The mixing matrices and the cubic equation

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This article begins by defining two sets of related constants: The first set is used to define a nonstandard cubic equation. The second set is used to define equations that constrain a pair of rotation matrices. Curiously, these sets each contain an angle, each defined much like the other, where these angles separately play a central role in identifying solutions to their respective, very different equations. In this way these similarly-defined angles provide good evidence of a non-coincidental relationship between the nonstandard cubic equation and the pair of matrices. Moreover, a special case of this nonstandard cubic neatly combines with these matrices to produce values that map over to multiple physical constants (the fine structure constant reciprocal, the Weinberg angle, and the quark and lepton mixing angles), providing further evidence of a non-coincidental relationship.

This article begins by defining two sets of related constants:

- The first set *without hats*

$$\{N,\, M,\, W,\, Z,\, \theta\}$$

  will be used to define a "nonstandard cubic equation."

- The second set *with hats*

$$\{\widehat{N},\, \widehat{M},\, \widehat{W},\, \widehat{Z},\, \widehat{\theta}\}$$

  will be used to define equations that place constraints on a pair of rotation matrices to be called "conjoined matrices."

Curiously, and perhaps surprisingly, these sets each contain an angle (θ and $\widehat{\theta}$) — each defined much like the other — where these angles separately play a central role in identifying solutions to their respective, very different equations. In this way the nonstandard cubic equation [1, 2] will be shown to relate to the conjoined matrices, an extension of earlier results [3, 4].

Note that the definition of a conjoined matrix will exploit the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements have a common absolute value, where this value is an intrinsic property of the rotation matrix. It is by constraining this value to be $\widehat{N}$ times larger for one matrix than the other *in three ways*, and requiring that a particular angle equal 45°, that the two rotation matrices become conjoined [4].

If we do not count $\widehat{N}$, the six angles of these conjoined matrices are subject to four constraints — hence, just two more constraints are needed to determine all six of the matrices' angles. As it turns out, the nonstandard cubic equation has an approximate solution that has a minimal case featuring $Z = 137.036$ and $\theta \approx 27.407\,157^\circ$, values that are close to the experimental fine structure constant reciprocal $1/\alpha$ and Weinberg angle θ_W (i.e., the weak mixing angle) [1]. Curiously, other values of this same minimal case can handily supply the above "missing" constraints in a way that determines conjoined matrices with angles fitting (within the limits of experimental error) five of the six quark and lepton mixing angles.

I. TWO SETS OF CONSTANTS

Define

$$N \ge 1 \tag{1a}$$

$$\widehat{N} \ge 1 \tag{1b}$$

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$$M = \frac{N^3}{3} + 1 \tag{2a}$$
$$\widehat{M} = 4\widehat{N} - 2 \tag{2b}$$

$$W = \frac{\left(M-1\right)^3}{N^3} + \left(M-1\right)^2 \tag{3a}$$
$$\widehat{W} = \frac{\left(\widehat{M}-1\right)^3}{\widehat{N}^3} + \left(\widehat{M}-1\right)^2 \tag{3b}$$

$$Z = \frac{M^3 - M^{-3}}{N^3} + M^2 - M^{-3} \tag{4a}$$
$$\widehat{Z} = \frac{\left(\widehat{M}^{+1} - \widehat{M}^{-1}\right)^3}{\widehat{N}^3} + \left(\widehat{M}^{+1} - \widehat{M}^{-1}\right)^2 \tag{4b}$$

$$\cos\theta = \sqrt{\frac{W}{Z}} \tag{5a}$$
$$\cos\widehat{\theta} = \sqrt{\frac{\widehat{W}}{\widehat{Z}}} \quad . \tag{5b}$$

Above, N, M, Z, W, θ, and their hatted counterparts are all constants. Observe that if

$$N = \widehat{N} = 3 \tag{6a}$$

then

$$M = \widehat{M} = 10 \quad . \tag{6b}$$

Consider as well that for these particular values it is only the small difference in how Z and $\widehat{Z}$ are defined that makes θ different from $\widehat{\theta}$. And, finally, note that Eq. (2a) tells us that $N = 3$ is the smallest N for which N and M are integers.

