Symmetry is not the source of unitary information in wave mechanics – context quantum randomness

Homogeneity of space is non-unitary

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Abstract The homogeneity symmetry is re-examined and shown to be non-unitary, with no requirement for the imaginary unit. This removes symmetry, as reason, for imposing unitarity (or self-adjointness) – *by Postulate.* The work here is part of a project researching logical independence in quantum mathematics, for the purpose of advancing a full and complete theory of quantum randomness.

Keywords foundations of quantum theory, quantum physics, quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space, unitary, non-unitary, unitarity, mathematical logic, formal system, elementary algebra, information, axioms, mathematical propositions, logical independence, quantum indeterminacy, quantum randomness.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [7], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al offer explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in (Boolean) mathematical propositions [8,9]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Tomasz Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

While those Boolean propositions *do* convey definitive information about quantum randomness, any insight they offer is obscure. In order to advance a full and complete theory of quantum randomness, understanding is needed of how this Boolean logical independence connects with *standard textbook quantum theory.*

In a related article by this author [4], logical independence in Elementary Algebra is discussed. This is the very familiar algebra upon which Applied Mathematics and Mathematical Physics rest. Logical independence in this algebra is well-known to Mathematical Logic [12]. Of particular interest is *logical independence* of the imaginary scalars, seen in contrast to *logical dependence* of the rational scalars – and – the possible prospect that these two types of logical information might pass into quantum mathematics. As it happens, the passage of that logical information is blocked. It is prevented by, not by unitarity itself, but because *unitarity* (or self-adjointness) is imposed axiomatically – *by Postulate.*

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Historically, the reason for unitarity is the universal need for preserved invariance of probability amplitude. Interpretationally, this universality is seen to imply that all fundamental symmetries in Nature are ontologically unitary [5, p109][6, p34]. This would indicate that unitarity should be a blanket condition covering the whole of quantum theory — and should be regarded as *axiomatic*. In short, unitarity is never in absence.

The findings of this paper contradicts the belief that symmetries are intrinsically unitary. It shows that the homogeneity symmetry, generally purported to imply the Canonical Commutation Relation [5, p115][11, p44], is not itself unitary, does not imply the Canonical Commutation Relation, indicating that the Relation's unitarity originates elsewhere.

This removes symmetry, as reason, for imposing unitarity −*by Postulate.*

2 The basic symmetry of wave mechanics: homogeneity of space

The *Canonical Commutation Relation*

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of wave mechanics. The professed significance of this relation is that it represents the homogeneity of space, with general acceptance by quantum theorists, as being unitary. In this paper, I re-examine and scrutinise the Canonical Relation's derivation and establish that the homogeneity symmetry is itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains other (unitary) information also.

Imposing homogeneity on a system is identical to imposing a null physical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *Form Invariance.* This is the concept, from relativity, that symmetry transformations leave (physical) formulae fixed in *form*, though *values* may alter [10]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x} \left| f_{\mathsf{x}}(x) \right\rangle = \mathsf{x} \left| f_{\mathsf{x}}(x) \right\rangle . \tag{1}$$

The san-serif x, here, is a label for f_{x} whose eigenvalue is x. The variable x (curly) is the function domain. The use of two different variables, here, may seem unusual and pointless. In fact, logically they are different. x is quantified *existentially* but x is quantified *universally.*

With form held fixed, as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally (Lie group and Lie algebra). To maintain the form of (1), under translation, the basis $|f_{\mathsf{x}}\rangle$ is cleverly managed: while the translation transforms the basis from $|f_{\mathsf{x}}\rangle$ to $|f_{\mathsf{x}-\epsilon}\rangle$, a similarity transformation is also applied, chosen to revert $|f_{\mathsf{x}-\epsilon}\rangle$ back to $|f_{\mathsf{x}}\rangle$. In this way $|f_{\mathsf{x}}\rangle$ is held static. We see below that, actually, similarity transforms can be found only for a certain class of functions: $\left\{\psi_{\mathsf{x}} \in L^1\right\} \subset \{f_{\mathsf{x}}\}$.

$\mathbb{F}$ is any infinite field.

The similarity transformations are the one-parameter subgroup of the general linear group, $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}(\mathbb{F})$, with the transformation parameter ϵ coinciding with the displacement parameter. The overall scheme of transformations is depicted in Figure 1.

