Eastern-European Journal of enterprise technologies 2/7 (74) April 2015, p.38-41.

TRANSFORMATION OF THE EQUATION OF MOTION IN STRESSES FOR AN INCOMPRESSIBLE FLUID

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The paper considers transformation of the equation of motion in stresses for an incompressible Newtonian fluid. The aim of the transformation is to obtain more detailed equations that account for the impact of vortex (rotational) and linear (forward) flows on the process of viscous friction. The transformation method is based on adding zero to the expressions for shear stresses with subsequent distinguishing of rotor velocity function and derivatives characterizing the linear flow. This approach as a form of recording the original equation does not require any additional restrictions. The transformation has resulted in new systems of equations for viscous vortex and vortex-free flows as well as three-dimensional vortex. The obtained equations allow obtaining the known exact solution for the laminar flow (Poiseuille's formula) and Euler's differential equation for an ideal fluid. We have shown that the Navier-Stokes equation is a separate case of a more general equation for Newtonian fluid motion. The obtained equations and connections between them allow improvement of the mathematical description of the incompressible fluid flow.
Keywords: general equation; Navier-Stokes; Euler; Poiseuille; vortex-free flow; 3D vortex.

1. Introduction

Mathematical description of liquids and gases behavior is an important and difficult task that has not been solved until now. The main difficulty consists in large deformations (deformation speed) of fluid environment that it is impossible to ignore unlike the deformed solid body. In spite of the identical initial system of solid and liquid equations obtained by Navier for a continuous environment, exact and numeral description of liquid behavior is less complete than that for a solid body.

Such a historically existing situation requires that theoretical and experimental researches be carried out that scientists from many countries of the world are being engaged in. A search for new exact equations of motion of fluid environment has a significant importance as they have much in common and their application domain is unlimited. The motion of transport vehicles in water and in mid air, in wind power engineering, in flows into power plants, in weather forecast improvements, in some aspects of astrophysics related to the flow of plasma can be referred to some of the fields [1, 2].

2. Analysis of literature and set of problem

Obtaining property of fluidity in liquid results in the final quantity of convective accelerations, and equations of motion become nonlinear. There is plenty of information on calculations of particular flows of liquid; however, there are just few exact solutions of equations of motion in this field of mechanics. For a turbulent flow, such solutions are unavailable that required the development of semi empiric theory of turbulence [3, 4]. For a laminar flow, there are a few exact tasks the most well-known one is named after Poiseuille. Within the framework of ideal liquid model, it is well-known Euler differential equation of motion including a few their exact solutions [5, 6]. All this resulted in large part of physical and numerical experiment while implementing any research in field of mechanics of liquid and allied subjects [7-9].

Besides the power field characterized by equations of motion, there exists the second physical field in an incompressible liquid - speeds of deformations, whose equations are not examined in this paper.

Equation of motion in tensions (Navier) that can be presented as [5, 6, 10] underlies mathematical description of flow of liquid.

$$X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)=\frac{du_x}{dt},$$

$$Y+\frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)=\frac{du_y}{dt}, \qquad (1)$$

$$Z+\frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)=\frac{du_z}{dt},$$

where p_{xx}, p_{yy}, p_{zz} are normal tensions, $\tau_{yx,}\tau_{zx,}\tau_{yz}$ are tangent tensions, X,Y,Z are specific mass force, u_x, u_y, u_z are speed projections , t is time.

One of the special cases of this equation for a Newtonian liquid is Navier-Stokes equation. There are a few methods of its derivation from which we would like to draw your attention to the one described in works [5, 6]. The characteristic feature of this derivation is a more clear formulation of accepted assumptions from which it is necessary to distinguish the following:

– average (AV) point pressure of Newtonian liquid can be found as an arithmetical mean value of projections

$$p=\frac{p_x+p_y+p_z}{3} \quad . \qquad (2)$$

As mentioned in some works, it is impossible to prove that the quantity from a formula (2) is a pressure in the thermodynamics sense [5, 11].

