BEAL's Conjecture: A Complete Proof

Abdelmajid Ben Hadj Salem

6,rue du Nil, Cité Soliman Er-Riadh, 8020 Soliman, Tunisia[1]

Abstract

In 1997, Andrew Beal [1] announced the following conjecture : *Let A, B, C, m, n, and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then A, B, and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l. We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers. Three numerical examples are given.

Keywords: Prime numbers, divisibility, roots of polynomials of third degree.

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O my Lord! Increase me further in knowledge.

(Holy Quran, Surah Ta Ha, 20:114.)

To my wife Wahida

1. Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1. *Let A, B, C, m, n, and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1}$$

[1] Email:abenhadjsalem@gmail.com

then $A, B,$ and C have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots A^m, B^n and $-C^l$ with the condition (1). The paper is organized as follows. In Section 2 of preliminaries, we begin with the trivial case where $A^m = B^n$. Then we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters ρ, θ that depend of A^m, B^n, C^l. The Section 3 is the main part of the paper. We write that $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2\dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$ where a, b are two positive coprime integers. We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending on each individual case, we obtain that A, B, C have or not a common factor. In the last Section, three numerical examples are presented. We finish with the conclusion.

2. Preliminaries

We begin with the trivial case when $A^m = B^n$. The equation (1) becomes:

$$2A^m = C^l \tag{2}$$

then $2|C^l \Longrightarrow 2|C \Longrightarrow \exists\, c \in N^*/\ C = 2c$, it follows $2A^m = 2^l c^l \Longrightarrow A^m = 2^{l-1}c^l$. As $l > 2$, then $2|A^m \Longrightarrow 2|A \Longrightarrow 2|B^n \Longrightarrow 2|B$. The conjecture (**??**) is verified.

We suppose in the following that $A^m > B^n$.

2.1. ***General Case***

Let $m, n, l \in N^* \ > 2$ and $A, B, C \in N^*$ such:

$$A^m + B^n = C^l \tag{3}$$

We call:

$$P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l)$$
$$+x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \tag{4}$$

Using the equation (3), $P(x)$ can be written:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n(A^m + B^n)} \tag{5}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n \tag{6}$$

$$q = A^m B^n(A^m + B^n) \tag{7}$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \tag{8}$$

Equation (5) becomes:

$$P(x) = x^3 - px + q \tag{9}$$

Using the equation (4), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{10}$$

To resolve (10) let:

$$x = u + v \tag{11}$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0 \tag{12}$$

To determine u and v, we obtain the conditions:

$$u^3 + v^3 = -q \tag{13}$$

$$uv = p/3 > 0 \tag{14}$$

Then u^3 and v^3 are solutions of the second ordre equation:

$$X^2 + qX + p^3/27 = 0 \tag{15}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27} \tag{16}$$

Let:

$$\begin{aligned} \bar{\Delta} = 27q^2 - 4p^3 &= 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m}B^{2n}(A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned} \tag{17}$$

Noting :

$$\alpha = A^m B^n > 0 \tag{18}$$

$$\beta = (A^m + B^n)^2 \tag{19}$$

we can write (17) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \tag{20}$$

As $\alpha \neq 0$, we can also rewrite (20) as :

$$\bar{\Delta} = \alpha^3 \left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3 \right) \tag{21}$$

We call t the parameter :

$$t = \frac{\beta}{\alpha} \tag{22}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t-1)^3) \tag{23}$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \tag{24}$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t) \tag{25}$$

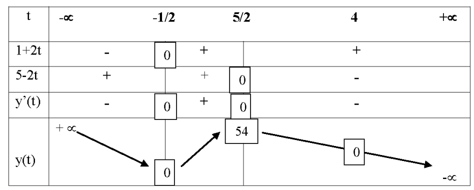

Figure 1: The table of variation

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^m B^n \tag{26}$$

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (15) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

2.2. ***Demonstration***

PROOF. The solutions of (15) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{27}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{28}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{29}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{30}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta} \tag{31}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \tag{32}$$

$$\text{and } \ sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \tag{33}$$

$$cos\theta = -\frac{q}{2\rho} < 0 \tag{34}$$

Then $\theta\,[2\pi] \in] + \frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{35}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{36}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{37}$$

Let:

$$u = re^{i\psi} \tag{38}$$

$$\text{and } \ j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{39}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \tag{40}$$

j is a complex cubic root of the unity $\Longleftrightarrow$ $j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{41}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{42}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{43}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{44}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{45}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{46}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{47}$$

$$v_2 = \overline{u_2} \tag{48}$$

$$v_3 = \overline{u_3} \tag{49}$$

We obtain as real solutions of the equation (12):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{50}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{51}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{52}$$

We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p}cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{53}$$

As $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$, then $sin\frac{\theta}{3}$ and $cos\frac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{54}$$

which is true since $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (10), we consider, as A^m is supposed great than B^n, the expressions:

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}} \tag{55}$$

3. Proof of the Main Theorem

Main Theorem: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{56}$$

then $A, B,$ and C have a common factor.

PROOF. $A^m = 2\sqrt[3]{\rho}cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3}$ is an integer. But:

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{57}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{58}$$

As A^{2m} is an integer, and p is an integer then $cos^2\frac{\theta}{3}$ must be written in the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{59}$$

with $b \in N^*$, for the last condition $a \in N^*$ and a, b coprime.

