An exact solution to the Navier Stokes Voight equation.

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Abstract

In this paper it is demonstrated that the Navier Stokes Voight equation has a smooth nontrivial exact solution in (2+1). This can be extended to (3+1). Smoothness of the solution is accomplished by connecting disconnected pieces of solution over a vanishingly small intervall. We show by example that the algebra of connecting coefficients is consitent.

I. INTRODUCTION

A. Preliminaries

In the present paper a simple exact solution to the (2+1), i.e. two space one time, Navier Stokes Voight (NSV) equation is presented. The solution observes the requirements of vanishing divergence, finite energy and bounded absolute differentials of velocity, pressure and force [1]. The claim is that the pair of exact solutions (u, p) exists that observe the requirements of the type A solution of [1] written down for the Navier Stokes (NS) equation. Despite the fact that we are dealing with the NSV we follow the requirements for the NS. The solution is initially associated to 4 quadrants of $\mathbb{R}^2$.For sufficiently small $0 < \epsilon \to 0$, the 4 quadrant NSV solutions are connected and the algebra of coefficients is demonstrated to be consistent. It is noted that for $x_k \in (-\epsilon, \epsilon)_{0<\epsilon\to 0}$, the NSV equation does not apply. In a physical sense we hence may claim to have obtained an exact modified type A solution. The NSV breaks down physically in $x_k \in (-\epsilon, \epsilon)_{0<\epsilon\to 0}$ because the absence of continuum mechanics beyond a certain length limit in a real fluid. Finite energy derives from a real physics fluid. It is noted that a real physics fluid consists of atoms. Beyond a certain length scale there is no continuum in a real fluid.

B. The equation

The velocity vector, u, $\{u_n\}_{n=1}^2$, is matched with a simultaneous solution for a constant pressure p. Generally we have for the n-th element $u_n = u_n(x_1, x_2, t)$, $(n = 1, 2)$ of the velocity vector and $p = p(x_1, x_2, t)$. The NSV equation is:

$$\frac{\partial u_n}{\partial t} + \sum_{j=1}^{2} u_j \frac{\partial u_n}{\partial x_j} - \nu \nabla^2 u_n - \eta^2 \nabla^2 \frac{\partial u_n}{\partial t} + \frac{\partial p}{\partial x_n} = f_n \tag{1}$$

with kinematic viscosity $\nu > 0$ and length scale $\eta > 1$. The function f_n is external. In type A, $f_n = 0$. Accordingly the solution, u_n in (1) must have finite energy [1]

$$\int_{\mathbb{R}^2} \sum_{n=1}^{2} u_n^2(x_1, x_2, t) d^2x \leq C(t) \tag{2}$$

and a vanishing divergence $\sum_{n=1}^{2}\frac{\partial}{\partial x_n}u_n = 0$. The challenge is to demonstrate that a non-trivial smooth exact solution is possible with the zero time initial conditions $u_{0,n}(x_1,x_2) = u_n(x_1,x_2,0)$. The pressure p and force f_n obey the requirements of type A. We demonstrate here that u_n follows the requirements of type A. The initial boundary conditions with irreducible scale $0<\epsilon\to 0$ represents the slight modification to A.

II. SOLUTION HEURISTICS

Let us define a heuristic solution for $u_n = u_n(x_1,x_2,t)$, with, $x=(x_1,x_2)$ and

$$u_n = c_n^{\iota}\exp\left[-at-\sum_{k=1}^{2}\alpha_k|x_k|\right](\lambda_{n1}^{\iota})^{H(\epsilon,x_1)}(\lambda_{n2}^{\iota})^{H(\epsilon,x_2)} \equiv u_n^{\iota} \tag{3}$$

with, $n=1,2$, $a>0$ real and $\alpha_k>0$ real, $k=1,2$, and $\lambda_{jn}^{\iota}\in\{-1,1\}$. The λ coefficients project in $\{-1,1\}$. Later we will enter into the details of the coefficients λ. The H exponents in (3) for the λ's are defined by

$$H(\epsilon,x_n) = \begin{cases} 1, & x_n\in(-\epsilon,\epsilon) \\ 0, & x_n\notin(-\epsilon,\epsilon)\end{cases} \tag{4}$$

The H functions are unequal to zero for an interval around zero with $0<\epsilon\to 0$ for $x_k\in(-\epsilon,\epsilon)$. Furthermore, $||\alpha||=1$ and $||.||$ the euclidean norm. The ι in the superscript is an index, with $\iota=(\iota_1,\iota_2)$ and $(x_1,x_2)\in\mathbb{R}_{\iota_1}\backslash\{0\}\times\mathbb{R}_{\iota_2}\backslash\{0\}$, with $x_n\in\mathbb{R}_{\iota_n}\backslash\{0\}$ such that $|x_n|>0$ and $\text{sgn}(x_n)=\iota_n=\pm1$. I.e. $\iota=\iota(x)=(\iota(x_1),\iota(x_2))=(\text{sgn}(x_1),\text{sgn}(x_2))$, with, $\text{sgn}(0)=0$. E.g. $\iota(x)=(+,-)$, for $x_1>0$ and $x_2<0$. The superscript indicates that we are looking at the solution related to that subsection, or quadrant, of $\mathbb{R}^2$ where in the example $x_1>0$ and $x_2<0$. The set $\{u^{\iota(x)}(x,t)|\iota(x)$ holds no zero's $\}$ contains u that are associated to the four quadrants excluding zero. Smoothness of the solution will be discussed later and is associated to the constants $c_n^{\iota(x)}$. Furthermore, it is assumed that the constants $\{c_n^{\iota(x)}\}_{n=1}^{2}$ and $\{\alpha_n\}_{n=1}^{2}$ are such that

