Transformation of second derivatives in weak formulation

Lenka Dubcova

December 16, 2009

1 Transformation of the first derivatives

Let me denote mapping from reference domain onto physical element by

$$\boldsymbol{x}_K = (x_{K,0}, x_{K,1}),$$

the Jacobi matrix of this mapping by

$$J = \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_{K,0}}{\partial \xi_1} & \frac{\partial x_{K,0}}{\partial \xi_2} \\ \frac{\partial x_{K,1}}{\partial \xi_1} & \frac{\partial x_{K,1}}{\partial \xi_2} \end{pmatrix}$$

and the inverse of transposed Jacobi matrix by

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = \frac{1}{\det(J)} \begin{pmatrix} J_{11} & -J_{10} \\ -J_{01} & J_{00} \end{pmatrix}$$

In this notation, the transformation of the first derivatives is the following

$$\frac{\partial u}{\partial \xi_1} = \frac{\partial u}{\partial x}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial u}{\partial y}\frac{\partial x_{K,1}}{\partial \xi_1} = \frac{\partial u}{\partial x} J_{00} + \frac{\partial u}{\partial y} J_{10},$$

$$\frac{\partial u}{\partial \xi_2} = \frac{\partial u}{\partial x}\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial u}{\partial y}\frac{\partial x_{K,1}}{\partial \xi_2} = \frac{\partial u}{\partial x} J_{01} + \frac{\partial u}{\partial y} J_{11}.$$

Therefore

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \xi_1} \\ \frac{\partial u}{\partial \xi_2} \end{pmatrix}$$

2 Second derivatives

Let me denote the matrix of the second derivatives of the reference mapping by

$$K = \begin{pmatrix} k_{00} & k_{01} & k_{02} \\ k_{10} & k_{11} & k_{12} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 x_{K,0}}{\partial \xi_1^2} & \frac{\partial^2 x_{K,0}}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 x_{K,0}}{\partial \xi_2^2} \\ \frac{\partial^2 x_{K,1}}{\partial \xi_1^2} & \frac{\partial^2 x_{K,1}}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 x_{K,1}}{\partial \xi_2^2} \end{pmatrix}$$

With this notation the second derivatives of u can be written as follows

$$\frac{\partial}{\partial \xi_1}\left(\frac{\partial u}{\partial \xi_1}\right) = \frac{\partial}{\partial \xi_1}\left(\frac{\partial u}{\partial x}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial u}{\partial y}\frac{\partial x_{K,1}}{\partial \xi_1}\right) =$$

$$= \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,1}}{\partial \xi_1}\right)\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial u}{\partial x}\frac{\partial^2 x_{K,0}}{\partial \xi_1^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial^2 u}{\partial y^2}\frac{\partial x_{K,1}}{\partial \xi_1}\right)\frac{\partial x_{K,1}}{\partial \xi_1} + \frac{\partial u}{\partial y}\frac{\partial^2 x_{K,1}}{\partial \xi_1^2}$$

$$= \left(\frac{\partial^2 u}{\partial x^2} J_{00} + \frac{\partial^2 u}{\partial x \partial y} J_{10}\right) J_{00} + \frac{\partial u}{\partial x} K_{00} + \left(\frac{\partial^2 u}{\partial x \partial y} J_{00} + \frac{\partial^2 u}{\partial y^2} J_{10}\right) J_{10} + \frac{\partial u}{\partial y} K_{10} =$$

$$= J_{00}^2 \frac{\partial^2 u}{\partial x^2} + 2 J_{00} J_{10} \frac{\partial^2 u}{\partial x \partial y} + J_{10}^2 \frac{\partial^2 u}{\partial y^2} + K_{00}\frac{\partial u}{\partial x} + K_{10}\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \xi_1}\left(\frac{\partial u}{\partial \xi_2}\right) = \frac{\partial}{\partial \xi_1}\left(\frac{\partial u}{\partial x}\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial u}{\partial y}\frac{\partial x_{K,1}}{\partial \xi_2}\right) =$$

$$= \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,1}}{\partial \xi_1}\right)\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial u}{\partial x}\frac{\partial^2 x_{K,0}}{\partial \xi_1 \partial \xi_2} + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,0}}{\partial \xi_1} + \frac{\partial^2 u}{\partial y^2}\frac{\partial x_{K,1}}{\partial \xi_1}\right)\frac{\partial x_{K,1}}{\partial \xi_2} + \frac{\partial u}{\partial y}\frac{\partial^2 x_{K,1}}{\partial \xi_1 \partial \xi_2}$$