3.1. ***Case*** $cos^2\frac{\theta}{3} = \frac{1}{b}$

We obtain :

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{60}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

3.1.1. Case $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

3.1.2. Case $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2.p' \tag{61}$$

But :

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{1}{2}\right) = \frac{p}{3} = \frac{3p'}{3} = p' \tag{62}$$

On the one hand:

$$A^{2m} = (A^m)^2 = 2p' \Rightarrow 2|p' \Rightarrow p' = 2p"^2 \Rightarrow A^{2m} = 4p"^2$$
$$\Rightarrow A^m = 2p" \Rightarrow 2|A^m \Rightarrow 2|A$$

On the other hand:

$B^nC^l = p' = 2p"^2 \Rightarrow 2|B^n$ or $2|C^l$. If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A^m$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : A,B and C solutions of (3) have a common factor. Also if $2|C^l$, we obtain the same result : A,B and C solutions of (3) have a common factor.

3.1.3. Case $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{3} = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Longrightarrow p'$ is not a prime. Let μ a prime with $\mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A$.

On the other hand:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 5p'$$

Then $\mu|B^n$ or $\mu|C^l$. If $\mu|B^n \Rightarrow \mu|B$. As $C^l = A^m + B^n$ and $\mu|A$ and $\mu|B$, it follows $\mu|A^m$ and $\mu|B^n$ then $\mu|(A^m + B^n) \Rightarrow \mu|C^l \Longrightarrow \mu|C$.

Then, we have : A,B and C solutions of (3) have a common factor. Also if $\mu|C^l$, we obtain the same result : A,B and C solutions of (3) have a common factor.

3.2. ***Case*** $a > 1$, $cos^2\frac{\theta}{3} = \frac{a}{b}$

That is to say:

$$cos^2\frac{\theta}{3} = \frac{a}{b} \tag{63}$$

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{64}$$

and a, b verify one of the two conditions:

$$\boxed{\{3|p \quad and \quad b|4p\}} \quad \text{or} \quad \boxed{\{3|a \quad and \quad b|4p\}} \tag{65}$$

and using the equation (36), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{66}$$

In these conditions, respectively, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (65).

3.2.1. ***Hypothesis:*** $\{3|p \quad and \quad b|4p\}$

3.2.1.1. Case $b = 2$ *and* $3|p$ *:.* $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a \tag{67}$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \tag{68}$$

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

3.2.1.2. Case $b = 4$ *and* $3|p$ *:.* We have $3|p \Longrightarrow p = 3p'$ with $p' \in N^*$, it follows:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \tag{69}$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \tag{70}$$

But a, b are coprime. Then the case $b = 4$ and $3|p$ is impossible.

3.2.1.3. Case: $b \neq 2, b \neq 4$, $b|p$ and $3|p$:. As $3|p$ then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3} cos^2 \frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b} \tag{71}$$

We consider the case: $b|p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (if $p" = 1$, then $p = 3b$, see sub-paragraph **II. Case k'=1** of paragraph **3.2.1.8**). Hence :

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{72}$$

Let us calculate B^nC^l:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = b.p".\frac{3b - 4a}{b} = p".(3b - 4a) \tag{73}$$

Finally, we have the two equations:

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{74}$$

$$B^nC^l = p".(3b - 4a) \tag{75}$$

I. Case $p"$ is prime:

From (74), $p"|A^{2m} \Rightarrow p"|A^m \Rightarrow p"|A$. From (75), $p"|B^n$ or $p"|C^l$. If $p"|B^n \Rightarrow p"|B$, as $C^l = A^m + B^n \Rightarrow p"|C^l \Rightarrow p"|C$. If $p"|C^l \Rightarrow p"|C$, as $B^n = C^l - A^m \Rightarrow p"|B^n \Rightarrow p"|B$.

Then A,B and C solutions of (3) have a common factor.

II. Case $p"$ is not prime:

Let λ one prime divisor of $p"$. From (74), we have :

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\, \lambda\, \text{is prime then} \quad \lambda|A \tag{76}$$

From (75), as $\lambda|p"$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or}\ \lambda|C^l \tag{77}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also :

$$\lambda|C^l \quad \text{as}\, \lambda\, \text{is prime, then} \quad \lambda|C \tag{78}$$

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$.

Then: A, B and C solutions of (3) have a common factor.

Let us verify the condition (66) given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{79}$$

The condition $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ is verified.
If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^n Y - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

<u>*3.2.1.4. Case $b = 3$ and $3|p$:.*</u> As $3|p \Longrightarrow p = 3p'$ and we write :

$$A^{2m} = \frac{4p}{3} cos^2 \frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3} \tag{80}$$

As A^{2m} is an integer and that a and b are coprime and $cos^2 \frac{\theta}{3}$ can not be one in reference to the equation (35), then we have necessary $3|p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3 \times 3p" = 9$ but $p = A^{2m} + B^{2n} + A^m B^n > 9$, the hypothesis $p" = 1$ is impossible, then $p" > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a \tag{81}$$

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2 \frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9 - 4a)}{3} = p".(9 - 4a) \tag{82}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow a = 2$ as $a > 1$. $a = 2$, we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a = 8p" \tag{83}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9-4a)}{3} = p" \tag{84}$$

The two last equations give that $p"$ is not prime. Then we use the same methodology described above for the case **3.2.1.3.**, and we have : A,B and C solutions of (3) have a common factor.

3.2.1.5. Case $3|p$ and $b = p$:. We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3} \tag{85}$$

As A^{2m} is an integer, this implies that $3|a$, but $3|p \Longrightarrow 3|b$. As a and b are coprime, hence the contradiction. Then the case $3|p$ and $b = p$ is impossible.

3.2.1.6. Case $3|p$ and $b = 4p$:. $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3|a \tag{86}$$

because A^{2m} is an integer. But $3|p \Longrightarrow 3|\left[(4p) = b\right]$, that is in contradiction with the hypothesis a, b are coprime. Then the case $b = 4p$ is impossible.

3.2.1.7. Case $3|p$ and $b = 2p$:. $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3|a \tag{87}$$

because A^{2m} is an integer. But $3|p \Longrightarrow 3|(2p) \Longrightarrow 3|b$, that is in contradiction with the hypothesis a, b are coprime. Then the case $b = 2p$ is impossible.