$$\sum_{j=1}^{2}\alpha_j c_j^{\iota(x)}\text{sgn}(x_j)=0 \tag{5}$$

With the sign of x_k in the index one can have different c. E.g., (x_1,x_2) such that, $x_1>0$, $x_2<0$ gives

$$c_1^{(+,-)}\alpha_1 - c_2^{(+,-)}\alpha_2 = 0 \tag{6}$$

while e.g. $x_1 < 0$, $x_2 < 0$

$$-c_1^{(-,-)}\alpha_1 - c_2^{(-,-)}\alpha_2 = 0 \tag{7}$$

etcetera, and $||\alpha||^2 = 1$. The reader may note that instead of e.g. $(+,+)$ we could have used $c_n^{1,1}$ or similar. The use of the superscript is to select the proper function from a family of 4 vector functions, associated to $x = (x_1, x_2)$, $x_k \neq 0$. This family is $\{u^{(-,-)}(x,t), u^{(-,+)}(x,t), u^{(+,-)}(x,t), u^{(+,+)}(x,t)\}$. The vector functions only differ by a constant c vector that is restricted by (5).

A. Finite energy

The requirement of finite energy is given in equation (2). The superscript $\iota(x)$ can be suppressed in the argument. The requirement can be expressed in subspaces of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ ($\times$ the Cartesian product),

$$\mathbb{R}^2 = (\mathbb{R}_- \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-) \cup (\mathbb{R}_+ \times \mathbb{R}_+)$$

Note, $\iota(x) = (\iota_1, \iota_2)$ with $(x_1, x_2) \in (\mathbb{R}_{\iota_1} \times \mathbb{R}_{\iota_2})$, and, $\forall_{n\in\{1,2\}} |x_n| > 0$. This implies (suppressing the use of d^2x for the moment)

$$C(t) \geq \int_{\mathbb{R}^2} ||u||^2 = \int_{\mathbb{R}_-\times\mathbb{R}_-} ||u||^2 + \int_{\mathbb{R}_-\times\mathbb{R}_+} ||u||^2 + \int_{\mathbb{R}_+\times\mathbb{R}_-} ||u||^2 + \int_{\mathbb{R}_+\times\mathbb{R}_+} ||u||^2 \tag{8}$$

and $C(t)$ finite. We have $||u||^2 = u_1^2 + u_2^2$. Because in the analysis of smoothness, a vanishingly small interval is excluded, the integration for e.g. $x_1 < 0$ and $x_2 < 0$ must be written as

$$E_1(\epsilon) = \int_{-\infty}^{-\epsilon}\int_{-\infty}^{-\epsilon} u_1^2 dx_1 dx_2 dx_2 \tag{9}$$

with $0 < \epsilon \to 0$. In effect we may take boundary in the integrations equal to zero and proceed in this way in our attempt to demonstrate finite energy. Hence, we may write the first term of (8) as

$$E_1 = \int_{-\infty}^{0}\int_{-\infty}^{0} u_1^2 dx_1 dx_2 \tag{10}$$

Then, looking at equation, (3), noting $x_k < 0$ in the first integral of (8).

$$E_1 = \int_{-\infty}^{0}\int_{-\infty}^{0} u_1^2 dx_1 dx_2 = \left\{c_1^{(-,-)}\right\}^2 e^{-2at} \int_{-\infty}^{0}\int_{-\infty}^{0} \exp\left[2\sum_{k=1}^{2}\alpha_k x_k\right] dx_1 dx_2 \tag{11}$$

Hence,

$$E_1 = \left\{c_1^{(-,-)}\right\}^2 e^{-2at} \int_{-\infty}^{0} \exp\left[2\alpha_1 x_1\right] dx_1 \int_{-\infty}^{0} \exp\left[2\alpha_2 x_2\right] dx_2 \tag{12}$$

such that

$$E_1 = \left\{c_1^{(-,-)}\right\}^2 e^{-2at} \int_{0}^{\infty} \exp\left[-2\alpha_1 x_1\right] dx_1 \int_{0}^{\infty} \exp\left[-2\alpha_2 x_2\right] dx_2 \tag{13}$$

which gives

$$E_1 = \left(\frac{1}{4\alpha_1\alpha_2}\right) \left\{c_1^{(-,-)}\right\}^2 e^{-2at} \tag{14}$$

