operators:

$$\min(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n)\leq INHFGOWA\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)\leq\max\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right).$$

(4) **Theorem 9 (Commutativity)**

Let $\left(\tilde{n}'_1,\tilde{n}'_2,\cdots,\tilde{n}'_n\right)$ be any permutation of $\left(\tilde{n}_1,\tilde{n}_2,\cdots\tilde{n}_n\right)$, then

$$\text{INHFGOWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right)=\text{INHFGOWA}\left(\tilde{n}'_1,\tilde{n}'_2,\cdots,\tilde{n}'_n\right)$$

Now we can consider some special cases of INHFGOWA operator:

(1) If $\lambda = 1$, then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted averaging operator (INHFOWA) operator :

$$\text{INHFOWA}\,(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \sum_{j=1}^{n} \omega_j \tilde{n}_{\sigma(j)}$$

$$= \left(\left[1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \gamma^L_{\sigma(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \gamma^L_{\sigma(j)}\right)^{\omega_j} \right], \left[\prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(\delta^L_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(\delta^U_{\sigma(j)}\right)^{\omega_j} \right], \right. \quad (38)$$

$$\left. \left[\prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(\eta^L_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(\eta^U_{\sigma(j)}\right)^{\omega_j} \right] \right)$$

(2) If $\lambda \to 0$, then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted geometric (INHFOWG)operator:

$$\text{INHFOWG}\,(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \prod_{j=1}^{n} (\tilde{n}_{\sigma(j)})^{\omega_j}$$

$$= \left(\left[\prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} (\gamma^L_{\sigma(j)})^{w_j}, \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} (\gamma^U_{\sigma(j)})^{w_j} \right], \left[1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} (1 - \delta^L_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} (1 - \delta^U_{\sigma(j)})^{w_j} \right], \right. \quad (39)$$

$$\left. \left[1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} (1 - \eta^L_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} (1 - \eta^U_{\sigma(j)})^{w_j} \right] \right)$$

(4) If $\lambda = 2$,then INHFGOWA operator becomes the interval neutrosophic hesitant fuzzy ordered weighted quadratic averaging (INHFOWQA) operator:

$$\text{INHFOWQA}\,(\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_n) = \left(\sum_{j=1}^{n} \omega_j \tilde{n}^2_{\sigma(j)} \right)^{1/2}$$

$$= \left\{ \left[\left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \left(\gamma^L_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2}, \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\gamma}_j \in \tilde{t}_j} \left(1 - \left(\gamma^U_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2} \right] \right.$$

$$\left[1 - \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - \left(1 - \delta^L_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\delta}_j \in \tilde{i}_j} \left(1 - \left(1 - \delta^U_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2} \right], \quad (40)$$

$$\left. \left[1 - \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - \left(1 - \eta^U_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2}, 1 - \left(1 - \prod_{j=1}^{n} \bigcup_{\tilde{\eta}_j \in \tilde{f}_j} \left(1 - \left(1 - \eta^U_{\sigma(j)}\right)^2 \right)^{\omega_j} \right)^{1/2} \right] \right\}$$

About the positional weight vector, we can get it according to the real application. In some cases, we can obtain it by mathematical method. O'Hagan[26, 27] developed a procedure to get the OWA weights which have a predefined degree of orness and maximize the entropy. They can be calculated by the following methods.

Maximize: $-\sum_{i=1}^{n} \omega_i \ln \omega_i$

Subject to: $\dfrac{1}{n-1}\sum_{i=1}^{n}(n-i)\omega_i = \alpha, 0 \le \alpha \le 1$

$\sum_{i=1}^{n}\omega_i = 1, 0 \le \omega_i \le 1, i = 1, 2, \cdots, n$ where, α is the predefined degree of orness.

In definitions 16 and 17, the attribute weight vector and the positional weight vector can be considered separately. However, in many decision making problems, we need take two kinds of weight into account. We can define the interval neutrosophic hesitant fuzzy generalized hybrid weighted averaging (INHFGHWA) operator to process the problems.