3.2.1.8. Case $3|p$ and $b \neq 3$ is a divisor of p :. We have $b = p' \neq 3$, and p is written as:

$$p = kp' \quad with \quad 3|k \Longrightarrow k = 3k' \tag{88}$$

and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4\times 3.k'p'}{3}\frac{a}{p'} = 4ak' \tag{89}$$

We calculate B^nC^l:

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) \tag{90}$$

I. Case $k' \neq 1$:

We suppose $k' \neq 1$, we use the same methodology described for the case **3.1.2.3.**, and we obtain: A, B and C solutions of (3) have a common factor.

II. Case $k' = 1$:

We have $k' = 1 \Longrightarrow p = 3b$, then we have:

$$A^{2m} = 4a \Longrightarrow a \quad \text{is even} \tag{91}$$

and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a \tag{92}$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{93}$$

The left member of (93) is an integer and b also, then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written in the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{94}$$

where k_1, k_2 are two coprime integers and $k_2|b \Longrightarrow b = k_2.k_3$.

II.1. Case $k_3 \neq 1$:

We suppose $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^mB^n = k_3.k_1 \tag{95}$$

Let μ is an prime integer such that $\mu|k_3$. If $\mu = 2 \Rightarrow 2|b$, but $2|a$ that is contradiction with a, b coprime. We suppose $\mu \neq 2$ and $\mu|k_3$, then:

$$\boxed{\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m \text{ or } \mu|(A^m + 2B^n)} \tag{96}$$

II.1.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|4a \Longrightarrow \mu|a$. As $\mu|k_3 \Longrightarrow \mu|b$ and that a, b are coprime hence the contradiction.

II.1.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m+2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m+2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{97}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{98}$$

As $p = 3b = 3k_2.k_3$ and $\mu|k_3$ hence $\mu|p \Longrightarrow \ p = \mu\mu'$, so we have :

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{99}$$

then:

$$\boxed{\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n \text{ or } \mu|(B^n - A^m)} \tag{100}$$

II.1.2.1. Case $\mu|B^n$:

If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with case **II.1.2.** above.

II.1.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \tag{101}$$

II.1.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$, using the result above of **II.1.2.1.** of this paragraph, it is impossible.

II.1.2.2.2. Case $\mu = 3$:

If $\mu = 3 \Longrightarrow 3|k_3 \Longrightarrow k_3 = 3k_3'$, and we have $b = k_2k_3 = 3k_2k_3'$, it follows $p = 3b = 9k_2k_3'$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then :

$$9k_2k_3' - 3A^mB^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k_3' - A^mB^n) = (A^m - B^n)^2 \tag{102}$$

hence :

$$\boxed{3|(3k_2k_3' - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow 3|A^m \text{ or } 3|B^n} \tag{103}$$

II.1.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3|4a \Longrightarrow 3|a$. As $b = 3k_2k_3'$ then $3|b$, but a, b are coprime hence the contradiction. Then $3 \nmid A$.

II.1.2.2.2.2. Case $3|B^n$:

If $3|B^n \Longrightarrow 3|B$, but the (102) gives $3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n) \Longrightarrow 3|A^m \Longrightarrow 3|(A^{2m} = 4a) \Rightarrow 3|a$. As $3|b$ then the contradiction with a, b coprime.

Then the hypothesis $k_3 \neq 1$ is impossible.

III. Case $k_3 = 1$:

Now we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{104}$$

with k_1, b coprime. We write (104) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \tag{105}$$

which implies that :

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{106}$$

III.1. Case $3|a$:

If $3|a$, as $A^{2m} = 4a \Longrightarrow 3|A^{2m} \Longrightarrow 3|A$ and $3|a$. But $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n)$. But $3|A$ hence $3|B^n \Longrightarrow 3|B$, as $m \geq 3 \Longrightarrow 3^2|p$, it follows $3|b$ then the contradiction with a, b coprime.

III.2. Case $3|(b-a)$:

Considering now that $3|(b-a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (95) and that $3|k_1 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \ or \ 3|(A^m + 2B^n)}$.

III.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A \Longrightarrow 3|A^{2m}$ then $3|4a \Longrightarrow 3|a$. But $3|(b-a) \Longrightarrow 3|b$ hence the contradiction with a, b are coprime.

III.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{107}$$

But $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^mB^n)$ or $9|(3b - 3A^mB^n)$, then $3|(b - A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $A^{2m} = 4a = (A^m)^2 \Longrightarrow \exists a' \in N^*$ and $a = a'^2 \Longrightarrow A^m = 2a'$. We arrive to:

$$\boxed{3|(a'^2 - 2a'B^n) \Rightarrow 3|a'(a' - 2B^n) \Rightarrow 3|a' \quad or \quad 3|(a' - 2B^n)} \tag{108}$$

III.2.2.1. Case $3|a'$:

If $3|a' \Rightarrow 3|a'^2 \Rightarrow 3|a$, but $3|(b-a) \Rightarrow 3|b$, then the contradiction with a, b coprime.

III.2.2.2. Case $3|(a'-2B^n)$:

Now if $3|(a'-2B^n) \Rightarrow 3|(2a'-4B^n) \Rightarrow 3|(A^m-4B^n) \Rightarrow 3|(A^m-B^n)$, we refind the case **III.2.2.**, equation (107), that has a solution given by the case **2.2.1.** above.

Then, the study of the case **3.2.1.8.** is finished.

3.2.1.9 Case $3|p$ and $b|4p$:. As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in N^*$ and $4p = 12p' = k_1 b$.

I. Case $k_1 = 1$:

If $k_1 = 1$, then $b = 12p'$, ($p' \neq 1$ if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{12p'}{3}\dfrac{a}{b} = \dfrac{4p'.a}{12p'} = \dfrac{a}{3} \Rightarrow 3|a$ because A^{2m} is an integer, then the contradiction with a, b coprime.