The last integration term for u_1 in (8) is

$$E_4 = \int_0^\infty \int_0^\infty u_1^2 dx_1 dx_2 \tag{15}$$

then, looking again at (3), noting $x_k > 0$ here,

$$E_4 = \left\{c_1^{(+,+)}\right\}^2 e^{-2at} \int_0^\infty \int_0^\infty \exp\left[-2\sum_{k=1}^{2} \alpha_k x_k\right] dx_1 dx_2 \tag{16}$$

This then gives

$$E_4 = \left(\frac{1}{4\alpha_1\alpha_2}\right) \left\{c_1^{(+,+)}\right\}^2 e^{-2at} \tag{17}$$

The second integral for u_1 is

$$E_2 = \int_{-\infty}^{0} \int_0^\infty u_1^2 dx_1 dx_2 \tag{18}$$

Hence, we may write

$$E_2 = \left\{c_1^{(-,+)}\right\}^2 e^{-2at} \int_{-\infty}^{0} \exp[2\alpha_1 x_1] dx_1 \int_0^\infty \exp[-2\alpha_2 x_2] dx_2 \tag{19}$$

This implies

$$E_2 = \frac{\left\{c_1^{(-,+)}\right\}^2 e^{-2at}}{4\alpha_1\alpha_2} \tag{20}$$

A similar form goes for E_3 the third term in (8). So,

$$E_3 = \frac{\left\{c_1^{(+,-)}\right\}^2 e^{-2at}}{4\alpha_1\alpha_2} \tag{21}$$

Because for u_1 we have $E = E_1 + E_2 + E_3 + E_4$ and for u_2 forms similar to, (14), (17), (20) and (21) can be derived, we may conclude that the energy is finite for this solution. Hence, it is possible to have

$$\infty > C(t) \geq e^{-at} \sum_{\iota_1 \in \{-,+\}} \sum_{\iota_2 \in \{-,+\}} \frac{||c^{(\iota_1,\iota_2)}||^2}{4\alpha_1\alpha_2}$$

B. Terms in the Navier Stokes equation

In the analysis we assume $x_k \notin (-\epsilon, \epsilon)$ for $k = 1, 2$, with $0 < \epsilon \to 0$.

1. *divergence*

From (3) observe that, if the dot denotes the time differentiation, then, $\dot{u}_n = -au_n$. Subsequently,

$$\frac{\partial u_n}{\partial x_n} = c_n^{\iota(x)} \frac{\partial}{\partial x_n} \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{22}$$

To be completely clear, the $\iota(x)$ in $c_n^{\iota(x)}$ is a superscript index, not a power. Then,

$$\frac{\partial u_n}{\partial x_n} = -c_n^{\iota(x)} \left(\alpha_n \frac{\partial}{\partial x_n} |x_n|\right) \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{23}$$

Furthermore, $\frac{\partial}{\partial x_n}|x_n| = \text{sgn}(x_n) + 2x_n\delta(x_n)$, with $\delta(x_n)$ the Dirac delta function. The term, $x_n\delta(x_n)$ can be ignored. We have, $\delta(x_n) \neq 0$ when $x_n = 0$, otherwise, $\delta(x_n) = 0$. The δ arises from $\frac{\partial}{\partial x_n}\text{sgn}(x_n) = \delta(x_n) + \delta(-x_n)$ and $\delta(-x_n) = \delta(x_n)$ noting $\text{sgn}(x_n) = \Theta(x_n) - \Theta(-x_n)$ and $\Theta(x_n) = 1$ for $x_n \geq 0$ and $\Theta(x_n) = 0$ for $x_n < 0$. From the equation (23) and $x_n \neq 0$, it follows that

$$\sum_{n=1}^{2} \frac{\partial u_n}{\partial x_n} = -\left(\sum_{n=1}^{2} c_n^{\iota(x)} \alpha_n \text{sgn}(x_n)\right) \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] = 0 \tag{24}$$

The exponent term in "exp" remains finite because $a > 0$, $\alpha_k > 0$ and $|x_k| > \epsilon$. From (5) the divergence of u, vanishes, i.e. $\nabla \cdot u = 0$, as required.

2. *u_j product differentiation & ∇^2*

In addition,

$$\frac{\partial u_n}{\partial x_j} = -c_n^{\iota(x)} \alpha_j \text{sgn}(x_j) \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{25}$$

Hence,

$$u_j \frac{\partial u_n}{\partial x_j} = -c_n^{\iota(x)} c_j^{\iota(x)} \alpha_j \text{sgn}(x_j) \exp\left[-2at - 2\sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{26}$$

