J
J. **x**, No. x (200x), pp. 1-9

A simple exact solution to the Navier Stokes equation.

Han Geurdes *

C vd Lijnstraat 164 2592 NN Den Haag Netherlands

Abstract. In this paper it is demonstrate, via construction, that the Navier Stokes equation has a smooth nontrivial exact solution in (2+1). The smoothness is accomplished by connecting disconnected pieces of solution over a vanishingly small intervall.We show by example that the algebra of connecting coefficients is consitent.

Key Words: exact solution Navier Stokes equation
AMS Subject Classifications: 25Q20, 76D05, 76M25

1 Introduction

In the present paper a simple solution to the (2+1), two space one time, Navier-Stokes equation is proposed that observes the requirements of vanishing divergence, finite energy and bounded absolute differentials of velocity and force [1]. The claim is that the pair of exact solutions (u,p) exists that observe the requirements. Here, the velocity vector, u, $\{u_i\}_{i=1}^2$, is matched with a simultaneous solution for pressure p. We have for the i-th element $u_i{=}u_i(x_1,x_2,t),(i{=}1,2)$ of the velocity vector and $p{=}p(x_1,x_2,t)$. The NS equation is:

$$\frac{\partial}{\partial t}u_i+\sum_{j=1}^{2}u_j\frac{\partial}{\partial x_j}u_i-\nu\nabla^2 u_i+\frac{\partial}{\partial x_i}p=f_i \tag{1.1}$$

Following [1] it is allowed to have $\nu{=}1$. The function f_i is external and we may assume to be able to select $f_i,(i{=}1,2)$ such that requirement (5) of [1] also applies. This assumption will be checked. The solution, u_i in (1.1) must have finite energy [1]

$$\int_{\mathbb{R}^2}\sum_{i=1}^{2}u_i^2(x_1,x_2,t)d^2x\leq C(t) \tag{1.2}$$

and a vanishing divergence $\sum_{i=1}^2\frac{\partial}{\partial x_i}u_i=0$. The challenge is to demonstrate that a nontrivial smooth exact solution (modified type A, [1]) is possible with the zero time initial conditions $u_{0,i}(x_1,x_2){=}u_i(x_1,x_2,0)$.

*Corresponding author. *Email address:* `han.geurdes@gmail.com` (J.F. Geurdes)

2 Solution heuristics

Let us define a heuristic solution for $u_i = u_i(x_1,x_2,t)$, with, $x=(x_1,x_2)$ and

$$u_i = c_i^{\iota(x)} \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] \left(\lambda_{i1}^{\iota(x_1)}\right)^{H(\epsilon,x_1)} \left(\lambda_{i2}^{\iota(x_2)}\right)^{H(\epsilon,x_2)} \equiv u_i^{\iota(x)} \tag{2.1}$$

with, $a>0$ real and $\alpha_k>0$ real, $k=1,2$, and $||\alpha||=1$ and $||.||$ the euclidean norm. The $\iota(x)$ in the superscript is intended as an index i.e. $\iota(x)=(\iota(x_1),\iota(x_2))=(\mathrm{sgn}(x_1),\mathrm{sgn}(x_2))$. E.g. $\iota(x)=(+,-)$, for $x_1>0$ and $x_2<0$. The superscript indicates that we are looking at the solution related to that subsection,or quadrant, of $\mathbb{R}^2$ where in the example $x_1>0$ and $x_2<0$. The family of vector functions $\{u^{\iota(x)}(x,t)|\iota(x)$ holds no zero's $\}$, are associated to the four quadrants that do not contain zero. Smoothness will be discussed later and is associated to the constants $c_i^{\iota(x)}$ and $\lambda_{ij}^{\iota(x_j)}$. The H exponents are defined by

$$H(\epsilon,x_i) = \begin{cases} 1, & x_i \in (-\epsilon,\epsilon) \\ 0, & x_i \notin (-\epsilon,\epsilon) \end{cases} \tag{2.2}$$