II. Case $k_1 = 3$:

If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{k_1.a}{3} = a$.

Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - \frac{a}{2} \tag{109}$$

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \tag{110}$$

The left member of the equation (110) is an integer and also p', then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written as :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3} \tag{111}$$

where k_2, k_3 are two coprime integers and:

$$k_3|p' \Longrightarrow \exists k_4 \in N^* \quad and \quad p' = k_3.k_4 \tag{112}$$

II.1. Case $k_4 \neq 1$:

We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \tag{113}$$

Let μ one prime integer with:

$$\mu | k_4 \tag{114}$$

Then :

$$\boxed{\mu | A^m(A^m + 2B^n) \Longrightarrow \mu | A^m \quad \text{or} \quad \mu | (A^m + 2B^n)} \tag{115}$$

II.1.1. Case $\mu | A^m$:

If $\mu | A^m \Longrightarrow \mu | A^{2m} \Longrightarrow \mu | a$. As $\mu | k_4 \Longrightarrow \mu | p' \Rightarrow \mu | (4p' = b)$. But a, b are coprime then the contradiction.

II.1.2. Case $\mu | (A^m + 2B^n)$:

If $\mu | (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu | (A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{116}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n(B^n - A^m) \tag{117}$$

As $p = 3p'$ and $\mu | p' \Rightarrow \mu | (3p') \Rightarrow \mu | p$, we can write :$\exists \mu' \in N^*$ and $p = \mu\mu'$, then we obtain :

$$\mu'\mu = \mu(\mu t'^2 - 2t' B^n) + B^n(B^n - A^m) \tag{118}$$

and:

$$\boxed{\mu | B^n(B^n - A^m) \Longrightarrow \mu | B^n \quad \text{or} \quad \mu | (B^n - A^m)} \tag{119}$$

II.1.2.1. Case $\mu|B^n$:

If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with the case **II.1.2.** above.

II.1.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\boxed{\mu|3B^n} \tag{120}$$

II.1.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ it is impossible, see the case **II.1.2.1.** above.

II.1.2.2.2 Case $\mu = 3$:

If $\mu = 3 \Longrightarrow 3|k_4 \Longrightarrow k_4 = 3k_4'$, and we obtain $p' = k_3k_4 = 3k_3k_4'$, it follows $p = 3p' = 9k_3k_4'$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then:

$$9k_4k_5' - 3A^mB^n = (A^m - B^n)^2$$

that we write :

$$3(3k_4k_5' - A^mB^n) = (A^m - B^n)^2 \tag{121}$$

then $3|(3k_4k_5' - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow \boxed{3|A^m \quad or \quad 3|B^n}$.

II.1.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b coprime. Then $3 \nmid A$.

II.1.2.2.2.2. Case $3|B^n$:

If $3|B^n$ and using (116), we have $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3|A^m \Rightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b coprime.

Then the hypothesis $k_4 \neq 1$ is impossible.

II.2. Case $k_4 = 1$:

We suppose that $\boxed{k_4 = 1} \Longrightarrow p' = k_3 k_4 = k_3$. Then we obtain:

$$2\sqrt{3} sin \frac{2\theta}{3} = \frac{k_2}{p'} \tag{122}$$

with k_2, p' coprime, we write (122) as :

$$4\sqrt{3} sin \frac{\theta}{3} cos \frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2 \frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2 \tag{123}$$

that implies:

$$\boxed{3|a \quad or \quad 3|(b-a)} \tag{124}$$

II.2.1. Case $3|a$:

If $3|a \Rightarrow 3|A^{2m} \Rightarrow 3|A$, as $p = (A^m - B^n)^2 + 3A^m B^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 9|(A^m - B^n)^2$. But $(A^m - B^n)^2 = p - 3A^m B^n = 3b - 3A^m B^n \Longrightarrow 3|(b - A^m B^n)$. As $3|A^m \Longrightarrow 3|b \Longrightarrow$ the contradiction with a, b coprime.

II.2.2. Case $3|(b-a)$:

We consider that $3|(b-a)$. As $k_2 = A^m(A^m + 2B^n)$ given by the equation (113) and that $3|k_2 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow \boxed{3|A^m \quad or \quad 3|(A^m + 2B^n)}$.

II.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with a, b coprime.

II.2.2.2. Case $3|(A^m + 2B^n)$:

If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{125}$$

but $p = A^{2m} + B^{2n} + A^m B^n = (A^m - B^n)^2 + 3A^m B^n$ then $p - 3A^m B^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^m B^n)$ or $9|(3p' - 3A^m B^n)$, then $3|(p' - A^m B^n) \Rightarrow$

$3|4(p' - 4A^mB^n) \Rightarrow 3|(b - 4A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $3|(A^{2m} - 4A^mB^n) \Rightarrow \boxed{3|A^m(A^m - 4B^n)}$.

II.2.2.2.1. Case $3|A^m$:

If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$ then the contradiction with a, b coprime.

II.2.2.2.2. Case $3|(A^m - 4B^n)$:

Now if $3|(A^m - 4B^n) \Longrightarrow 3|(A^m - B^n)$, we refind the hypothesis of the beginning (125) above, that has a solution **II.2.2.2.1.**.

III. Case $k_1 \neq 3$ and $3|k_1$:

We suppose $k_1 \neq 3$ and $3|k_1 \Rightarrow k_1 = 3k'1$ with $k'_1 \neq 1$. We have $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{3k'_1 b}{3}\frac{a}{b} = k'_1 a \tag{126}$$

and B^nC^l:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{k'_1}{4}(3b - 4a) \tag{127}$$

As B^nC^l is an integer, we must have $\boxed{4|(3b-4a) \quad \text{or} \quad 4|k'_1}$.