Because,(5) we see that

$$\sum_{j=1}^{2} u_j \frac{\partial u_n}{\partial x_j} = 0 \tag{27}$$

From equation (25) it also follows that

$$\frac{\partial^2 u_n}{\partial x_j^2} = c_n^{\iota(x)}\{(\alpha_j)^2 - 2\alpha_j \delta(x_j)\} \exp\left[-at + \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{28}$$

with $||\alpha||^2 = 1$. Note $\delta(x_j) = 0$ for $x_j \neq 0$, then $\nabla^2 u_n = u_n$. From the previous we also may see that the Voight term is equal to

$$-\eta^2 \nabla^2 \frac{\partial u_n}{\partial t} = a\eta^2 u_n.$$

Hence, the NSV equation reduces for $x_k \neq 0$ with $k = 1, 2$, to $(\nu > 0, \eta > 1)$

$$-(a + \nu - a\eta^2)u_n + \frac{\partial p}{\partial x_n} = f_n \tag{29}$$

For $n = 1, 2$ in type A we have $f_n = 0$. If, $a = \frac{\nu}{\eta^2 - 1}$ for $\eta > 1$, then $a > 0$. This, together with $p = \textit{constant}$ and $f_n = 0$ gives a complete type A solution provided u_n is smooth.

C. Smoothness of u_n arguments

The following sections will be devoted to the connection between the solutions of the four $\iota(x) \neq 0$ subspaces via x_n in $(-\epsilon, \epsilon)$ intervals. The left and right hand limit of u_n at each (x_1, x_2) must be equal in order to claim a smooth solution. The smoothness of u_1 and of u_2 is inspected for the coefficients $c_1^{\iota(x)}$ and $c_2^{\iota(x)}$ separately. The u_1 needs to smoothly connect for index $n = 1$ and limits. Similarly for u_2 and index $n = 2$ plus limits. It isn't necessary to do algebra for connecting u_1 with u_2 in the limits.

1. *limits*

The limits employ c coefficients in the u_n, $n = 1, 2$, functions with one or both spatial variables, x_1, and/or, x_2 inside $(-\epsilon, \epsilon)$. The $0 < \epsilon \to 0$ warrants that this 'not valid' interval is vanishingly small. In physics, a real fluid will no longer be continuous if the ϵ decreases beyond the size of the particles that constitute the fluid. Boundary values in $x_k = 0$, i.e. along the axes, are initial givens to the Navier Stokes Voight equation. The axes can be arbitrarily projected in the fluid that is supposed to fill $\mathbb{R}^2$. Extension to $\mathbb{R}^3$ will be discussed later.

2. coefficients

In the first place let us look at the following limit, where $x_2 \notin (-\epsilon, \epsilon)$, where $H(\epsilon, x_1) = 1$, for a certain given small $\epsilon > 0$ that can be decreased to 0 in a "later" limit process.

$$\lim_{\substack{0 > x_1 \to 0^- \\ x_2 \notin (-\epsilon,\epsilon)}} u_n^{\iota(x)}(x_1, x_2, t) = \lambda_{1n}^{(-,\pm)} c_n^{(-,\pm)} w(0^-, x_2, t) \tag{30}$$

with $w(x_1, x_2, t) = \exp[-at - \alpha_1|x_1| - \alpha_2|x_2|]$. Note that, for limits, $w(0^-, x_2, t) = w(0, x_2, t)$. The second limit we need to look at is

$$\lim_{\substack{0 < x_1 \to 0^+ \\ x_2 \notin (-\epsilon,\epsilon)}} u_n^{\iota(x)}(x_1, x_2, t) = \lambda_{1n}^{(+,\pm)} c_n^{(+,\pm)} w(0^+, x_2, t) \tag{31}$$

Here, $w(0^+, x_2, t) = w(0, x_2, t)$. For $x_1 = 0$ and $|x_2| > 0$, we have

$$u_n^{\iota(x)}(0, x_2, t) = \lambda_{1n}^{(0,\pm)} c_n^{(0,\pm)} w(0, x_2, t) \tag{32}$$

For $x_2 = 0$ and $|x_1| > 0$, we have

$$u_n^{\iota(x)}(x_1, 0, t) = \lambda_{2n}^{(\pm,0)} c_n^{(\pm,0)} w(x_1, 0, t) \tag{33}$$

For $x_1 = x_2 = 0$ we have $u_n^{(0,0)}(0,0,t) = c_n^{(0,0)} \lambda_{1n}^{(0,0)} \lambda_{2n}^{(0,0)} w(0,0,t)$. Here, $c_n^{(0,0)} \lambda_{1n}^{(0,0)} \lambda_{2n}^{(0,0)} = 1$, and where limits from either quadrant show discontinuity, a jump is replaced by a smooth connection in $x_1 \in (-\epsilon, \epsilon)$ and $x_2 \in (-\epsilon, \epsilon)$, with, $0 < \epsilon \to 0$.