Definition 18. An interval neutrosophic hesitant fuzzy generalized hybrid weighted averaging (INHFGHWA) operator of dimension n is a mapping I : $\Omega^n \to \Omega$, defined by an associated weighting vector $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$ such that $\omega_j \in [0,1]$, $\sum_{j=1}^{n}\omega_j = 1$ and parameter $\lambda \in (0, +\infty)$, so,

$$\mathrm{INHFGHWA}(\tilde{n}_1, \tilde{n}_2 \cdots, \tilde{n}_n) = \left(\sum_{j=1}^{n}\omega_j \tilde{b}_{\sigma(j)}^{\lambda}\right)^{1/\lambda}. \tag{41}$$

where, the $\tilde{b}_{\sigma(j)}$ is the jth largest of the $nw_j\tilde{n}_j$ $(j = 1,2,\cdots, n)$,and $w = (w_1, w_2, \cdots w_n)^T$ is the weighting vector of the $\tilde{n}_j$ $(j = 1,2,\cdots,n)$, $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$,and $\tilde{n}_i$ $(i = 1,2,\cdots, n)$ are interval neutrosophic hesitant fuzzy elements (INHFE),and $\tilde{n}_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j)(j = 1,2,\cdots,n)$.

Theorem 10. Let $\tilde{n}_j = ([\gamma_j^L, \gamma_j^U], [\delta_j^L, \delta_j^U], [\eta_j^L, \eta_j^U])$ $(j = 1,2,\cdots,n)$ be the collection of INHEs, $\tilde{b}_{\sigma(j)} = \left(\left[\alpha_{\sigma(j)}^L, \alpha_{\sigma(j)}^U\right], \left[\beta_{\sigma(j)}^L, \beta_{\sigma(j)}^U\right], \left[\chi_{\sigma(j)}^L, \chi_{\sigma(j)}^U\right]\right)$ is the jth largest of the $nw_j\tilde{n}_j$ $(j = 1,2,\cdots, n)$ then the result aggregated from Definition 18 is still an INHEs, and even

$$\left(\sum_{j=1}^{n}\omega_j\tilde{b}_{\sigma(j)}^{\lambda}\right)^{1/\lambda} = \left\{\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\alpha_{\sigma(j)}^L\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, \left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\alpha_{\sigma(j)}^U\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right]\right.$$
$$\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\beta_{\sigma(j)}^L\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\beta_{\sigma(j)}^U\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right], \tag{42}$$
$$\left.\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\chi_{\sigma(j)}^U\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\chi_{\sigma(j)}^U\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right]\right\}$$

Where

$$\tilde{b}_j = \left\{\left[\alpha_j^L,\alpha_j^U\right],\left[\beta_j^L,\beta_j^U\right],\left[\chi_j^L,\chi_j^U\right]\right\}$$
$$= \left\{\left[1\text{-}\left(1\text{-}\left(1\text{-}\left(1\text{-}\gamma_j^L\right)^{w_j}\right)\right)^n, 1\text{-}\left(1\text{-}\left(1\text{-}\left(1\text{-}\gamma_j^U\right)^{w_j}\right)\right)^n\right], \left[\left(\left(\delta_j^L\right)^{w_j}\right)^n, \left(\left(\delta_j^U\right)^{w_j}\right)^n\right], \left[\left(\left(\eta_j^L\right)^{w_j}\right)^n, \left(\left(\eta_j^U\right)^{w_j}\right)^n\right]\right\} \quad (43)$$

The proof of this theorem is similar with theorem 1, it's omitted here.

Now we can consider some special cases of INHFGHWA operator:

(1) If $\lambda = 1$,then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid weighted averaging (INHFHWA) operator :

$$\text{INHFHWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right) = \sum_{j=1}^{n}\omega_j\tilde{b}_{\sigma(j)}$$

$$= \left\{\begin{matrix} \left[1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\alpha_{\sigma(j)}^L\right)^{\omega_j}, 1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\alpha_{\sigma(j)}^L\right)^{\omega_j}\right], \left[\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{t}_j}\left(\beta_{\sigma(j)}^L\right)^{\omega_j}, \prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{t}_j}\left(\beta_{\sigma(j)}^U\right)^{\omega_j}\right], \\ \left[\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(\chi_{\sigma(j)}^L\right)^{\omega_j}, \prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(\chi_{\sigma(j)}^U\right)^{\omega_j}\right] \end{matrix}\right\} \quad (44)$$

Where, $\tilde{b}_j$ can be obtained by (43).