III.1. Case $4|(3b - 4a)$:

We suppose that $4|(3b - 4a) \Rightarrow \dfrac{3b - 4a}{4} = c \in N^*$, and we obtain:

$$A^{2m} = k'_1 a$$
$$B^nC^l = k'_1 c$$

III.1.1. Case k'_1 is prime:

If k'_1 is prime, then $k'_1|A^{2m} \Rightarrow k'_1|A$ and $k'_1|B^nC^l \Rightarrow k'_1|B^n$ or $k'_1|C^l$. If $k'_1|B^n \Rightarrow k'_1|B$, then $k'_1|C^l \Rightarrow k'_1|C$. With the same method if $k'_1|C^l$, we arrive to $k'_1|B$.

We obtain: A,B and C solutions of (3) have a common factor.

III.1.2. Case k'_1 not a prime:

We suppose k'_1 not a prime. Let μ a prime divisor of k'_1, as described in **III.1.1.** above, we obtain : A,B and C solutions of (3) have a common factor.

III.2. Case $4|k'_1$:

Now, we suppose that $4|k'_1$.

III.2.1. Case $k'_1 = 4$:

We suppose $k'_1 = 4$, then $A^{2m} = 4a$ and $B^nC^l = 4c$, It is easy to verify that 2 is a common factor of A, B, C.

We obtain: A,B and C solutions of (3) have a common factor.

III.2.2. Case $k'_1 = 4k"_1$:

If $k'_1 = 4k"_1$ with $k"_1 > 1$. Then, we have:

$$A^{2m} = 4k"_1 a \tag{128}$$

$$B^nC^l = k"_1(3b - 4a) \tag{129}$$

III.2.2.1. Case $k"_1$ prime:

If $k"_1$ is prime, then $k"_1|A^{2m} \Rightarrow k"_1|A$ and $k"_1|B^nC^l \Rightarrow k"_1|B^n$ or $k"_1|C^l$. If $k"_1|B^n \Rightarrow k"_1|B$, then $k"_1|C^l \Rightarrow k"_1|C$. With the same method if $k"_1|C^l$, we arrive to $k"_1|B$.

We obtain: A,B and C solutions of (3) have a common factor.

III.2.2.2. Case $k"_1$ not a prime:

If $k"_1$ not a prime. Let μ a prime divisor of $k"_1$, as described in case **III.2.2.1.** above, we obtain : A,B and C solutions of (3) have a common factor.

3.2.2. ***Hypothesis*** *:* $\{3|a \quad and \quad b|4p\}$

We have :

$$3|a \Longrightarrow \exists a' \in N^* \ / \ a = 3a' \tag{130}$$

3.2.2.1. Case $b = 2$ *and* $3|a$ *:.* A^{2m} is written as :

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{131}$$

Using the equation (130), A^{2m} becomes:

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{132}$$

But $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

3.2.2.2. Case $b = 4$ *and* $3|a$ *:.* A^{2m} is written as :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{133}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{134}$$

which is impossible.

Then the case $b = 4$ is impossible.

3.2.2.3. Case $b = p$ *and* $3|a$ *:.* Then:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{135}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{136}$$

$$\exists a" \in N^* \ / \ a' = a"^2 \tag{137}$$

We calculate A^mB^n, hence:

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{138}$$

The left member of (138) is an integer and p is also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{139}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = b = k_2.k_3,\ k_3 \in N^*$.

I. Case $k_3 \neq 1$:

We suppose that $k_3 \neq 1$. We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{140}$$

Let us μ a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n)$. Hence:

$$\boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)} \tag{141}$$

I.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu|4a' \Longrightarrow (\mu = 2$ but $2|a')$ or $\mu|a'$. Then $\mu|a$ hence the contradiction with a, b coprime.

I.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{142}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{143}$$

Since $p = b = k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists\mu' \in N^*$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{144}$$

From the last equation, we get $\mu|B^n(B^n - A^m) \Longrightarrow \boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

I.2.1. Case $\mu|B^n$:

If $\mu|B^n$ which is contradiction with $\mu \nmid B^n$.

I.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we arrive to:

$$\mu|3B^n \Longrightarrow \begin{cases} \boxed{\mu|B^n} \\ or \\ \boxed{\mu = 3} \end{cases} \tag{145}$$

I.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ which is contradiction with $\mu \nmid B$ from **I.2. Case** $\mu|(A^m + 2B^n)$.

I.2.2.2. Case $\mu = 3$:

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ then the contradiction with a, b coprime.

II. Case $k_3 = 1$:

We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1 \tag{146}$$

$$b = k_2 \tag{147}$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{b} \tag{148}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p - a) = k_1^2 \tag{149}$$

but $a' = a"^2$ then $p - a$ is a square. Let us:

$$\lambda^2 = p - a \tag{150}$$

The equation (149) becomes:

$$4^2 a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{151}$$

taking the positive square root. Using (146), we get :

$$k_1 = 4a"\lambda \tag{152}$$

But $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda \tag{153}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not, $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2 \Longrightarrow 2|(p-a)$ but a is even, then $2|p \Longrightarrow 2|b$ which is contradiction with a, b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|\lambda^2 \quad and \quad \lambda_1|(A^m + 2B^n) \tag{154}$$

$$\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{155}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(p = b) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{156}$$

hence the contradiction with a, b coprime.

II.1. Case $\lambda_1 \nmid A^m$ and $\lambda_1|(A^m + 2B^n)$:

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \Longrightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(p - a)$ hence by difference, we obtain $\lambda_1|(a - 3B^{2n})$ or $\lambda_1|(3a' - 3B^{2n}) \Longrightarrow \lambda_1|3(a' - B^{2n})$, Then:

$$\boxed{\lambda_1 = 3 \quad or \quad \lambda_1|(a' - B^{2n})} \tag{157}$$

II.1.1. Case $\lambda_1 = 3$:

If $\lambda_1 = 3$ but $3|a$, as $\lambda_1|(p-a) \Longrightarrow 3|(p=b)$ hence the contradiction with a, b coprime.