II. THE NONSTANDARD CUBIC EQUATION AND ITS SOLUTIONS

The constants without hats will now be used to define and solve a nonstandard cubic equation.

Define **The Nonstandard Cubic Equation**

$$\frac{\left(M-x\right)^3}{N^3} + (M-x)^2 = Z \quad , \tag{7a}$$

where x is a variable, and all of the above definitions of unhatted constants apply *except* that of Z. For Z we merely require that it be a positive constant. Then,

$$x = (1-M)\left(\sqrt[3]{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt[3]{\frac{1-\sin\theta}{1+\sin\theta}}\right) + 2M - 1 \tag{7b}$$

solves Eq. (7a) for any positive Z. But now suppose that Z *is* constrained by Eq. (4a); then, if $M \geq 10$, a surprisingly simple, but accurate, **Approximate Solution** to Eq. (7a) becomes possible; that is,

$$x \approx \frac{1}{3 \times M^4} \quad . \tag{7c}$$

Note that Eqs. (7b) and (7c) are an obvious consequence of Theorems I and II, respectively, proved in [2].

TABLE I: When Z is determined by Eq. (4a), then The Approximate Solution, Eq. (7c), can closely estimate x for The Nonstandard Cubic, Eq. (7a). Below, this x approximation and its related values are calculated for the two smallest N for which M is also an integer. The first row represents The Minimal Case.

N	M^a	Z	$\frac{(M-x)^3}{N^3}$	$(M-x)^2$
3	10	137.036	$\left(\frac{10}{3} - \frac{1}{3 \times 29\,999.932\ldots}\right)^3$	$\left(\frac{10}{1} - \frac{1}{29\,999.932\ldots}\right)^2$
6	73	7130.004 ...	$\left(\frac{73}{6} - \frac{1}{6 \times 85\,194\,722.990\ldots}\right)^3$	$\left(\frac{73}{1} - \frac{1}{85\,194\,722.990\ldots}\right)^2$

[a] $M = \frac{N^3}{3} + 1$.

III. THE APPROXIMATE SOLUTION'S MINIMAL CASE

To show off the accuracy of The Approximate Solution (which only improves as N grows larger) let $N = 3$, so that Eq. (2a) gives $M = 10$. It follows that Eq. (4a) gives

$$\begin{aligned} Z &= \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} \\ &= \frac{999.999}{27} + 99.999 \\ &= 137.036 \quad , \end{aligned} \tag{8a}$$

so that Eq. (7a) gives

$$\begin{aligned} Z &= \left(\frac{10}{3} - \frac{1}{3 \times 29\,999.932\ldots}\right)^3 \\ &\quad + \left(\frac{10}{1} - \frac{1}{29\,999.932\ldots}\right)^2 \\ &= 137.036 \quad , \end{aligned} \tag{8b}$$

where $1/29\,999.932\ldots$ is closely approximated by $1/(3 \times M^4) = 1/(3 \times 10^4)$. These values occupy the first row of Table I. Moreover, from Eq. (3a) we know that $W = 108$, so from Eq. (5a) we find that

$$\begin{aligned} \theta &\approx \arccos\sqrt{\frac{108}{137.036}} \\ &\approx 27.407\,157^\circ \quad . \end{aligned} \tag{8c}$$

Readers familiar with the fundamental constants of physics will recognize that $Z = 137.036$ and $\theta \approx 27.407\,157^\circ$ are close to the experimental fine structure constant reciprocal and the Weinberg angle, respectively [1]; moreover, $N = 3$ equals the number of quark colors. Because $N = 3$ is the smallest N for which N and M are integers, the above values for N, M, x, W, Z, and θ will be termed **The Minimal Case**. These values occupy the first row of Table II.