(2)If $\lambda \to 0$, then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid weighted geometric (INHFHWG) operator.

$$\text{INHFHWG}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right) = \prod_{j=1}^{n}\left(\tilde{b}_{\sigma(j)}\right)^{\omega_j}$$

$$= \left\{\begin{matrix} \left[\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}(\alpha_{\sigma(j)}^L)^{\omega_j}, \prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}(\alpha_{\sigma(j)}^U)^{\omega_j}\right], \left[1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}(1-\beta_{\sigma(j)}^L)^{\omega_j}, 1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}(1-\beta_{\sigma(j)}^U)^{\omega_j}\right] \\ \left[1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}(1-\chi_{\sigma(j)}^L)^{\omega_j}, 1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}(1-\chi_{\sigma(j)}^U)^{\omega_j}\right] \end{matrix}\right\} \quad (45)$$

Where, $\tilde{b}_j$ can be obtained by (43).

(3)If $\lambda = 2$,then INHFGHWA operator becomes the interval neutrosophic hesitant fuzzy hybrid quadratic weighted averaging (INHFHQWA)operator:

$$\text{INHFHQWA}\left(\tilde{n}_1,\tilde{n}_2,\cdots,\tilde{n}_n\right) = \left(\sum_{j=1}^{n}\omega_j\tilde{b}_{\sigma(j)}^2\right)^{1/2}$$

$$
=\left\{\left[\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\alpha_{\sigma(j)}^{L}\right)^{2}\right)^{\omega_j}\right)^{1/2},\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\gamma}_j\in\tilde{t}_j}\left(1-\left(\alpha_{\sigma(j)}^{U}\right)^{2}\right)^{\omega_j}\right)^{1/2}\right]\right.
$$

$$
\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\beta_{\sigma(j)}^{L}\right)^{2}\right)^{\omega_j}\right)^{1/2},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\delta}_j\in\tilde{i}_j}\left(1-\left(1-\beta_{\sigma(j)}^{U}\right)^{2}\right)^{\omega_j}\right)^{1/2}\right], \tag{46}
$$

$$
\left.\left[1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\chi_{\sigma(j)}^{U}\right)^{2}\right)^{\omega_j}\right)^{1/2},1-\left(1-\prod_{j=1}^{n}\bigcup_{\tilde{\eta}_j\in\tilde{f}_j}\left(1-\left(1-\chi_{\sigma(j)}^{U}\right)^{2}\right)^{\omega_j}\right)^{1/2}\right]\right\}
$$

Where, $\tilde{b}_j$ can be obtained by (43).

5.A decision-making method based on the INHFGHWA operator

In this section, we will use the proposed aggregation operators of the interval neutrosophic hesitant fuzzy numbers to the multiple attribute decision making problems in which attribute values take the form of the interval neutrosophic hesitant fuzzy information.

For a multiple attribute decision making problem, let $X=\{x_1,x_2,\cdots,x_m\}$ be a set of alternatives, $C=\{c_1,c_2,\cdots,c_n\}$ be a set of attributes, and $w=(w_1,w_2,\cdots w_n)^T$ be the weighting vector of attributes such that $w_j\in[0,1]$. Suppose that $\tilde{n}_{ij}=\left(\tilde{t}_{ij},\tilde{i}_{ij},\tilde{f}_{ij}\right)$ is the evaluation information of the criteria C_j on the alternative A_i which is represented by the form of the interval neutrosophic hesitant fuzzy information. Where $\tilde{t}_{ij}=\left\{\tilde{\gamma}_{ij}\middle|\tilde{\gamma}_{ij}\in\tilde{t}_{ij}\right\}$, $\tilde{i}_{ij}=\left\{\tilde{\delta}_{ij}\middle|\tilde{\delta}_{ij}\in\tilde{i}_{ij}\right\}$, and $\tilde{f}_{ij}=\left\{\tilde{\eta}_{ij}\middle|\tilde{\eta}_{ij}\in\tilde{f}_{ij}\right\}$ are three sets of some interval values in real unit interval $[0,1]$, which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees, and satisfies these limits : $\tilde{\gamma}_{ij}=\left[\gamma_{ij}^L,\gamma_{ij}^U\right]\subseteq[0,1]$, $\tilde{\delta}_{ij}=\left[\delta_{ij}^L,\delta_{ij}^U\right]\subseteq[0,1]$, $\tilde{\eta}_{ij}=\left[\eta_{ij}^L,\eta_{ij}^U\right]\subseteq[0,1]$ and $0\le\sup\tilde{\gamma}^{+}+\sup\tilde{\delta}^{+}+\sup\tilde{\eta}^{+}\le 3$, where $\tilde{\gamma}^{+}=\bigcup_{\tilde{\gamma}_{ij}\in\tilde{t}_{ij}}\max\left\{\gamma_{ij}^U\right\}$, $\tilde{\delta}^{+}=\bigcup_{\tilde{\delta}_{ij}\in\tilde{i}_{ij}}\max\left\{\delta_{ij}^U\right\}$, and $\tilde{\eta}^{+}=\bigcup_{\tilde{\eta}_{ij}\in\tilde{f}_{ij}}\max\left\{\eta_{ij}^U\right\}$. Then we can rank the order of the alternatives. The steps are shown as follows.