II.1.2. Case $\lambda_1|(a' - B^{2n})$:

If $\lambda_1|(a' - B^{2n}) \Longrightarrow \lambda_1|(a"^2 - B^{2n}) \Longrightarrow \boxed{\lambda_1|(a" - B^n)(a" + B^n)} \Longrightarrow \lambda_1|(a" + B^n)$ or $\lambda_1|(a" - B^n)$, because $(a" - B^n) \neq 1$, if not, we obtain $a"^2 - B^{2n} = a" + B^n \Longrightarrow a"^2 - a" = B^n - B^{2n}$. The left member is positive and the right member is negative, then the contradiction.

II.1.2.1. Case $\lambda_1|(a" - B^n)$:

If $\lambda_1|(a" - B^n) \Longrightarrow \lambda_1|2(a" - B^n) \Longrightarrow \lambda_1|(A^m - 2B^n)$ but $\lambda_1|(A^m + 2B^n)$ hence $\lambda_1|2A^m \Longrightarrow \lambda_1|A^m$ as $\lambda_1 \neq 2$, it follows $\lambda_1|A^m$ hence the contradiction with (155).

II.1.2.2. Case $\lambda_1|(a" + B^n)$:

If $\lambda_1|(a" + B^n) \Longrightarrow \lambda_1|2(a" + B^n) \Longrightarrow \lambda_1|(2a" + 2B^n) \Rightarrow \lambda_1|(A^m + 2B^n)$. We find the case **II.1.** that has solutions.

Then the case $k_3 = 1$ is impossible.

3.2.2.4. Case $b|p \Rightarrow \ p = b.p', p' > 1,\ b \neq 2,\ b \neq 4$ and $3|a$:.

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{158}$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{159}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a') \tag{160}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^n C^l = p'(b - 4a') \tag{161}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{162}$$

I. Case λ a prime divisor of p':

Let λ a prime divisor of p' (we suppose p' not prime). From (162), we have:

$$\lambda | A^{2m} \Rightarrow \lambda | A^m \quad \text{as}\,\lambda\,\text{is a prime, then} \;\; \lambda | A \tag{163}$$

From (161), as $\lambda | p'$ we have:

$$\lambda | B^n C^l \Rightarrow \lambda | B^n \quad \text{or } \lambda | C^l \tag{164}$$

If $\lambda | B^n$, λ is a prime $\lambda | B$, but $C^l = A^m + B^n$, then we have also :

$$\lambda | C^l \quad \text{as}\,\lambda\,\text{is a prime, then } \lambda | C \tag{165}$$

By the same way, if $\lambda | C^l$, we obtain $\lambda | B$. then : A, B and C solutions of (3) have a common factor.

II. Case p' is a prime number:

We suppose now that p' is prime, from the equations (161) and (162), we obtain that:

$$p' | A^{2m} \Rightarrow p' | A^m \Rightarrow p' | A \tag{166}$$

and:

$$p' | B^n C^l \Rightarrow p' | B^n \quad \text{or } p' | C^l \tag{167}$$

$$\text{If} \quad p' | B^n \Rightarrow p' | B \tag{168}$$

$$\text{As} \quad C^l = A^m + B^n \quad \text{and that} \;\; p' | A, p' | B \Rightarrow p' | A^m, p' | B^n \Rightarrow p' | C^l$$
$$\Rightarrow p' | C \tag{169}$$

By the same way, if $p' | C^l$, we arrive to $p' | B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.5. Case $b = 2p$ and $3|a$:. We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' \Longrightarrow 2|A^m \Longrightarrow 2|a \Longrightarrow 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with a, b coprime.

3.2.2.6. Case $b = 4p$ and $3|a$:. We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a'$$

Calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$

$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} \tag{170}$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{171}$$

The left member of (171) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{172}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = k_2.k_3$.

I. Case $k_3 \neq 1$:

Firstly, we suppose that $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^mB^n = k_3.k_1 \tag{173}$$

Let μ a prime integer and $\mu|k_3$, then $\mu|A^m(A^m+2B^n) \Longrightarrow \boxed{\mu|A^m \quad or \quad \mu|(A^m + 2B^n)}$.

I.1. Case $\mu|A^m$:

If $\mu|A^m \Longrightarrow \mu|(A^{2m} = a') \Rightarrow \mu|(3a' = a)$. As $\mu|k_3 \Longrightarrow \mu|p \Rightarrow \mu|(4p = b)$. Then the contradiction with a, b coprime.

I.2. Case $\mu|(A^m + 2B^n)$:

If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{174}$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t' \quad t' \in N^* \tag{175}$$

Then :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{176}$$

As $b = 4p = 4k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu' \in N^*$ that $b = \mu\mu'$, we obtain:

$$\mu'\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m) \tag{177}$$

The last equation implies $\mu|4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\boxed{\mu|B^n \quad or \quad \mu|(B^n - A^m)}$.

I.2.1. Case $\mu|B^n$:

If $\mu|B^n$ then the contradiction with (174).

I.2.2. Case $\mu|(B^n - A^m)$:

If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\boxed{\mu|3B^n \Longrightarrow \mu|B^n \quad or \quad \mu = 3} \tag{178}$$

I.2.2.1. Case $\mu|B^n$:

If $\mu|B^n$ it is contradiction with (174).

I.2.2.2. Case $\mu = 3$:

If $\mu = 3$, then $b = 3\mu'$, but $3|a$ which is contradiction with a, b coprime.