– viscosity influences not only on tangents but also on normal tensions(p_{xx}, p_{yy}, p_{zz}), that is illustrated by the system of equations [5, 6, 10].

$$p_{xx}=-p+2\mu\frac{\partial u_x}{\partial x},$$
$$p_{yy}=-p+2\mu\frac{\partial u_y}{\partial y} \quad . \qquad (3)$$
$$p_{zz}=-p+2\mu\frac{\partial u_z}{\partial z}.$$

The system (3) does not comply with the Newton's law for a viscous friction according to which viscosity influences on tangent tensions only [5, 6, 10].

Navier-Stokes equation has not been solved in a general view, and all their particular exact solutions possess one general property -
the calculations results correspond to the supervisions in the number range of Reynolds
$Re \approx (10^{-3} \ldots 5)$.

Over the past decades, many critical remarks have been spoken out concerning this equation; nevertheless it was widely spread not only in mechanics of liquid, but also in other sciences. Reason for such situation is absence of other equations that would have been more reasonable from the physical and mathematical point of view [5-7, 8, 12].

3. Aim and research tasks

The aim of the current research paper is to obtain more detailed equations of motion by transformation (1) at the minimum number of the well-known limitations characteristic of Newtonian nonviscous and ideal liquid.

In order to gain the goal, the well-known method of transformation that consists in adding zero to mathematical expression with its subsequent presentation in a form of two identical elements with different signs has been selected. Such method of transformation does not require the use of additional conditions and their proof.

4. Transformation of equations of motion and special cases

We will transform the system (1) by dissociating pressure from tangent tensions. As $p_{xx}=-p_x \;\; p_{yy}=-p_y \; p_{zz}=-p_z$ (where p_x, p_y, p_z – are projections of pressure).

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)=\frac{du_x}{dt},$$

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{zy}}{\partial z}\right)=\frac{du_y}{dt}, \qquad (4)$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}\right)=\frac{du_z}{dt}.$$

Let us transform the first line of equation (4) putting expressions for tangent tensions in a Newtonian liquid

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt}.$$

We will add a zero to the derivatives into brackets presenting it in a form of two identical components with a different sign.

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}-\frac{\partial u_x}{\partial y}+\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}-\frac{\partial u_z}{\partial x}+\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt},$$

As a result, we will obtain

$$X-\frac{1}{\rho}\frac{\partial p_x}{\partial x}+\nu\left[\frac{\partial}{\partial y}\left(\left(\mathrm{rot}\, u\right)_z+2\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\left(\mathrm{rot}\, u\right)_y+2\frac{\partial u_z}{\partial x}\right)\right]=\frac{du_x}{dt}.$$

After analogical transformations

$$Y-\frac{1}{\rho}\frac{\partial p_y}{\partial y}+\nu\left[\frac{\partial}{\partial x}\left(\left(\mathrm{rot}\, u\right)_z+2\frac{\partial u_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\left(\mathrm{rot}\, u\right)_x+2\frac{\partial u_y}{\partial z}\right)\right]=\frac{du_y}{dt}, \qquad (5)$$

$$Z-\frac{1}{\rho}\frac{\partial p_z}{\partial z}+\nu\left[\frac{\partial}{\partial x}\left(\left(\mathrm{rot}\, u\right)_y+2\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left(\mathrm{rot}\, u\right)_x+2\frac{\partial u_y}{\partial z}\right)\right]=\frac{du_z}{dt}.$$

In equations (4, 5) the following expressions are used:

-for tangent tensions in a Newtonian liquid [5, 6] $\tau_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$, $\tau_{xz} = \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$, $\tau_{yz} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$;

-for the rotor of speed [5] $(rot\ u)_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}$, $(rot\ u)_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$, $(rot\ u)_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$.

The characteristic feature of the system (5) is an influence of vortical (rotatory) and linear (forward) flow of liquid to be recorded, and also direct dependence of pressure from position of surface element.

From equations (5) by the data $(rot\ u)_i = 0$ follow equations of viscous irrotational (noncirculatory) flow.