Step 1. Utilize the INHFGHWA operator

$$
\tilde{I}_i=\text{INHFGHWA}\left(\tilde{n}_{i1},\tilde{n}_{i2}\cdots,\tilde{n}_{in}\right)=\left(\sum_{j=1}^{n}\omega_j\tilde{b}_{i\sigma(j)}^{\lambda}\right)^{1/\lambda}\quad(i=1,2,\cdots,m). \tag{47}
$$

to derive the collective overall preference values $\tilde{I}(i=1,2,\cdots,m)$. Where, $\tilde{b}_{ij}=nw_j\tilde{n}_{ij}$.

Step 2. Utilize the score function expressed by (26) to calculate the ranking values. Because the score function is an interval value, it can be compared by possibility degrees of the interval numbers defined in (1). We can use a simple average method to improve the score function as follows.

$$s_i = s\left(\tilde{I}_i\right) = \left[\frac{1}{l_i}\sum_{i=1}^{l_i}\left(\frac{\gamma_i^L+\gamma_i^U}{2}\right) + \left(\frac{\sum_{i=1}^{p_i}\left(1-\frac{\delta_i^L+\delta_i^U}{2}\right)}{p_i}\right) + \left(\frac{\sum_{i=1}^{q_i}\left(1-\frac{\eta_i^L+\eta_i^U}{2}\right)}{q_i}\right)\right] \Bigg/ 3 \qquad (48)$$

Where l_i, p_i, and q_i are the numbers of interval values in $\tilde{t}_i, \tilde{i}_i, \tilde{f}_i$.

Step 3. Rank all the alternatives $x_i\,(i=1,2,\cdots,m)$ in descending order and select the best one(s) in accordance with the values of $s_i\,(i=1,2,\cdots,m)$.

Step 4. End.

6. An numerical example

In the section, we will provide an example to illustrate the application of INHFGHWA operator.

Suppose that an investment company wants to invest an amount of money in the best selection .There is a panel with four possible alternatives to which to invest the money: A_i (i=1,2,3,4) The investment company must consider the three attributes: (1) C1 (the risk index); (2) C2 (the growth index); (3) C3 (environmental impact index). We suppose the weight vector of the attributes is $w=(0.35,0.25,0.4)^T$. We can evaluate the alternatives by using the three attributes form INHFGHWA operators, and construct the following matrix R shown in the Table 1.

Table 1 interval neutrosophic hesitant generalized aggregation operator decision matrix

	C1	C2	C3
A1	{{[0.3,0.4],[0.4,0.4],[0.4, 0.5]},{[0.1,0.2]}, {[0.3,0.4]}}	{{[0.4,0.5],[0.5,0.6]},{[0.2,0. 3]},{[0.3,0.3], [0.3,0.4]}}	{{[0.2,0.3]},{[0.1,0.2]}, {[0.4,0.5],[0.5,0.6]}}
A2	{{[0.6,0.7]}, {[0.1,0.2]}, {[0.1,0.2], [0.2,0.3]}}	{{[0.6,0.7]}, {[0.1,0.1]}, {[0.2,0.3]}}	{{[0.6,0.7]},{[0.1,0.2]}, {[0.1,0.2]}}
A3	{{[0.3,0.4],[0.5,0.6]},{[0 .2,0.4]},{[0.2,0.3]}}	{{[0.5,0.6]}, {[0. 2,0.3]}, {[0.3,0.4]}}	{{[0.5,0.6]},{[0.1,0.2], [0.2,0.3]},{[0.2,0.3]}}
	{{[0.7,0.8]},	{{[0.6,0.7]},	{{[0.3,0.5]},{[0.2,0.3]},