3. example

Suppose, $\alpha_1 = \frac{1}{2} > 0$, $\alpha_2 = \frac{1}{2}\sqrt{3} > 0$. Hence, $||\alpha||^2 = 1$. The $s^\iota = \pm$ is not important for the relation of α and c. The next step is to see if c vectors with given α are possible. Let us define the c coefficients with unequal signs in the superscript. Hence,

$$\begin{aligned} c_1^{(+,-)} &= \sqrt{3}, & c_2^{(+,-)} &= 1 \\ c_1^{(-,+)} &= -\sqrt{3}, & c_2^{(-,+)} &= -1 \end{aligned} \tag{34}$$

So,

$$\alpha_1 c_1^{(+,-)} - \alpha_2 c_2^{(+,-)} = \left(\frac{1}{2} * \sqrt{3}\right) - \left(\frac{1}{2}\sqrt{3} * 1\right) = 0$$

and,

$$-\alpha_1 c_1^{(-,+)} + \alpha_2 c_2^{(-,+)} = -\left(-\frac{1}{2} * \sqrt{3}\right) + \left(-\frac{1}{2}\sqrt{3} * 1\right) = 0.$$

For equal sign coefficients

$$\begin{aligned} c_1^{(+,+)} &= \sqrt{3}, \qquad & c_2^{(+,+)} &= -1 \\ c_1^{(-,-)} &= -\sqrt{3}, \qquad & c_2^{(-,-)} &= 1 \end{aligned} \tag{35}$$

Subsequently, we have to check consistency

$$\alpha_1 c_1^{(+,+)} + \alpha_2 c_2^{(+,+)} = \left(\frac{1}{2} * \sqrt{3}\right) + \left(-\frac{1}{2}\sqrt{3} * 1\right) = 0$$

together with

$$-\alpha_1 c_1^{(-,-)} - \alpha_2 c_2^{(-,-)} = -\left(-\frac{1}{2} * \sqrt{3}\right) - \left(\frac{1}{2}\sqrt{3} * 1\right) = 0$$

4. further breakdown of the c coefficients

The result of the previous paragraph is given by

$$\begin{aligned} &c_1^{(+,-)} = \sqrt{3},\ c_2^{(+,-)} = 1;\ c_1^{(-,+)} = -\sqrt{3},\ c_2^{(-,+)} = -1 \\ &c_1^{(+,+)} = \sqrt{3},\ c_2^{(+,+)} = -1;\ c_1^{(-,-)} = -\sqrt{3},\ c_2^{(-,-)} = 1 \end{aligned} \tag{36}$$

When we check for smoothness, or connectedness, algebraic consistency checks are necessary for equal lower (n-) indexed $c_n^{(\iota_1,\iota_2)}$. Suppose, furthermore, that the c coefficients can be broken down into indexed factors e, d, f, g and h

$$c_n^{(\iota_1,\iota_2)} = e_n^{\iota_1} d_n^{\iota_2} f_n g^{\iota_1} h^{\iota_2} \tag{37}$$

Subsequently let us inspect equal lower indexed c. For, $n = 1, 2$ let us inspect quotients of c. Firstly, we take a look at unequal signed upper indices

$$\frac{c_n^{(-,+)}}{c_n^{(+,-)}} = \frac{e_n^- d_n^+ g^- h^+}{e_n^+ d_n^- g^+ h^-} = -1 \tag{38}$$

Secondly, the equal upper indexes and $n = 1, 2$,

$$\frac{c_n^{(-,-)}}{c_n^{(+,+)}} = \frac{e_n^- d_n^- g^- h^-}{e_n^+ d_n^+ g^+ h^+} = -1 \tag{39}$$

Thirdly we take a look at the other upper indexes

$$\begin{aligned} \frac{c_1^{(+,-)}}{c_1^{(+,+)}} &= \frac{e_1^+ d_1^- g^+ h^-}{e_1^+ d_1^+ g^+ h^+} = \frac{d_1^- h^-}{d_1^+ h^+} = 1 \\ \frac{c_2^{(+,-)}}{c_2^{(+,+)}} &= \frac{e_2^+ d_2^- g^+ h^-}{e_2^+ d_2^+ g^+ h^+} = \frac{d_2^- h^-}{d_2^+ h^+} = -1 \end{aligned} \tag{40}$$

together with

$$\begin{aligned}\frac{c_1^{(-,+)}}{c_1^{(-,-)}} &= \frac{e_1^- d_1^+ g^- h^+}{e_1^- d_1^- g^- h^-} = \frac{d_1^+ h^+}{d_1^- h^-} = 1 \\ \frac{c_2^{(-,+)}}{c_2^{(-,-)}} &= \frac{e_2^- d_2^+ g^- h^+}{e_2^- d_2^- g^- h^-} = \frac{d_2^+ h^+}{d_2^- h^-} = -1\end{aligned} \tag{41}$$