Having just seen the usefulness of the constants without hats in solving The Nonstandard Cubic, we can now turn our attention to a related role performed by their hatted counterparts for conjoined matrices:

- In Sec. IV an intrinsic property of rotation matrices will be defined.
- In Sec. V this property will be used to define conjoined matrices.
- In Sec. VI some approximate minima of conjoined matrices will be found using $\widehat{\theta}$.

TABLE II: When Z is determined by Eq. (4a), then The Approximate Solution, Eq. (7c), can closely estimate x for The Nonstandard Cubic, Eq. (7a). Below, this x approximation and its related values are calculated for the two smallest N for which M is also an integer. The first row represents The Minimal Case.

N	M^a	x	W	Z	θ
3	10	$1/29\,999.932\ldots^b$	108	137.036^c	$27.407\,157\ldots^{\circ\,d}$
6	73	$1/85\,194\,722.991\ldots^e$	6912	$7130.004\ldots^f$	$10.070\,451\ldots^{\circ\,g}$

[a] $M = \dfrac{N^3}{3} + 1$.

[b] Approximately $\dfrac{1}{3 \times M^4} = \dfrac{1}{3 \times 10^4} = \dfrac{1}{30\,000}$.

[c] Given that $\dfrac{M^3 - M^{-3}}{N^3} + M^2 - M^{-3} = \dfrac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = 137.036$.

[d] Where $\cos 27.407\,157^\circ \approx \sqrt{\dfrac{W}{Z}} = \sqrt{\dfrac{108}{137.036}}$, and $\sin^2 27.407\,157^\circ \approx 0.211\,886$.

[e] Approximately $\dfrac{1}{3 \times M^4} = \dfrac{1}{3 \times 73^4} = \dfrac{1}{85\,194\,723}$.

[f] Given that $\dfrac{M^3 - M^{-3}}{N^3} + M^2 - M^{-3} = \dfrac{73^3 - 73^{-3}}{6^3} + 73^2 - 73^{-3} \approx 7130.004$.

[g] Where $\cos 10.070\,451^\circ \approx \sqrt{\dfrac{W}{Z}} = \sqrt{\dfrac{6912}{7130.004\ldots}}$, and $\sin^2 10.070\,451^\circ \approx 0.030\,576$.

IV. AN INTRINSIC PROPERTY OF ROTATION MATRICES

Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

be a rotation matrix, where squaring its elements gives

$$S = \begin{bmatrix} r_{11}^2 & r_{12}^2 & r_{13}^2 \\ r_{21}^2 & r_{22}^2 & r_{23}^2 \\ r_{31}^2 & r_{32}^2 & r_{33}^2 \end{bmatrix} \quad .$$

Given that a rotation matrix with its elements squared has rows and columns that sum to one, the above matrix can be rewritten

$$S = \begin{bmatrix} r_{11}^2 & 1 - r_{11}^2 - r_{13}^2 & r_{13}^2 \\ 1 - r_{11}^2 - r_{31}^2 & r_{11}^2 + r_{13}^2 + r_{31}^2 + r_{33}^2 - 1 & 1 - r_{13}^2 - r_{33}^2 \\ r_{31}^2 & 1 - r_{31}^2 - r_{33}^2 & r_{33}^2 \end{bmatrix} \quad .$$

For the difference of S and its transpose S^T we get

$$S - S^T = \begin{bmatrix} 0 & r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 \\ r_{13}^2 - r_{31}^2 & 0 & r_{31}^2 - r_{13}^2 \\ r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 & 0 \end{bmatrix} \quad ,$$

a matrix whose non-diagonal elements all equal

$$\pm \left(r_{31}^2 - r_{13}^2 \right) \quad .$$

It follows that $|r_{31}^2 - r_{13}^2|$ is an intrinsic property of rotation matrix R. Note that, in the next section, $|q_{31}^2 - q_{13}^2|$ will represent this same property for rotation matrix Q, and $|l_{31}^2 - l_{13}^2|$ for rotation matrix L.