A4	{[0,0.1]},{[0.1,0.2]}}	{[0,0.1]}, {[0.2,0.2]}}	{[0.1,0.2],[0.3,0.3]}}

6.1 The evaluation steps by the proposed method

Step 1. We can get the INHFGHWA operators based on Eq.(47) (Suppose the positional weight ($\omega = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $\lambda = 1$):

$$\tilde{I}_1 = \left\{ \begin{array}{l} \{(0.2895, 0.3903)(0.3568, 0.4234)(0.3568, 0.4590)\}, \\ \{(0.1189, 0.2213)\}, \{(0.3041, 0.4070)(0.3680, 0.4704)\} \end{array} \right\};$$

$$\tilde{I}_2 = \{\{0.6, 0.7\}, \{(0.1, 0.1682)\}, \{(0.1189, 0.2213)(0.1516, 0.2551)\}\};$$

$$\tilde{I}_3 = \left\{ \begin{array}{l} \{(0.4375, 0.5390)(0.5, 0.6)\}, \{(0.1516, 0.2821)(0.2, 0.3318)\}, \\ \{(0.2213, 0.3224)\} \end{array} \right\};$$

$$\tilde{I}_4 = \{\{0.5476, 0.6807\}, \{(0, 0.1552)\}, \{(0.1189, 0.2)(0.1845, 0.2352)\}\}.$$

Step 2 .calculate the score function according to Eq.(48):

$$s_1 = 1.0273;\ s_2 = 1.402;\ s_3 = 1.1339;\ s_4 = 1.3969.$$

So, $s_2 > s_4 > s_3 > s_1$.

Step 3. Rank all the alternatives $A_j\,(j = 1,2,3,4)$ in descending order and select the best one(s)in accordance with the values of $s_j\,(j = 1,2,3,4)$,we can get $A_2 > A_4 > A_3 > A_1$ and A_2 is the best option.

Step 4. End.

6.2 The influence of the parameter λ on decision making of this example

In order to illustrate the influence of the parameter λ on decision making of this example, we use the different value λ in step 1 to rank the alternatives. The results are shown in Table 2.

Table 2 Ranking of the alternatives by utilizing the different λ in INHFGHWA operator

λ	Score function values $s_j\,(j = 1,2,3,4)$	Ranking
$\lambda = 0.001$	$s_1 = 0.66646$, $s_2 = 0.66661$ $s_3 = 0.66662$, $s_4 = 0.66651$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\lambda = 0.1$	$s_1 = 0.64800$, $s_2 = 0.66180$	$A_3 \succ A_2 \succ A_4 \succ A_1$

	$s_3=0.66318$, $s_4=0.65267$	
$\lambda=0.7$	$s_1=0.61207$, $s_2=0.64704$ $s_3=0.67125$, $s_4=0.59764$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\lambda=0.8$	$s_1=0.61471$, $s_2=0.64662$ $s_3=0.67620$, $s_4=0.59233$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\lambda=1.0$	$s_1=0.6245$, $s_2=0.6742$ $s_3=0.6882$, $s_4=0.5843$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda=2.0$	$s_1=0.71978$, $s_2=0.67197$ $s_3=0.77333$, $s_4=0.57952$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda=5.9$	$s_1=1.05053$, $s_2=0.88072$ $s_3=1.08634$, $s_4=0.71573$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda=15$	$s_1=1.26617$, $s_2=1.18636$ $s_3=1.29611$, $s_4=0.90952$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\lambda=39$	$s_1=1.32926$, $s_2=1.32323$ $s_3=1.33213$, $s_4=0.99365$	$A_3 \succ A_1 \succ A_2 \succ A_4$

As we can see from Table 2, the ordering of the alternatives may be different for the different value λ in INHFGHWA operator.

(1) When $0.001 \le \lambda \le 39$, the best alternative is A_3.

(2) When $0.001 < \lambda < 39$, the ranking of the alternatives is different with respect to different λ. When $0.001 \le \lambda \le 0.1$, the ranking of the alternatives is $A_3 \succ A_2 \succ A_4 \succ A_1$. When $0.7 \le \lambda \le 0.8$, the ranking of the alternatives is $A_3 \succ A_2 \succ A_1 \succ A_4$. When $1.0 \le \lambda \le 39$, the ranking of the alternatives is $A_3 \succ A_1 \succ A_2 \succ A_4$.