So, if we take

$$\frac{h^-}{h^+} = 1 \tag{42}$$

and,

$$d_2^- = -1, \;\; d_2^+ = d_1^- = d_1^+ = 1 \tag{43}$$

then, (40) and (41) are ok. Subsequently, in this section we inspect the, e, coefficients. So,

$$\begin{aligned}\frac{c_1^{(-,+)}}{c_1^{(+,+)}} &= \frac{e_1^- d_1^+ g^- h^+}{e_1^+ d_1^+ g^+ h^+} = \frac{e_1^- g^-}{e_1^+ g^+} = -1 \\ \frac{c_2^{(-,+)}}{c_2^{(+,+)}} &= \frac{e_2^- d_2^+ g^- h^+}{e_2^+ d_2^+ g^+ h^+} = \frac{e_2^- g^-}{e_2^+ g^+} = 1\end{aligned} \tag{44}$$

and

$$\begin{aligned}\frac{c_1^{(+,-)}}{c_1^{(-,-)}} &= \frac{e_1^+ d_1^- g^+ h^-}{e_1^- d_1^- g^- h^-} = \frac{e_1^+ g^+}{e_1^- g^-} = -1 \\ \frac{c_2^{(+,-)}}{c_2^{(-,-)}} &= \frac{e_2^+ d_2^- g^+ h^-}{e_2^- d_2^- g^- h^-} = \frac{e_2^+ g^+}{e_2^- g^-} = 1\end{aligned} \tag{45}$$

Similarly we can derive

$$\frac{g^-}{g^+} = -1 \tag{46}$$

and,

$$e_2^- = -1, \;\; e_2^+ = e_1^- = e_1^+ = 1 \tag{47}$$

Fourthly, let us check the statements in (38) and (39) with the results for d and e coefficients in (43) and (47). We have

$$-1 = \frac{c_1^{(-,+)}}{c_1^{(+,-)}} = \frac{e_1^- d_1^+ g^- h^+}{e_1^+ d_1^- g^+ h^-} = \frac{g^- h^+}{g^+ h^-} \tag{48}$$

With, (42) and (46), $\frac{g^-/g^+}{h^-/h^+} = -1$, and $e_1^- = e_1^+ = 1$ and $d_1^+ = d_1^- = 1$, (48) is verified. Subsequently,

$$-1 = \frac{c_2^{(-,+)}}{c_2^{(+,-)}} = \frac{e_2^- d_2^+ g^- h^+}{e_2^+ d_2^- g^+ h^-} = \frac{-g^- h^+}{-g^+ h^-} \tag{49}$$

which with $e_2^- d_2^+ = -1$ and $e_2^+ d_2^- = -1$, then completes the verification of the claim in (38). Further, we have

$$\frac{c_1^{(-,-)}}{c_1^{(+,+)}} = \frac{e_1^- d_1^- g^- h^-}{e_1^+ d_1^+ g^+ h^+} = \frac{g^- h^-}{g^+ h^+} = -1 \tag{50}$$

with, $e_1^- d_1^- = e_1^+ d_1^+ = 1$.

$$\frac{c_2^{(-,-)}}{c_2^{(+,+)}} = \frac{e_2^- d_2^- g^- h^-}{e_1^+ d_1^+ g^+ h^+} = \frac{(-1)\times(-1)\times g^- h^-}{g^+ h^+} = -1 \tag{51}$$

because, $\frac{g^-}{g^+} = -1$, and, $\frac{h^-}{h^+} = 1$. So we may use the broken down expression for the c coefficients in (37) with the given coefficients in (36).

5. λ coefficients first coordinate

Let us assume in the first place that $\lambda_{jn}^{(0,\pm)} = \lambda_{jn}^{(\pm,0)} = 1$, for $j, n = 1, 2$. Let us, in the second place, look at

$$c_n^{(0,\pm)} = c_n^{(+,\pm)} \lambda_{1n}^{(+,\pm)} = c_n^{(-,\pm)} \lambda_{1n}^{(-,\pm)} \tag{52}$$

Let us first look at $n = 1$ and the second ι coordinate $\iota_2 = +$. We note from (36), $c_1^{(+,+)} = \sqrt{3}$ and $c_1^{(-,+)} = -\sqrt{3}$. So in order to bring those two coefficients in balance, we may take, $\lambda_{11}^{(+,+)} = 1$ and $\lambda_{11}^{(-,+)} = -1$. For $n = 2$ we have, $c_2^{(+,+)} = -1$ and $c_2^{(-,+)} = -1$. Then (52) for $n = 2$ and $\iota_2 = +$, is consistent with, $\lambda_{12}^{(+,+)} = \lambda_{12}^{(-,+)} = 1$. Subsequently let us take $\iota_2 = -$ and look at $n = 1$ again. This then gives from (36), $c_1^{(+,-)} = \sqrt{3}$ and $c_1^{(-,-)} = -\sqrt{3}$. Hence, (52) can be made consistent with $\lambda_{11}^{(+,-)} = 1$ and $\lambda_{11}^{(-,-)} = -1$. If we then take a look at $n = 2$ and $\iota_2 = -$, this gives first looking at (36), $c_2^{(+,-)} = 1$ and $c_2^{(-,-)} = 1$ the necessity to have $\lambda_{12}^{(+,-)} = \lambda_{12}^{(-,-)} = 1$ for consistency in (52).