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + 2\nu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) = \frac{du_x}{dt},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} + 2\nu\left(\frac{\partial^2 u_y}{\partial z^2} + \frac{\partial^2 u_x}{\partial x \partial y}\right) = \frac{du_y}{dt}, \qquad (6)$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + 2\nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial z}\right) = \frac{du_z}{dt}.$$

Using cylindrical coordinates(*r, z*), it is possible to get the special case of equations(6) for a round pipe in the form of $\frac{d^2 u_z}{dr^2} = \frac{1}{2\mu} \cdot grad\ p$ from which it follows Poiseuille equation for the laminar flow mode [5, 6, 10].

The system (5) has another special case for the three-dimensional revolved vortex without forward motion. Excepting linear speeds in the left part (5), we will obtain:

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + \nu\left[\frac{\partial}{\partial y}(\text{rot}\,u)_z + \frac{\partial}{\partial z}(\text{rot}\,u)_y\right] = \frac{du_x}{dt},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} + \nu\left[\frac{\partial}{\partial x}(\text{rot}\,u)_z + \frac{\partial}{\partial z}(\text{rot}\,u)_x\right] = \frac{du_y}{dt}, \qquad (7)$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + \nu\left[\frac{\partial}{\partial x}(\text{rot}\,u)_y + \frac{\partial}{\partial y}(\text{rot}\,u)_x\right] = \frac{du_z}{dt}.$$

Equation (7) can be presented in other form that uses rotary particle velocity.

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} + 2\nu\left(\frac{\partial \omega_z}{\partial y} + \frac{\partial \omega_y}{\partial z}\right) = \frac{du_x}{dt},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} + 2\nu\left(\frac{\partial \omega_z}{\partial x} + \frac{\partial \omega_x}{\partial z}\right) = \frac{du_y}{dt},$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} + 2\nu\left(\frac{\partial \omega_y}{\partial x} + \frac{\partial \omega_x}{\partial y}\right) = \frac{du_z}{dt}.$$

where $(\mathrm{rot}\ u)_i = 2\omega_i$, $/\omega_i$ - is an rotary velocity of particle rotation /.

Comparing expressions in brackets with formulas for tangent tensions, one can notice their analogy. These expressions (formula), in accordance with a dimension, characterize the force of friction attributed to the quantity of the revolved volume.

The examined approach allows to derive equation of motion within the framework of the nonviscous liquid model. Supposing $\nu = 0$, we will obtain from (5) -(7) :

$$X - \frac{1}{\rho}\frac{\partial p_x}{\partial x} = \frac{du_x}{dt},$$

$$Y - \frac{1}{\rho}\frac{\partial p_y}{\partial y} = \frac{du_y}{dt}, \qquad (8)$$

$$Z - \frac{1}{\rho}\frac{\partial p_z}{\partial z} = \frac{du_z}{dt}.$$

From (8) it follows that the projections of pressure can differ in the absence of viscosity influence (impact). This conclusion conflicts with the well-known point of view that $p_x \neq p_y \neq p_z$ is possible only under influence of viscosity [5, 6, 10].

If we consider the hydrostatical law of distribution of pressure ($p = p_x = p_y = p_z$), then we will obtain Euler equation from the dynamics of ideal liquid [5, 12]. From (1) it is possible to derive Navier-Stokes equation making use of a few well-known assumptions [5-7].

5. Short analysis of transformation

Connection between the considered equations can be presented in the form of the following chart (fig. 1).