$$= J_{00} J_{01} \frac{\partial^2 u}{\partial x^2} + (J_{01} J_{10} + J_{00} J_{11}) \frac{\partial^2 u}{\partial x \partial y} + J_{10} J_{11} \frac{\partial^2 u}{\partial y^2} + K_{01}\frac{\partial u}{\partial x} + K_{11}\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \xi_2}\left(\frac{\partial u}{\partial \xi_2}\right) = \frac{\partial}{\partial \xi_2}\left(\frac{\partial u}{\partial x}\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial u}{\partial y}\frac{\partial x_{K,1}}{\partial \xi_2}\right) =$$

$$= \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,1}}{\partial \xi_2}\right)\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial u}{\partial x}\frac{\partial^2 x_{K,0}}{\partial \xi_2^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x_{K,0}}{\partial \xi_2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial x_{K,1}}{\partial \xi_2}\right)\frac{\partial x_{K,1}}{\partial \xi_2} + \frac{\partial u}{\partial y}\frac{\partial^2 x_{K,1}}{\partial \xi_2^2}$$

$$= J_{01}^2 \frac{\partial^2 u}{\partial x^2} + 2 J_{01} J_{11} \frac{\partial^2 u}{\partial x \partial y} + J_{11}^2 \frac{\partial^2 u}{\partial y^2} + K_{02}\frac{\partial u}{\partial x} + K_{12}\frac{\partial u}{\partial y}$$

In the matrix notation

$$\begin{pmatrix} \frac{\partial^2 u}{\partial \xi_1^2} \\ \frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 u}{\partial \xi_2^2} \end{pmatrix} = \begin{pmatrix} J_{00}^2 & 2 J_{00} J_{10} & J_{10}^2 \\ J_{00} J_{01} & J_{00} J_{11} + J_{01} J_{10} & J_{10} J_{11} \\ J_{01}^2 & 2 J_{01} J_{11} & J_{11}^2 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} + \begin{pmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \\ K_{02} & K_{12} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

For the transformation of expressions in a weak formulation we need to express derivatives with respect to x, y with aid of only derivatives with respect to ξ_1, ξ_2.

$$\begin{pmatrix} \frac{\partial^2 u}{\partial \xi_1^2} \\ \frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 u}{\partial \xi_2^2} \end{pmatrix} = A \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} + K^T \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial^2 u}{\partial \xi_1^2} \\ \frac{\partial^2 u}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 u}{\partial \xi_2^2} \end{pmatrix} - A^{-1} K^T M \begin{pmatrix} \frac{\partial u}{\partial \xi_1} \\ \frac{\partial u}{\partial \xi_2} \end{pmatrix}$$

where

$$A^{-1} = \frac{1}{\det^2(J)} \begin{pmatrix} J_{11}^2 & -2J_{10}J_{11} & J_{10}^2 \\ -J_{01}J_{11} & J_{00}J_{11} + J_{01}J_{10} & -J_{00}J_{10} \\ J_{01}^2 & -2J_{00}J_{01} & J_{00}^2 \end{pmatrix} = \begin{pmatrix} m_{00}^2 & 2m_{01}m_{00} & m_{01}^2 \\ m_{10}m_{00} & m_{00}m_{11} + m_{01}m_{10} & m_{11}m_{01} \\ m_{10}^2 & 2m_{11}m_{10} & m_{11}^2 \end{pmatrix}$$

Notice, that for elements whose Jacobian is constant (triangles, nice quadrilaterals) the matrix K is zero matrix and transforamtions can be simplified (it does not content the first derivatives of u with respect to $\boldsymbol{\xi}$ at all).