6. λ coefficients second coordinate

Let us look at the $|x_1| > 0$ and $x_2 \in (-\epsilon, \epsilon)$, so $H(\epsilon, x_2) = 1$ and $H(\epsilon, x_1) = 0$.

$$c_n^{(\pm,0)} = c_n^{(\pm,+)} \lambda_{2n}^{(\pm,+)} = c_n^{(\pm,-)} \lambda_{2n}^{(\pm,-)} \tag{53}$$

Let us first inspect $n = 1$ and $\iota_1 = +$. From (36) we see, $c_1^{(+,+)} = \sqrt{3}$ and $c_1^{(+,-)} = \sqrt{3}$. Hence, (53) is consistent when $\lambda_{21}^{(+,+)} = \lambda_{21}^{(+,-)} = 1$. We have $\iota_1 = -$, with, $c_1^{(-,+)} = -\sqrt{3}$ and $c_1^{(-,-)} = -\sqrt{3}$. Hence, $\lambda_{21}^{(-,+)} = \lambda_{21}^{(-,-)} = 1$. Subsequently, we take $n = 2$ and $c_2^{(+,+)} = -1$ together with $c_2^{(+,-)} = 1$. Consistency in (53) can be concluded via $\lambda_{22}^{(+,+)} = -1$ and $\lambda_{22}^{(+,-)} = 1$. Subsequently, $n = 1$ and $\iota_1 = -$ gives $c_1^{(-,+)} = -\sqrt{3}$ and $c_1^{(-,-)} = -\sqrt{3}$. Hence, $\lambda_{21}^{(-,+)} = \lambda_{21}^{(-,-)} = 1$ gives (53) consistency in this case. For $n = 2$, $\iota_1 = -$ we see $c_2^{(-,+)} = -1$ and $c_2^{(-,-)} = 1$. Hence $\lambda_{22}^{(-,+)} = -1$ and $\lambda_{22}^{(-,-)} = 1$ in (53).

7. verification

In the previous two sections the λ and c coefficients were obtained. In this section the combination of the form in (37) will be verified for all the λ coefficients to check system consistency. In the first place we inspect $c_1^{(0,+)}$. We have

$$c_1^{(0,+)} = \lambda_{11}^{(-,+)} c_1^{(-,+)} = \lambda_{11}^{(+,+)} c_1^{(+,+)} \tag{54}$$

With (37) we write $\lambda_{11}^{(-,+)} e_1^- d_1^+ f_1 g^- h^+ = \lambda_{11}^{(+,+)} e_1^+ d_1^+ f_1 g^+ h^+$. Hence

$$\lambda_{11}^{(-,+)} = \lambda_{11}^{(+,+)} \frac{e_1^+ g^+}{e_1^- g^-} \tag{55}$$

With $\lambda_{11}^{(+,+)} = 1$ and $\lambda_{11}^{(-,+)} = -1$ this is consistent. The $\frac{e_1^+ g^+}{e_1^- g^-} = -1$. For, $c_1^{(0,-)}$ we write

$$c_1^{(0,-)} = \lambda_{11}^{(-,-)} c_1^{(-,-)} = \lambda_{11}^{(+,-)} c_1^{(+,-)} \tag{56}$$

With, (37) we then may note that, $\lambda_{11}^{(-,-)} e_1^- d_1^- f_1 g^- h^- = \lambda_{11}^{(+,-)} e_1^+ d_1^- f_1 g^+ h^-$. This implies,,

$$\lambda_{11}^{(-,-)} = \lambda_{11}^{(+,-)} \frac{e_1^+ g^+}{e_1^- g^-} \tag{57}$$

And so, with $\lambda_{11}^{(-,-)} = -1$ and $\lambda_{11}^{(+,-)} = 1$, there is also consistency in this case. Subsequently we look at $c_2^{(0,+)}$. Hence,

$$c_2^{(0,+)} = \lambda_{12}^{(-,+)} c_2^{(-,+)} = \lambda_{12}^{(+,+)} c_2^{(+,+)} \tag{58}$$

This implies, $\lambda_{12}^{(-,+)} e_2^- d_2^+ f_2 g^- h^+ = \lambda_{12}^{(+,+)} e_2^+ d_2^+ f_2 g^+ h^+$, such that

$$\lambda_{12}^{(-,+)} = \lambda_{12}^{(+,+)} \frac{e_2^+ g^+}{e_2^- g^-} \tag{59}$$

We have, $\frac{e_2^+ g^+}{e_2^- g^-} = 1$, and with, $\lambda_{12}^{(+,+)} = \lambda_{12}^{(-,+)} = 1$ hence consistency. Furthermore,

$$c_2^{(0,-)} = \lambda_{12}^{(-,-)} c_2^{(-,-)} = \lambda_{12}^{(+,-)} c_2^{(+,-)} \tag{60}$$

This leads to

$$\lambda_{12}^{(-,-)} = \lambda_{12}^{(+,-)} \frac{e_2^+ g^+}{e_2^- g^-} \tag{61}$$