In standard theory, textbook understanding is that $\mathsf{S}(\epsilon)$ is intrinsically and necessarily unitary, and it is in *that* unitarity where the Canonical Commutation

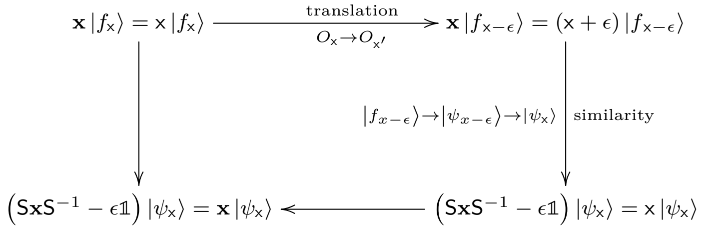

Figure 1 Scheme of transformations. The bottom left hand formula is the resulting group relation.

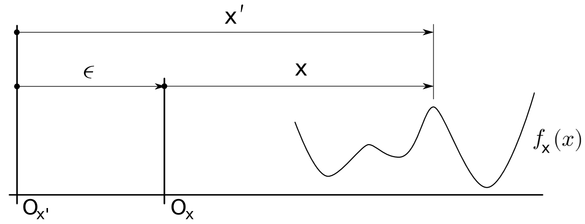

Figure 2 Passive translation of a function Two reference systems, O_x and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, individually act as reference systems for position of a function f_x. If the x-space is homogeneous, then regardless of the value of ϵ, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; f_x is non-observable and O_x and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

Relation finds its unitary origins. And so, because its presence is thought *intrinsically necessary*, unitarity is imposed axiomatically on the theory, *by Postulate*. The upshot is that standard theory *imposes* Hilbert space on vectors $|f_\mathsf{x}\rangle$. This imposed unitarity is added information, extra to the information of homogeneity. In consequence, the underlying symmetry beneath wave mechanics is not homogeneity of space, but instead, a unitary subgroup.

As an experiment, we proceed, in this paper, by treating unitarity as a purely separate issue from homogeneity, allowing $\mathsf{S}(\epsilon)$ it's widest generality, so that the *whole information* of homogeneity (upto the general linear similarity transformation) is faithfully and genuinely conveyed through the theory.

The experiment begins with the eigenvalue equation for position (1) being rewritten, as the eigenformula in the quantified proposition (2). From here on, all informal assumptions are to be shed, with the Dirac notation dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator $\mathbf{x}$, eigenfunctions f_x and eigenvalues x, seen from the reference frame O_x:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_\mathsf{x} \mid \quad \mathbf{x} f_\mathsf{x}(x) = \mathsf{x} f_\mathsf{x}(x) \tag{2}$$

Translation: Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through ϵ. See Figure 2. *Form Invariance* guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_x in (2), thus:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_\mathsf{x} \mid \quad \mathbf{x} f'_\mathsf{x}(x') = \mathsf{x}' f'_\mathsf{x}(x') \tag{3}$$

A relation for $\mathbf{x}$ is to be evaluated, so $\mathbf{x}$ is held static for all reference frames. The translation transforms position, thus:

$$\forall \epsilon \forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{4}$$

and transforms the function, thus:

$$\forall \epsilon \forall x' \forall f'_\mathsf{x} \exists f_\mathsf{x} \exists x \mid \quad f_\mathsf{x}(x) \mapsto f'_\mathsf{x}(x') = f_{\mathsf{x}-\epsilon}(x - \epsilon) \tag{5}$$

Substituting (4) and (5) into (3) gives the translated formula:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists f_\mathsf{x} \mid \quad \mathbf{x} f_{\mathsf{x}-\epsilon}(x - \epsilon) = (\mathsf{x} + \epsilon) f_{\mathsf{x}-\epsilon}(x - \epsilon) \,. \tag{6}$$

Substitution involving quantified variables

$$\begin{aligned} \forall\beta\underline{\forall\gamma}\exists\alpha \mid \quad & \alpha = \beta + \gamma \\ \forall\beta\underline{\exists\gamma} \mid \quad & \gamma = \beta + \beta \\ \Rightarrow \forall\beta\exists\alpha \mid \quad & \alpha = \beta + \beta + \beta \end{aligned}$$

For logically dependent substitution, an existential quantifier of one proposition should be matched with a universal quantifier of the other. This is because, for this type of substitution coincidence is certain and not accidental. In the example given, matching quantifiers are underlined.

Similarity: Applying the similarity transformation. This involves the one parameter linear operator $\mathsf{S}_{(\epsilon)}$. Such an $\mathsf{S}_{(\epsilon)}$ exists only if there exists a space of functions ψ_x, that is complete, normalisable,and not restricted to separable[1] functions, that is also a subset of the translatable functions f_x. Logically, the act of assuming such an $\mathsf{S}_{(\epsilon)}$ hypothesises that such a class of functions does indeed exist. No such function space is guaranteed. Accordingly, the assertion of proposition (7) is newly assumed information entering the system.