V. CONJOINED MATRICES

Define two rotation matrices Q and L such that

$$Q = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{9a}$$

where $s_{12} \equiv \sin\theta^Q_{12}$, $c_{12} \equiv \cos\theta^Q_{12}$, etc., and

$$L = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{9b}$$

where $s_{12} \equiv \sin\theta^L_{12}$, $c_{12} \equiv \cos\theta^L_{12}$, etc. Also, let

$$\left.\begin{aligned} \sin\theta^L_{12} &= \sqrt{g_{12}\widehat{N}^{+1}} \\ \sin\theta^Q_{13} &= \sqrt{g_{13}\widehat{N}^{-1}} \\ \\ \sin\theta^Q_{12} &= \sqrt{g_{12}} \times \sin\theta^L_{23\text{ offset}} \\ \sin\theta^L_{13} &= \sqrt{g_{13}} \times \sin\theta^Q_{23\text{ offset}} \\ \\ \sin\theta^L_{23} &= \theta^L_{23\text{ offset}} \\ \sin\theta^Q_{23} &= \theta^Q_{23\text{ offset}} + 90^\circ \end{aligned}\right\} \tag{9c}$$

where

$$\left.\begin{aligned} &\widehat{N} \geq 1 \\ &0 \leq g_{12} \leq \widehat{N}^{-1} \\ &0 \leq g_{13} \leq \widehat{N}^{+1} \\ &0 \leq \theta^Q_{23\text{ offset}} \leq 90^\circ \end{aligned}\right\} \tag{9d}$$

assures that θ^L_{12} and θ^Q_{13} are real (note that $\widehat{N}$ was defined the same way at the outset).

At this point the reader perhaps has noticed that to determine all six angles of Q and L we need only know $\theta^L_{23\text{ offset}}$ and $\theta^Q_{23\text{ offset}}$. With this in mind, let

$$\theta^L_{23\text{ offset}} = 45^\circ \quad , \tag{9e}$$

and choose $\theta^Q_{23\text{ offset}}$ so that

$$\frac{|l^2_{31} - l^2_{13}|}{|q^2_{31} - q^2_{13}|} = \widehat{N} \quad , \tag{9f}$$

where there will sometimes be two values for $\theta^Q_{23\text{ offset}}$ that fulfill Eq. (9f), as shown in Fig. 1.

> *Definition:* If Q and L fulfill Eqs. (9a)–(9f) they are said to be **Conjoined Matrices**.

A parenthetical note concerning the definition of conjoined matrices: For conjoined matrices L and Q the value $|l^2_{31} - l^2_{13}|$ is $\widehat{N}$ times $|q^2_{31} - q^2_{13}|$, a consequence of Eq. (9f). But, as already noted, for conjoined matrices there are actually *three* ways that $|l^2_{31} - l^2_{13}|$ is $\widehat{N}$ times $|q^2_{31} - q^2_{13}|$; this is a consequence of Eqs. (9a)–(9f), as explained in detail here [4].

TABLE III: The red square and blue circle solution points of Figs. 1 and 2.

	Red Square "Near Minima" at $\widehat{\theta}$ [a]		Blue Circle "Actual Minima" [b]	
$\widehat{N}$	g_{13}	$\theta^{Q}_{23\text{ offset}}$ [c]	g_{13}	$\theta^{Q}_{23\text{ offset}}$ [c]
1	1/1.994 634 . . .	53.395 723 . . .	1/2	54.735 610 . . .
2	1/5.999 659 . . .	33.806 195 . . .	1/6	33.690 067 . . .
3[d]	1/9.999 986 . . .	26.111 924 . . .	1/10	26.100 138 . . .
4	1/13.997 941 . . .	21.912 321 . . .	1/14	22.001 713 . . .
5	1/17.990 017 . . .	19.224 863 . . .	1/18	19.359 648 . . .
6	1/21.977 237 . . .	17.330 707 . . .	1/22	17.480 170 . . .
10	1/37.910 754 . . .	13.118 720 . . .	1/38	13.244 942 . . .
100	1/397.576 274 . . .	4.057 747 . . .	1/398	4.064 948 . . .
1 000	1/3 997.508 239 . . .	1.281 360 . . .	1/3 998	1.281 599 . . .
10 000	1/39 997.500 830 . . .	0.405 148 . . .	1/39 998	0.405 155 . . .
100 000	1/399 997.500 083 . . .	0.128 117 . . .	1/399 998	0.128 117 . . .
1 000 000	1/3 999 997.500 008 . . .	0.040 514 . . .	1/3 999 998	0.040 514 . . .