We have, $\lambda_{12}^{(-,-)} = \lambda_{12}^{(+,-)} = 1$ and hence, consistency. Then we continue with $n = 1$ and $c_1^{(+,0)}$. This gives

$$c_1^{(+,0)} = \lambda_{21}^{(+,+)} c_1^{(+,+)} = \lambda_{21}^{(+,-)} c_1^{(+,-)} \tag{62}$$

Using (37) this reduces to

$$\lambda_{21}^{(+,+)} = \lambda_{21}^{(+,-)} \frac{d_1^- h^-}{d_1^+ h^+} \tag{63}$$

From the first equation in (41) it follows that $\frac{d_1^- h^-}{d_1^+ h^+} = 1$. hence, when $\lambda_{21}^{(+,+)} = \lambda_{21}^{(+,-)} = 1$ there is consistency. Similarly, the expression for $c_1^{(-,0)}$ leads to

$$\lambda_{21}^{(-,+)} = \lambda_{21}^{(-,-)} \frac{d_1^- h^-}{d_1^+ h^+} \tag{64}$$

This is consistent because in the previous section we found $\lambda_{21}^{(-,+)} = \lambda_{21}^{(-,-)} = 1$. Because, $\lambda_{22}^{(\pm,+)} = -1$ and $\lambda_{22}^{(\pm,-)} = 1$ together with $\frac{d_2^- h^-}{d_2^+ h^+} = -1$, the expressions for $c_2^{(\pm,0)}$ are consistent too. Hence, we have verified our system of coefficients and demonstrated its consistency. For decreasingly positive ϵ the system of connecting coefficients that connect the u in separate quadrants, are consistent.

III. CONCLUSION AND DISCUSSION

In the previous section it was demonstrated that the NSV equation has a nontrivial exact type A solution for the 4 quadrants of $\mathbb{R}^2$ and $x_k \neq 0$. The 4 quadrants show $u^{(\iota_1,\iota_2)}$, for, $(\mathbb{R}_{\iota_1} \backslash \{0\}) \times (\mathbb{R}_{\iota_2} \backslash \{0\})$ and $\iota_n \in \{+,-\}$, with $n = 1, 2$. The algebra for smoothly connecting solutions with $x_1 = 0$ and/or $x_2 = 0$ is given in the paper.

For sufficiently small $0 < \epsilon \to 0$, the 4 quadrant NSV solutions are connected and the algebra of coefficients is, by giving an example, demonstrated to be consistent. It must be noted that connecting the u_n^ι functions over the interval between $-\epsilon$ and ϵ may change the sign. For certain ι and n we can have outside the interval $u_n^\iota = c_n^\iota w(x_1, x_2, t)$ and inside the interval $u_n^\iota = -c_n^\iota w(x_1, x_2, t)$, because there a λ is -1. We may assume that a smooth but fast change can always replace the jump in sign. It is also noted that for $x_k \in (-\epsilon, \epsilon)_{0<\epsilon\to 0}$, the NSV equation does not apply. In a physical sense we may claim to have obtained an exact modified type A solution. The NSV breaks down physically in $x_k \in (-\epsilon, \epsilon)_{0<\epsilon\to 0}$ and this can be explained by the absence of continuum mechanics beyond a certain length limit in a real fluid. Note that the requirement of finite energy derives from physics. If this is used, then should irreducible length scales also not be a part of the solution considering that a fluid in real physics also consists of atoms in addition to having finite energy. This irreducible discreteness effect shows at the axes of the coordinate system. It occurs throughout the fluid however, because the origin of the coordinate system is arbitrary.

The algebraic construction of $\sum_{n=1}^{2} c_n^{\iota(x)} \alpha_n \text{sgn}(x_n) = 0$ is basic to the solution. Use is made of $\frac{\partial}{\partial x_n}|x_n| = \text{sgn}(x_n)$ thereby ignoring the $x_n \delta(x_n)$ term. For $|x_n| > 0$ the $\delta(x_n)$ from $\frac{\partial}{\partial x_k}\text{sgn}(x_k)$ is ignored. Moreover, select a pair (x_1, x_2), with e.g. $|x_2| > \epsilon$ and $(\exists_{\epsilon>0})\, x_1 \in (-\epsilon, \epsilon)$, $|x_1| > 0$, then there always will be an $0 < \epsilon' < \epsilon$ such that $x_1 \notin (-\epsilon', \epsilon')$ and (x_1, x_2) is included in the solution space. In fact for all $\{(x_1, x_2) \in \mathbb{R}^2 : \iota(x_n) \neq 0, n = 1, 2\}$ an exact solution is found. With the $c_n^{(0,\pm)}, c_n^{(\pm,0)}, c_n^{(0,0)}$ we can identify initial boundary values of the problem. Hence, we claim a slightly modified type A solution for NSV with connection to boundary (initial) values in $x_1 = 0$ and/or $x_2 = 0$. The (2+1) can be extended to (3+1) by having $u_3 = 0$, which means no contribution to finite energy, and $f_3 = 0$. In effect this means that in this paper we looked at a laminar NSV.

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[1] C.L. Fefferman, Existence and smoothness of the Navier Stokes equation, (2000), Clay Institute.