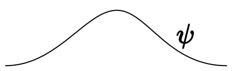

Figure 3 The linear transformations S exist only for bounded ψ_x.

$$\forall x \forall \epsilon \forall \psi_{\mathsf{x}-\epsilon} \exists \mathsf{S} \exists \psi_\mathsf{x} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} \psi_\mathsf{x}(x) = \psi_{\mathsf{x}-\epsilon}(x - \epsilon) \,. \tag{7}$$

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

In standard theory, $\mathsf{S}_{(\epsilon)}$ is set unitary by the mathematician. Doing that restricts the space of functions ψ_{x} to the Hilbert space L^2. Here, $\mathsf{S}_{(\epsilon)}$ is a member of the one parameter subgroup of the infinite dimensional, (non-unitary) general linear group $\mathsf{GL}(\mathbb{F})$. This restricts ψ_{x} not to the Hilbert space L^2 but to the Banach space L^1.

The similarity transformation is formed, thus:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} \psi_{\mathsf{x}}(x) = (\mathsf{x} + \epsilon)\, \psi_{\mathsf{x}}(x).$$

Introducing the trivial eigenformula: $\forall \psi_{\mathsf{x}} \forall x \forall \epsilon \mid \epsilon \mathbb{1} \psi_{\mathsf{x}}(x) = \epsilon \psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbb{1} \right) \psi_{\mathsf{x}}(x) = \mathsf{x} \psi_{\mathsf{x}}(x). \tag{8}$$

Now comparing the original position eigenformula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathbf{x} \psi_{\mathsf{x}}(x) = \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbb{1} \right) \psi_{\mathsf{x}}(x). \tag{9}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(\mathbb{F})$, there exists a unique linear operator $\mathbf{g}$ for real parameters ϵ, such that:

$$\forall \mathsf{S} \exists \mathbf{g} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \tag{10}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extracted, thus: $\forall \mathbf{g} \forall \eta \exists \mathbf{k} : \mathbf{g} = \eta \mathbf{k}$, implying:

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta \epsilon \mathbf{k}} \tag{11}$$

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)}^{-1} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta \epsilon \mathbf{k}} \tag{12}$$

Substitution of (11) and (12) into (9) gives:

$$\begin{aligned}
& \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \exp(+\eta \epsilon \mathbf{k})\, \mathbf{x} \exp(-\eta \epsilon \mathbf{k})\, \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbb{1} + \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right) \right] \mathbf{x} \left[\mathbb{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right) \right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} + \mathcal{O}\left(\epsilon^2\right) \right] \left[\mathbb{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right) \right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} - \eta \epsilon \mathbf{x} \mathbf{k} + \mathcal{O}\left(\epsilon^2\right) \right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && [\mathbf{k} \mathbf{x} - \mathbf{x} \mathbf{k}]\, \psi_{\mathsf{x}}(x) = \left[\eta^{-1} \mathbb{1} - \mathcal{O}(\epsilon) \right] \psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{k}, \mathbf{x}]\, \psi_{\mathsf{x}}(x) = \eta^{-1} \mathbb{1} \psi_{\mathsf{x}}(x) \tag{13}$$

And by a similar proof, conditional on the existence of eigenfunctions $\chi(k)$, of $\mathbf{k}$:

$$\forall k \forall \eta \exists \mathsf{k} \exists \chi_{\mathsf{k}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{x}, \mathbf{k}]\, \chi_{\mathsf{k}}(k) = \eta^{-1} \mathbb{1} \chi_{\mathsf{k}}(k). \tag{14}$$

Importantly, we see (13) and (14) are $\forall \eta$, rather than the particular case of $\eta^{-1} = -i$ that we see in the unitary subalgebra we know as the Canonical Commutation Relation:

$$[\mathbf{k}, \mathbf{x}] = -i \mathbb{1} \quad \text{or} \quad [\mathbf{p}, \mathbf{x}] = -i \hbar \mathbb{1} \tag{15}$$

Conclusion

The above establishes that the homogeneity symmetry is not a source of unitary information in wave mechanics. And therefore, if the reason given for postulating that quantum theory should be unitary or self-adjoint, is that symmetries in Nature are ontologically unitary, then a different reason must be found, and the postulate must be withdrawn.

This opens up the possibility of a logical modification to quantum theory, where quantum theory remains open to unitarity, but where, that unitarity (or self-adjointness) is not axiomatically imposed *by Postulate.*

As a further possibility, that modified quantum theory would be open to logical independence, well-known to Mathematical Logic, entering from elementary algebra into quantum mathematics.

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