[a]Knowing $\widehat{N}$, we can easily use Eq. (5b) to calculate the angles of the third column, which, as $\widehat{N}$ grows, grow closer to those of the fifth.
[b]Knowing $\widehat{N}$, we can easily use Eq. (10) to calculate both the angles of the fifth column and the values for g_{13} of the fourth column.
[c]In degrees.
[d]Observe that the red square and blue circle solution points coincide much more closely for $\widehat{N} = 3$ (used to produce the mixing angles) than for either $\widehat{N} = 2$ or $\widehat{N} = 4$. When $\widehat{N} = 1.589\,722\,118\ldots$ or $\widehat{N} = 3.090\,198\,575\ldots$ the blue circles and red squares coincide exactly.

VI. FINDING MINIMA OF CONJOINED MATRICES

Just as *unhatted* θ — in Eq. (7b) — helped solve The Nonstandard Cubic, so *hatted* $\widehat{\theta}$ can help find some important minima of conjoined matrices. To see how, consider Fig. 1, whose six graphs (for $\widehat{N} = 1, 2, 3, 4, 5, 6$) show how $\theta^{Q}_{23\text{ offset}}$ and g_{13} interact for **Simplified Conjoined Matrices**, which are merely conjoined matrices having

$$g_{12} = \frac{1}{\widehat{N}} \quad .$$

Each of this figure's six graphs also has a *red square marker* at $\theta^{Q}_{23\text{ offset}} = \widehat{\theta}$, where $\widehat{\theta}$ is easily calculated from $\widehat{N}$ using Eq. (5b). Inspection shows these red markers to be "Near Minima" for g_{13}. But just how close are these red square solution points to the *actual* minima? Figure 2 helps answer this question by combining the above red square markers with the *blue circle markers* of the following "Actual Minima" of g_{13}:

$$\left.\begin{aligned} \theta^{Q}_{23\text{ offset}} &= \arcsin\sqrt{\frac{2\widehat{N}}{(2\widehat{N}-1)^2 + 2\widehat{N}}} \\ g_{13} &= \frac{1}{\widehat{M}} = \frac{1}{4\widehat{N}-2} \end{aligned}\right\} \tag{10}$$

These red and blue markers are especially close when $\widehat{N} = 3$, the point magnified in the figure. But the small differences that separate these solution points can best be judged in Table III, in which $\widehat{N}$ ranges from one to one million. The essential points to take away from this table are:

- That for wide-ranging $\widehat{N}$ the minima of g_{13} consistently occur near $\widehat{\theta}$.
- That this is surprising because hatted $\widehat{\theta}$ is merely a variant of The Nonstandard Cubic's unhatted θ.

One immediately suspects that there must exist an important relationship between The Nonstandard Cubic and conjoined matrices. This expectation is reinforced when we note that if $x = M^{-1}$ for The Nonstandard Cubic, then $N = \widehat{N} = 3$ gives $M = \widehat{M}$, $W = \widehat{W}$, $Z = \widehat{Z}$, and $\theta = \widehat{\theta}$, which underscores the similarity of the unhatted and hatted definitions.

FIG. 1: Solutions to Simplified Conjoined Matrices; that is, conjoined matrices having $g_{12} = 1/\widehat{N}$. Red squares mark solution points having $\theta^Q_{23\text{ offset}} = \widehat{\theta}$, which inspection suggests are *close* to g_{13} minima.