Furthermore, it is assumed that the constants $\{c_i^{\iota(x)}\}_{i=1}^2$ and $\{\alpha_i\}_{i=1}^2$ are such that

$$\sum_{j=1}^{2} \alpha_j c_j^{\iota(x)} \mathrm{sgn}(x_j) = 0 \tag{2.3}$$

With the sign of x_k in the index one can have different c. E.g., (x_1,x_2) such that, $x_1 > 0, x_2<0$ gives

$$c_1^{(+,-)}\alpha_1 - c_2^{(+,-)}\alpha_2 = 0 \tag{2.4}$$

or e.g. $x_1<0, x_2<0$

$$-c_1^{(-,-)}\alpha_1 - c_2^{(-,-)}\alpha_2 = 0 \tag{2.5}$$

etcetera, while $||\alpha||=1$. The reader may note that instead of e.g. $(+,+)$ we could have used $c_i^{1,1}$ or similar. The use of the superscript is to select the proper form from a family of 4 vector functions, associated to $x=(x_1,x_2)$, $x_k \neq 0$ and explicitly written as, $\{u^{(-,-)}(x,t), u^{(-,+)}(x,t), u^{(+,-)}(x,t), u^{(+,+)}(x,t)\}$ that differ by a constant c vector that gives (2.3). We take $\mathrm{sgn}(0)=0$. If, e.g. $x_1=0$, and, $x_2>0$, then we have $c_2^{(0,+)}$ in (2.1). Note in passing that a (1+1) NS is not a (completely) valid problem in terms of vanishing divergence and nontrivial solution. So, $c_i^{(0,+)}$ for i=1,2 is a valid characteristic of the (2+1) solution but is not a part of a nontrivial (1+1) solution.

2.1 Finite energy

The requirement of finite energy is given in equation (1.2). The superscript $\iota(x)$ can be suppressed in the argument. The requirement can be expressed in subspaces of $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$

($\times$ the Cartesian product),

$$\mathbb{R}^2 = (\mathbb{R}_- \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-) \cup (\mathbb{R}_+ \times \mathbb{R}_+)$$

This implies (suppressing the use of d^2x for the moment)

$$C(t) \geq \int_{\mathbb{R}^2} ||u||^2 = \int_{\mathbb{R}_- \times \mathbb{R}_-} ||u||^2 + \int_{\mathbb{R}_- \times \mathbb{R}_+} ||u||^2 + \int_{\mathbb{R}_+ \times \mathbb{R}_-} ||u||^2 + \int_{\mathbb{R}_+ \times \mathbb{R}_+} ||u||^2 \tag{2.6}$$

and $C(t)$ finite. We have $||u||^2 = u_1^2 + u_2^2$. Because in the analysis of smoothness, a vanishingly small interval is excluded, the integration for e.g. $x_1 < 0$ and $x_2 < 0$ must be written as

$$E_1(\epsilon) = \int_{-\infty}^{-\epsilon} \int_{-\infty}^{-\epsilon} u_1^2 dx_1 dx_2 dx_2 \tag{2.7}$$

with $0 < \epsilon \to 0$. In effect we may take the, in this case upper limit, of the integrations equal to zero and proceed in this way in our attempt to demonstrate finite energy. Hence, we may write the first term of (2.6) as

$$E_1 = \int_{-\infty}^{0} \int_{-\infty}^{0} u_1^2 dx_1 dx_2 \tag{2.8}$$

Then, looking at equation, (2.1), noting $x_k < 0$ in the first integral of (2.6).

$$E_1 = \int_{-\infty}^{0} \int_{-\infty}^{0} u_1^2 dx_1 dx_2 = \left\{ c_1^{(-,-)} \right\}^2 e^{-2at} \int_{-\infty}^{0} \int_{-\infty}^{0} \exp\left[2 \sum_{k=1}^{2} \alpha_k x_k \right] dx_1 dx_2 \tag{2.9}$$

Hence,

$$E_1 = \left\{ c_1^{(-,-)} \right\}^2 e^{-2at} \int_{-\infty}^{0} \exp[2\alpha_1 x_1] dx_1 \int_{-\infty}^{0} \exp[2\alpha_2 x_2] dx_2 \tag{2.10}$$

such that

$$E_1 = \left\{ c_1^{(-,-)} \right\}^2 e^{-2at} \int_{0}^{\infty} \exp[-2\alpha_1 x_1] dx_1 \int_{0}^{\infty} \exp[-2\alpha_2 x_2] dx_2 \tag{2.11}$$

which gives

$$E_1 = \left(\frac{1}{4\alpha_1 \alpha_2} \right) \left\{ c_1^{(-,-)} \right\}^2 e^{-2at} \tag{2.12}$$

The last integration term for u_1 in (2.6) is

$$E_4 = \int_{0}^{\infty} \int_{0}^{\infty} u_1^2 dx_1 dx_2 \tag{2.13}$$

then, looking again at (2.1), noting $x_k > 0$ here,

$$E_4 = \left\{ c_1^{(+,+)} \right\}^2 e^{-2at} \int_{0}^{\infty} \int_{0}^{\infty} \exp\left[-2 \sum_{k=1}^{2} \alpha_k x_k \right] dx_1 dx_2 \tag{2.14}$$