3 Final expressions

From previous it follows how to transform $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ in weak formulations. Expressions are in terms of entries of M and K.

$$\frac{\partial^2 u}{\partial x^2} = m_{00}^2 \frac{\partial^2 u}{\partial \xi_1^2} + 2m_{01}m_{00} \frac{\partial^2 u}{\partial \xi_1 \xi_2} + m_{01}^2 \frac{\partial^2 u}{\partial \xi_2^2} \tag{1}$$

$$- \left[\left(m_{00}^2 k_{00} + 2m_{01}m_{00}k_{01} + m_{01}^2 k_{02} \right) m_{00} + \left(m_{00}^2 k_{10} + 2m_{01}m_{00}k_{11} + m_{01}^2 k_{12} \right) m_{10} \right] \frac{\partial u}{\partial \xi_1}$$

$$- \left[\left(m_{00}^2 k_{00} + 2m_{01}m_{00}k_{01} + m_{01}^2 k_{02} \right) m_{01} + \left(m_{00}^2 k_{10} + 2m_{01}m_{00}k_{11} + m_{01}^2 k_{12} \right) m_{11} \right] \frac{\partial u}{\partial \xi_1}$$

$$
\frac{\partial^2 u}{\partial x \partial y} = m_{00}m_{10}\frac{\partial^2 u}{\partial \xi_1^2} + (m_{00}m_{11} + m_{10}m_{01})\frac{\partial^2 u}{\partial \xi_1 \xi_2} + m_{01}m_{11}\frac{\partial^2 u}{\partial \xi_2^2} \tag{2}
$$

$$
- \left[(m_{00}m_{10}k_{00} + (m_{00}m_{11} + m_{10}m_{01})k_{01} + m_{01}m_{11}k_{02})\, m_{00} + (m_{00}m_{10}k_{10} + (m_{00}m_{11} + m_{10}m_{01})k_{11} + m_{01}m_{11}k_{12})\, m_{10}\right] \frac{\partial u}{\partial \xi_1}
$$

$$
- \left[(m_{00}m_{10}k_{00} + (m_{00}m_{11} + m_{10}m_{01})k_{01} + m_{01}m_{11}k_{02})\, m_{01} + (m_{00}m_{10}k_{10} + (m_{00}m_{11} + m_{10}m_{01})k_{11} + m_{01}m_{11}k_{12})\, m_{11}\right] \frac{\partial u}{\partial \xi_1}
$$

$$
\frac{\partial^2 u}{\partial y^2} = m_{10}^2\frac{\partial^2 u}{\partial \xi_1^2} + 2m_{11}m_{10}\frac{\partial^2 u}{\partial \xi_1 \xi_2} + m_{11}^2\frac{\partial^2 u}{\partial \xi_2^2} \tag{3}
$$

$$
- \left[\left(m_{10}^2 k_{00} + 2m_{11}m_{10}k_{01} + m_{11}^2 k_{02}\right) m_{00} + \left(m_{10}^2 k_{10} + 2m_{11}m_{10}k_{11} + m_{11}^2 k_{12}\right) m_{10}\right] \frac{\partial u}{\partial \xi_1}
$$

$$
- \left[\left(m_{10}^2 k_{00} + 2m_{11}m_{10}k_{01} + m_{11}^2 k_{02}\right) m_{01} + \left(m_{10}^2 k_{10} + 2m_{11}m_{10}k_{11} + m_{11}^2 k_{12}\right) m_{11}\right] \frac{\partial u}{\partial \xi_1}
$$

4 Example: Laplace

$$
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =
$$

$$
= (m_{00}^2 + m_{10}^2)\frac{\partial^2 u}{\partial \xi_1^2} + (2m_{01}m_{00} + 2m_{10}m_{11})\frac{\partial^2 u}{\partial \xi_1 \xi_2} + (m_{01}^2 + m_{11}^2)\frac{\partial^2 u}{\partial \xi_2^2}
$$

$$
- \Big[\Big((m_{00}^2 + m_{10}^2)k_{00} + (2m_{01}m_{00} + 2m_{10}m_{11})k_{01} + (m_{01}^2 + m_{11}^2)k_{02}\Big) m_{00} + \Big((m_{00}^2 + m_{10}^2)k_{10} + (2m_{01}m_{00} + 2m_{10}m_{11})k_{11} + (m_{01}^2 + m_{11}^2)k_{12}\Big) m_{10}\Big] \frac{\partial u}{\partial \xi_1}
$$

$$
- \Big[\Big((m_{00}^2 + m_{10}^2)k_{00} + (2m_{01}m_{00} + 2m_{10}m_{11})k_{01} + (m_{01}^2 + m_{11}^2)k_{02}\Big) m_{01} + \Big((m_{00}^2 + m_{10}^2)k_{10} + (2m_{01}m_{00} + 2m_{10}m_{11})k_{11} + (m_{01}^2 + m_{11}^2)k_{12}\Big) m_{11}\Big] \frac{\partial u}{\partial \xi_1}
$$