7.Conclusion

The HFS can allow the membership function of an element to a set represented by several possible values, however, it cannot handle indeterminate and inconsistent information, while the NS can easily represent uncertainty, incomplete, and inconsistent information. In this paper, combined the IVHFS and INS, we further propose the concept of interval neutrosophic hesitant fuzzy sets (INHFSs), which extends truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of an element to a given set to IVHFS. The INHFS is a generalization of fuzzy set, IFS, IVIFS, NS, INS, HS, IVHFS, and so on. Then, we proposed the score function and comparison method of NHFSs, and developed some new aggregation operators for the interval neutrosophic hesitant fuzzy information, including interval neutrosophic hesitant fuzzy generalized weighted (INHFGWA) operator, interval neutrosophic hesitant fuzzy generalized ordered weighted (INHFGOWA) operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted (INHFGHWA) operator, and discuss some properties. Furthermore, we propose the decision making method for multiple attribute group decision making (MAGDM) problems with interval neutrosophic hesitant fuzzy information, and give the detail decision steps. A significant characteristic of the proposed method is that it can process many kinds of fuzzy information. In the future, we can apply the proposed operators to expend the scope of application, such as selection of supplier, science-technology assessment, the performance evaluation.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8(1965)338- 356.

[2] K.T. A tanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.

[3] K.T. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33(1989)37-46.

[4] K.T. Atanassov, G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 3(1989)343-349.

[5] K.T. Atanassov, Operators over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64(1994)159-174.

[6] X. Zhang, P.D. Liu, Method for aggregating triangular intuitionistic fuzzy information and its application to decision making. Technological and economic development of economy 16(2010) 280-290

[7] J.Q. Wang, Overview on fuzzy multi-criteria decision-making approach. Control and decision 23(2008) 601-606.

[8] H. Wang, F. Smarandache, Y. Zhang R. Sunderraman, Single valued neutrosophic sets, Proc Of 10th 476 Int Conf on Fuzzy Theory and Technology, Salt Lake City, 477 Utah, 2005.

[9] F Smarandache, A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth , 1999.

[10] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, AZ. 2005.

[11] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets. Multispace and Multistructure 4(2010) 410-413.

[12] J. Ye, Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems 42(4) (2013) 386-394.

[13] J. Ye, Single valued neutrosophic cross-entropy for multicriteria decision making problems. Applied Mathematical Modelling 38(2014) 1170-1175.

[14] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision. in: The 18th IEEE International

Conference on Fuzzy Systems, Jeju Island, Korea, (2009), pp. 1378－1382.

[15] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25(2010) 529－539.

[16] N. Chen, Z.S. Xu, M.M. Xia, Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Applied Mathematical Modelling 37(2013) 2197－2211.

[17] X.F. Zhao, R. Lin, G.W. Wei, Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making, Expert Systems with Applications 41(4)(2014) 1086-1094.

[18] F.Y. Meng, X.H. Chen, Q. Zhang, Multi-attribute decision analysis under a linguistic hesitant fuzzy environment, Information Sciences 267(2014) 287-305.

[19] B. Farhadinia, Correlation for Dual Hesitant Fuzzy Sets and Dual Interval-Valued Hesitant Fuzzy Sets, International Journal of Intelligent Systems 29(2)(2014) 184-205.

[20] J. Ye, Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making, Applied Mathematical Modelling 38(2014) 659-666.

[21] J.J. Peng, J.Q. Wang, J. Wang, X.H. Chen, Multicriteria Decision-Making Approach with Hesitant Interval-Valued Intuitionistic Fuzzy Sets, Scientific World Journal 2014(2014)1-18.

[22] R.R. Yager, Generalized OWA aggregation operators, Fuzzy Optimization and Decision Making 3(2004) 93－107.

[23] Z.S. Xu, Q.L. Da, An overview of operators for aggregating information, International Journal of Intelligent Systems 18 (2003) 953－969.

[24] N. Chen, Z.S. Xu, M.M. Xia, Interval-valued hesitant preference relations and their applications to group decision making. Knowledge-Based Systems 37 (2013) 528－540.

[25] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making. International Journal of Approximate Reasoning 52(2011) 395－407.

[26] M. O’Hagan, Fuzzy decision aids. In: Proc 21st Asilomar Conference on Signal, Systems and Computers, vol II. Pacific Grove, CA: IEEE and Maple Press; 1987. pp 624–628.

[27] M. O’Hagan, Aggregating template rule antecedents in real-time expert systems with fuzzy set logic. In: Proc 22ndAnnual IEEE Asilomar Conference on Signals, Systems and Computers. Pacific Grove, CA: IEEE and Maple Press; 1988. pp 681–689.