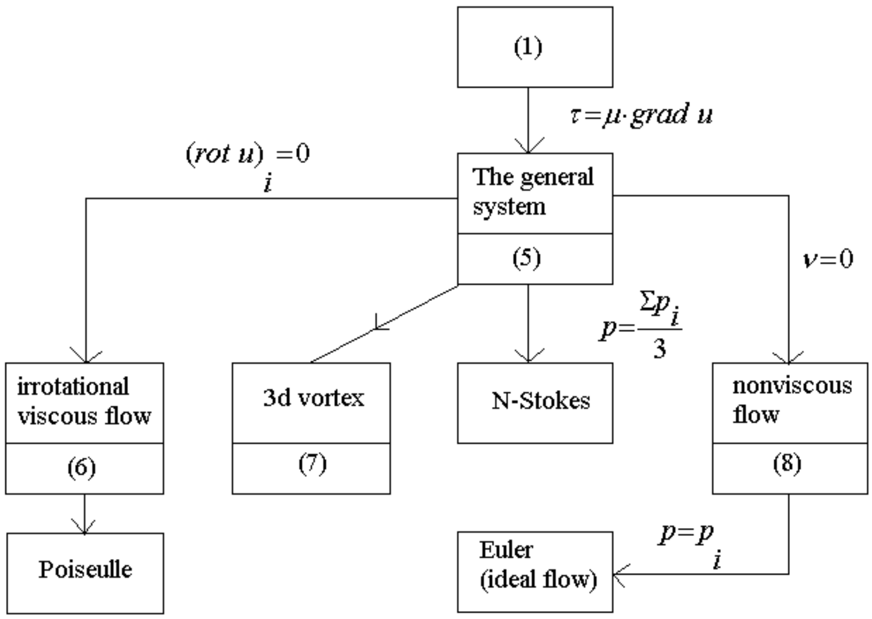

Fig.1. Chart of connections between equations.

In equalization (5) there are elements characterizing all types of incompressible liquid flow: forward and vortex. It allows to suppose that it can be used for the calculation of turbulent flow.

Using the well-known definition for an irrotational flow, from (5) it is possible to derive equations for a viscous irrotational flow, and also for viscous three-dimensional vortex without linear speed.

Equation in a form (1) or (3) used establish Navier-Stokes equation with the use of two basic assumptions [5, 6]:

Linear equalization $\left(p = \frac{p_x + p_y + p_z}{3}\right)$ is correct to obtain mean pressure of nonlinear function $p = f(x, y, z, t)$. This standard assumption is executed at the small interval of averaging only and must be considered approximate.

Point pressure changes under influence of viscosity. This assumption does not comply with the Newton's law for a viscid friction and conflicts with equation of motion for the nonviscous liquid (8).

Taking into account both assumptions Navier-Stokes equation must be considered approximate for the laminar flow mode.

The approach being considered in the paper towards mathematical description of flow of incompressible liquid allows us to find another way of deriving the well-known formula for a laminar flow (Poiseuille) and differential Euler equation (fig.1).

6. Discussion of equations and connections between them

The system of motion equations of Newtonian liquid (5) is obtained from (1) while using only one assumption based on the Newton's law for a viscous friction. It allows considering that (5) is the most general system of motion equations of Newtonian liquid. A presence of elements that characterize two possible types of motion (linear and vortex) in these equations gives hope to the description of the turbulent flow mode.

As the exact solutions of particular tasks for this mode are absent, it is necessary to undertake additional theoretical and experimental studies to clarify the possibilities of this equations system.

The system (6) is obtained from (5) except for function of rotor of speed that characterizes vortex formation. By simplifying the system (6) it is possible to obtain the well-known Poiseuille formula. It allows to suppose that the well-known definition for an irrotational flow refers to the laminar mode.

At the same time, simplification (6) requires a large number of elements be eliminated whose contribution to the general result of calculation is not clear enough. It requires that a more detailed analysis of elements be conducted depending on viscosity.

The system (7) is obtained by eliminating from (5) elements that characterize linear motion, and does not have clear definition in literature, and also exact solutions. Nevertheless, the flows of such type (stand-up air and aquatic vortex) are observed in nature and technology that gives hope to perfection of their mathematical description [1, 6].

Found systems of equations (5) - (8) are open and their exact solutions are possible for some simple tasks only.

7. Conclusions

The applied method of transformation of the equations system of motion in tensions allowed to distinguish (single out) two types of components characteristic of a viscous friction in a Newtonian liquid at a vortical and linear (forward) flow. It allowed making up three systems of equations for three varieties of flow process: flow at joint influence of friction in two types of flow; linear flow with a friction only; vortical flow with a friction only.

The special cases of the found general equations system's comparison was conducted with the well-known exact solution for the laminar flow mode (Poiseuille formula).

The new way of differential Euler equation derived in which the successive change of terms and equations takes place has been described according to a chart: viscous liquid $\rightarrow$ non-viscous liquid$\rightarrow$ ideal liquid (fig. 1).

8. References

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