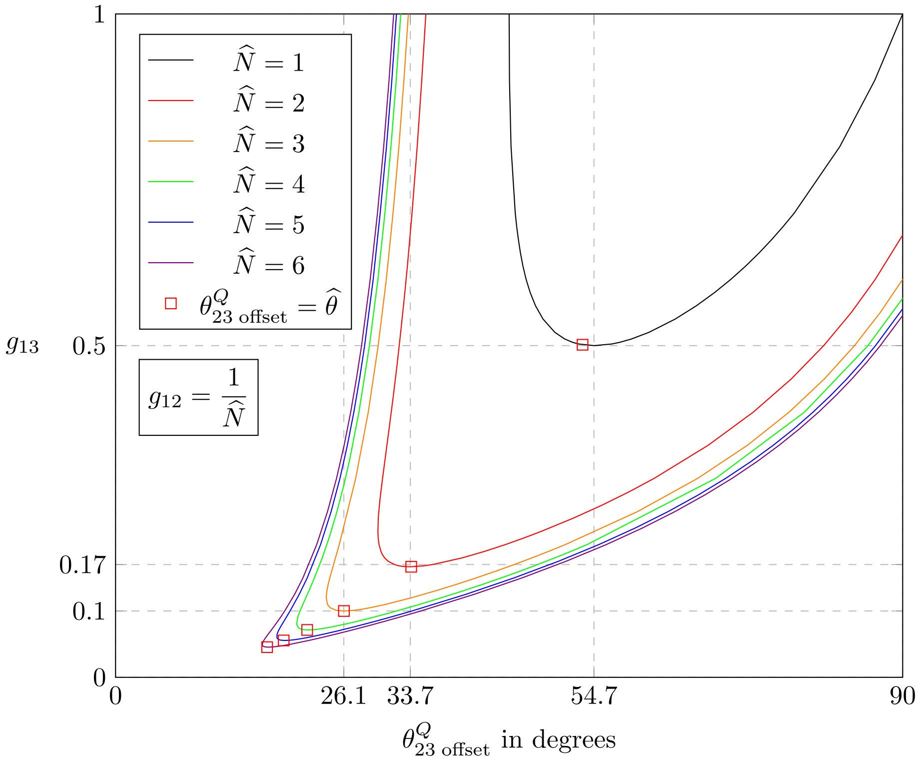
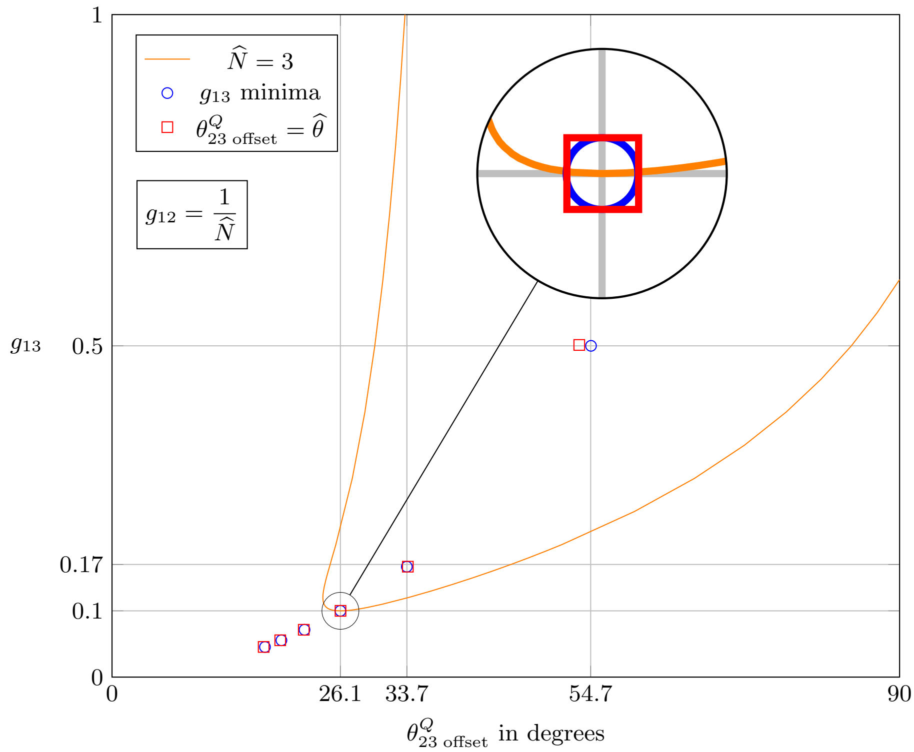