II. Case $k_3 = 1$:

We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \tag{179}$$

$$p = k_2 \tag{180}$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{p} \tag{181}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{182}$$

but $a' = a"^2$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \tag{183}$$

The equation (182) becomes:

$$a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = a" \lambda \tag{184}$$

taking the positive square root. Using (179), we get :

$$k_1 = a" \lambda \tag{185}$$

But $k_1 = A^m (A^m + 2B^n) = a"(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \tag{186}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2 | \lambda \Longrightarrow 2 | \lambda^2$. As $2 | (b = 4p) \Longrightarrow 2 | (a = 3a')$ which is contradiction with a, b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|(A^m + 2B^n) \tag{187}$$

$$\Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{188}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(b = 4p) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a\text{''} \Longrightarrow \lambda_1|a \tag{189}$$

hence the contradiction with a, b coprime.

II.1. Case $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$ and $\lambda_1|(A^m + 2B^n)$:

We assume now $\lambda_1 \nmid A^m$, $\lambda_1 \nmid B^n$. $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^m B^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^m B^n + 3B^{2n}) \Longrightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Longrightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(4p - a)$ hence by difference, we obtain $\lambda_1|(a - 3(B^{2n} + p))$ or $\lambda_1|(3a' - 3(B^{2n} + p)) \Longrightarrow \lambda_1|3(a' - B^{2n} - p) \Longrightarrow \boxed{\lambda_1 = 3 \text{ or } \lambda_1|(a' - (B^{2n} + p))}$.

II.1.1. Case $\lambda_1 = 3$:

If $\lambda_1 = 3|\lambda \Rightarrow 3|\lambda^2 \Rightarrow 3|b - a$ but $3|a \Longrightarrow 3|(p = b)$ hence the contradiction with a, b coprime.

II.1.2. Case $\lambda_1|(a' - (B^{2n} + p))$:

If $\lambda_1 \neq 3$ and $\lambda_1|(a' - B^{2n} - p) \Longrightarrow \lambda_1|(A^m B^n + B^{2n}) \Longrightarrow \lambda_1|B^n(A^m + 2B^n) \Longrightarrow \boxed{\lambda_1|B^n \quad or \quad \lambda_1|(A^m + 2B^n)}$.

II.1.2.1. Case $\lambda_1|B^n$:

If $\lambda_1|B^n$ that is in contradiction with the hypothesis $\lambda_1 \nmid B$ cited above case II.1.

II.1.2.2. Case $\lambda_1|(A^n + 2B^n)$:

If $\lambda_1|(A^n + 2B^n)$. We refind this condition in the case II.1.

Then the case $k_3 = 1$ is impossible.

3.2.2.7. Case $3|a$ and $b = 2p'$ $b \neq 2$ with $p'|p$:. $3|a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, hence:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \tag{190}$$

Calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{191}$$

But $\sqrt[3]{\rho^2} = \frac{p}{3}$ hence en using $cos^2\frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a') \tag{192}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = k(p' - 2a') \tag{193}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{194}$$

I. Case λ is a prime divisor of k:

We suppose that λ is a prime divisor of k (we suppose k not a prime). From (194), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{195}$$

From (193), as $\lambda|k$, we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l \tag{196}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|C \tag{197}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (3) have a common factor.

II. Case k is prime:

Now, we suppose now that k is prime, from the equations (193) and (194), we obtain:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{198}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{199}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{200}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\ k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{201}$$

By the same way, if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$:. $3|a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$, if not, $b = 4p$ a case that has been studied (paragraph **3.2.2.6**), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \tag{202}$$

Writing B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{203}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence en using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a') \tag{204}$$

As $p = b.p'$, and $p' > 1$, we have:

$$B^nC^l = k(p' - 2a') \tag{205}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{206}$$

I. Case λ a prime divisor of k:

Let λ a prime divisor of k (we suppose k not a prime). From (206), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{207}$$

From (205), as $\lambda|k$ we obtain:

$$\lambda|B^nC^l \Rightarrow \boxed{\lambda|B^n \quad or \quad \lambda|C^l} \tag{208}$$

I.1 Case $\lambda|B^n$ or $\lambda|C^n$:

If $\lambda|B^n$, λ is a prime, then $\lambda|B$, and as $\lambda|A \Rightarrow \lambda|(A^m + B^n = C^l) \Rightarrow \lambda|C$. By the same way if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (3) have a common factor.

II. Case k is prime:

We suppose now that k is prime, from the equations (205) and (206), we have:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{209}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or}\ k|C^l \tag{210}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{211}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that}\ k|A, k|B \Rightarrow k|A^m,\, k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{212}$$

By the same way if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (3) have a common factor.

3.2.2.9. Case $3|a$ and $b|4p$:. $a = 3a'$ and $4p = k_1b$ with $k_1 \in N^*$. As $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3} = \dfrac{4p}{3}\dfrac{3a'}{b} = k_1a'$ and B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b - 4a') \tag{213}$$

As B^nC^l is an integer, we must have $\boxed{4|k_1 \quad or \quad 4|(b - 4a')}$.

I. Case $k_1 = 1$:

If $k_1 = 1 \Rightarrow b = 4p$: it is the case (**3.2.2.6**) above.

II. Case $k_1 = 4$:

If $k_1 = 4 \Rightarrow p = b$: it is the case (**3.2.2.3**) above.

III. Case $4|k_1$:

We suppose that $4|k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k_1'$, then we have:

$$A^{2m} = 4k_1'a'$$

$$B^nC^l = k_1'(b - 4a')$$

By discussing k_1' is a prime integer or not, we arrive easily to: A ,B and C solutions of (3) have a common factor.

III.1. Case $4 \nmid (b - 4a')$ and $4 \nmid k_1'$:

If $4 \nmid (b - 4a')$ and $4 \nmid k_1'$ it is impossible.