FIG. 2: The red square markers of Fig. 1 with blue circle solution points marking *actual* g_{13} minima.



VII. THE MINIMAL CASE AND THE MIXING ANGLES

In fact the "unexpected effectiveness" of $\widehat{\theta}$ in helping identify minima in conjoined matrices is not the only reason to suspect a relationship between The Nonstandard Cubic and these matrices. An additional reason is provided by the constants of The Minimal Case — the first row of Table II — which can be used to determine conjoined matrices having angles close to the experimental quark and lepton mixing angles. Specifically,

$$\left.\begin{aligned} N &= \mathbf{3} \\ M &= \mathbf{10} \\ x &= \mathbf{1/29\,999.932\ldots} \\ W &= 108 \\ Z &= 137.036 \\ \theta &= 27.407\,157\ldots^\circ \end{aligned}\right\} \text{The Minimal Case for Eq. (7a)} \tag{11a}$$

has values (in boldface) for N, M, and x that can be used to determine conjoined matrices having angles close to the mixing angles. Accordingly, if we make the following assignments

$$\left.\begin{aligned} \widehat{N} &= \mathbf{3} \\ g_{12} &= \mathbf{1/10} \\ g_{13} &= \mathbf{1/29\,999.932\ldots} \end{aligned}\right\} \tag{11b}$$

then Eq. (9c) gives

$$\left.\begin{aligned} \theta_{12}^{Q} &\approx 12.920\,966^\circ \\ \theta_{13}^{Q} &\approx 0.190\,987^\circ \\ \theta_{23}^{Q} &\approx 2.367\,444^\circ + 90^\circ \end{aligned}\right\} \tag{12a}$$

and

$$\left.\begin{aligned} \theta_{12}^{L} &\approx 33.210\,911^\circ \\ \theta_{13}^{L} &\approx 0.013\,665^\circ \\ \theta_{23}^{L} &= 45^\circ \end{aligned}\right\} \tag{12b}$$

for the conjoined matrices' angles. Five of these angles fit the quark and lepton mixing angles within the limits of experimental error. (For the treatment of θ_{13}^{L}, the angle that does not *directly* map over, see [4].)

VIII. SUMMARY

In the above way, The Nonstandard Cubic's approximation and conjoined matrices directly produce the following approximations of these physical constants:

- The fine structure constant reciprocal (137.036).
- The Weinberg angle (27.407 157°).
- The three quark mixing angles (12.920 966°, 0.190 987°, and 2.367 444°).
- The two largest leptonic mixing angles (33.210 911° and 45°).

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[1] J. S. Markovitch, "The cubic equation's relation to the fine structure constant, the mixing angles, and Weinberg angle" (2014) `http://www.vixra.org/abs/1406.0033`.

[2] J. S. Markovitch, "A nonstandard cubic equation" (2015) `http://www.vixra.org/abs/1412.0253`.

[3] J. S. Markovitch, "A mathematical model of the quark and lepton mixing angles (2011 update)" (2011) `http://www.vixra.org/abs/1102.0021`.

[4] J. S. Markovitch, "Underlying symmetry among the quark and lepton mixing angles (Seven year update)" (2014) `http://www.vixra.org/abs/1412.0259`.