III.2. Case $4|(b - 4a')$:

If $4|(b - 4a') \Rightarrow (b - 4a') = 4c$, with $c \in N^*$, then we obtain:

$$A^{2m} = k_1a'$$

$$B^nC^l = k_1c$$

By discussing k_1 is a prime integer or not, we arrive easily to: A ,B and C solutions of (3) have a common factor.

The main theorem is proved.

4. Numerical Examples

4.1. Example 1:

We consider the example:

$$6^3 + 3^3 = 3^5 \tag{214}$$

with $A^m=6^3$, $B^n=3^3$ and $C^l=3^5$. With the notations used in the paper, we obtain:

$$p=3^6\times 73, \tag{215}$$

$$q=8\times 3^{11}, \tag{216}$$

$$\bar{\Delta}=4\times 3^{18}(3^7\times 4^2-73^3)<0, \tag{217}$$

$$\rho=\frac{p\sqrt{p}}{3\sqrt{3}}=\frac{3^8\times 73\sqrt{73}}{3}, \tag{218}$$

$$cos\theta=-\frac{4\times 3^3\times\sqrt{3}}{73\sqrt{73}} \tag{219}$$

As $A^{2m}=\frac{4p}{3}.cos^2\frac{\theta}{3}\Longrightarrow cos^2\frac{\theta}{3}=\frac{3A^{2m}}{4p}=\frac{3\times 2^4}{73}=\frac{a}{b}\Longrightarrow a=3\times 2^4,\ b=73;$ then:

$$cos\frac{\theta}{3}=\frac{4\sqrt{3}}{\sqrt{73}} \tag{220}$$

$$p=3^6 b \tag{221}$$

Let us verify the equation (219) using the equation (220):

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{4\sqrt{3}}{\sqrt{73}}\right)^3-3\frac{4\sqrt{3}}{\sqrt{73}}=-\frac{4\times 3^3\times\sqrt{3}}{73\sqrt{73}} \tag{222}$$

That's OK. For this example, we can use the two conditions of (65) as $3|p$,$b|4p$ and $3|a$. The cases **3.2.1.3** and **3.2.2.4** are respectively used. We find for both cases that A^m, B^n and C^l of the equation (214) have a common prime factor which is true.

4.2. Example 2:

Let the second example:

$$7^4+7^3=14^3\Rightarrow 2401+343=2744 \tag{223}$$

With the notations of the paper, we take:

$$A^m=7^4 \tag{224}$$

$$B^n=7^3 \tag{225}$$

$$C^l=14^3 \tag{226}$$

We obtain:

$$p = 57 \times 7^6 = 3 \times 19 \times 7^6 \tag{227}$$

$$q = 8 \times 7^{10} \tag{228}$$

$$\overline{\Delta} = 27q^2 - 4p^3 = 27 \times 4 \times 7^{18}(16 \times 49 - 19^3)$$
$$= -27 \times 4 \times 7^{18} \times 6075 < 0 \tag{229}$$

$$\rho = \tfrac{p\sqrt{p}}{3\sqrt{3}} = 19 \times 7^9 \times \sqrt{19} \tag{230}$$

$$cos\theta = \frac{-q}{2\rho} = -\frac{4 \times 7}{19\sqrt{19}} \tag{231}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{7^2}{4 \times 19} = \dfrac{a}{b} \Longrightarrow a = 7^2,\ b = 4 \times 19;$ then:

$$cos\frac{\theta}{3} = \frac{7}{2\sqrt{19}} \tag{232}$$

$$3|p \quad and \quad b|(4p) \tag{233}$$

Let us verify the equation (231) using the equation (232):

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{7}{2\sqrt{19}}\right)^3 - 3\frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}} \tag{234}$$

It is the same value of (231)!

Now, from (233), we have $3|p \Rightarrow p = 3p'$, $b|(4p)$ with $b \neq 2, 4$ then $12p' = k_1 b = 3 \times 7^6 b$. It concerns the paragraph **3.2.1.9.** of the first hypothesis. As $k_1 = 3 \times 7^6 = 3k'_1$ with $k'_1 = 7^6 \neq 1$. It is the case **III.**, with the two conditions: $4|(3b - 4a)$ or $4|k'_1$. We take $4|(3b - 4a)$. Let us calculate $3b - 4a$:

$$3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \Longrightarrow 4|(3b - 4a) \tag{235}$$

Then it is the sous-case **III.1.** with $A^{2m} = 7^8 = 7^6 \times 7^2 = k'_1.a$ with k'_1 not a prime, we find the sous-case **III.1.2** with the result that A, B and C have a common factor namely the prime number 7 a divisor of $k'_1 = 7^6$!.

4.3. Example 3:

Let the third example:

$$7^2 + 2^5 = 3^4 \tag{236}$$

with:

$$A^m = 7^2; B^n = 2^5; C^l = 3^4$$

We obtain:

$$p = 4999 \quad a\, prime\, number \tag{237}$$

$$q = 2^5 \times 7^2 \times 3^4 = 127008 \gg p \tag{238}$$

As $q \gg p$, we find that :

$$\overline{\Delta} = 27q^2 - 4p^3 > 0 \tag{239}$$

Then we cannot use the results of our proof because in this example, $m = 2 < 3$. We remark that in all the proof, we don't encountered that m, n or l must be great than 2. Then the condition that $m, n, l > 2$ is important in (1).

5. Conclusion

As seen above, the examples confirm the results of the proof. In conclusion, we can announce the theorem:

Theorem 1. ***(A. Ben Hadj Salem, A. Beal, 2016):*** *Let A, B, C, m, n, and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{240}$$

then A, B, and C have a common factor.

References

R. Daniel Mauldin. *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem.* Notice of AMS, Vol44, $n°11$, 1997, pp1436-1437.