This then gives

$$E_4=\left(\frac{1}{4\alpha_1\alpha_2}\right)\left\{c_1^{(+,+)}\right\}^2 e^{-2at} \tag{2.15}$$

The second integral for u_1 is

$$E_2=\int_{-\infty}^{0}\int_{0}^{\infty} u_1^2 dx_1 dx_2 \tag{2.16}$$

Hence, we may write

$$E_2=\left\{c_1^{(-,+)}\right\}^2 e^{-2at}\int_{-\infty}^{0}\exp[2\alpha_1 x_1]dx_1\int_{0}^{\infty}\exp[-2\alpha_2 x_2]dx_2 \tag{2.17}$$

This implies

$$E_2=\frac{\left\{c_1^{(-,+)}\right\}^2 e^{-2at}}{4\alpha_1\alpha_2} \tag{2.18}$$

A similar form goes for E_3 the third term in (2.6). So,

$$E_3=\frac{\left\{c_1^{(+,-)}\right\}^2 e^{-2at}}{4\alpha_1\alpha_2} \tag{2.19}$$

Because for u_1 we have $E=E_1+E_2+E_3+E_4$ and for u_2 forms similar to, (2.12), (2.15), (2.18) and (2.19) can be derived, we may conclude that the energy is finite for this solution. Hence, it is possible to have

$$\infty > C(t) \geq e^{-at}\sum_{\iota_1\in\{-,+\}}\sum_{\iota_2\in\{-,+\}}\frac{||c^{(\iota_1,\iota_2)}||^2}{4\alpha_1\alpha_2}$$

2.2 Terms in the Navier Stokes equation

In the analysis we assume $x_k \notin (-\epsilon,\epsilon)$ for $k=1,2$, with $0<\epsilon\to 0$.

2.2.1 divergence

From (2.1) observe that, if the dot denotes the time differentiation, then, $\dot{u}_i=-au_i$. Subsequently,

$$\frac{\partial u_i}{\partial x_i}=c_i^{\iota(x)}\frac{\partial}{\partial x_i}\exp\left[-at-\sum_{k=1}^{2}\alpha_k|x_k|\right] \tag{2.20}$$

Note that the $\iota(x)$ in $c_i^{\iota(x)}$ is an index, not a power. Then,

$$\frac{\partial u_i}{\partial x_i}=-c_i^{\iota(x)}\left(\alpha_i\frac{\partial}{\partial x_i}|x_i|\right)\exp\left[-at-\sum_{k=1}^{2}\alpha_k|x_k|\right] \tag{2.21}$$

Furthermore, $\frac{\partial}{\partial x_i}|x_i| = \text{sgn}(x_i) + 2x_i\delta(x_i)$, with $\delta(x_i)$ the Dirac delta function. The term, $x_i\delta(x_i)$ can be ignored. We have, $\delta(x_i) \neq 0$ when $x_i = 0$. The δ arises from $\frac{\partial}{\partial x_i}\text{sgn}(x_i) = \delta(x_i) + \delta(-x_i)$ and $\delta(-x_i) = \delta(x_i)$ noting $\text{sgn}(x_i) = \Theta(x_i) - \Theta(-x_i)$ and $\Theta(x_i) = 1$ for $x_i \geq 0$ and $\Theta(x_i) = 0$ for $x_i < 0$. From the equation (2.21) and $x_i \neq 0$, it follows that

$$\sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} = -\left(\sum_{i=1}^{2} c_i^{\iota(x)} \alpha_i \text{sgn}(x_i)\right) \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] = 0 \tag{2.22}$$

The exponent term remains finite because $a > 0$, $\alpha_k > 0$ and $|x_k| > \epsilon$. From (2.3) the divergence of u, vanishes, i.e. $\nabla \cdot u = 0$, as required.

2.2.2 u_j product differentiation & ∇^2

In addition,

$$\frac{\partial u_i}{\partial x_j} = -c_i^{\iota(x)} \alpha_j \text{sgn}(x_j) \exp\left[-at - \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{2.23}$$

Hence,

$$u_j \frac{\partial u_i}{\partial x_j} = -c_i^{\iota(x)} c_j^{\iota(x)} \alpha_j \text{sgn}(x_j) \exp\left[-2at - 2\sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{2.24}$$

Because,(2.3) we see that

$$\sum_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} = 0 \tag{2.25}$$

From equation (2.23) it also follows that

$$\frac{\partial^2 u_i}{\partial x_j^2} = c_i^{\iota(x)} \{(\alpha_j)^2 - 2\alpha_j \delta(x_j)\} \exp\left[-at + \sum_{k=1}^{2} \alpha_k |x_k|\right] \tag{2.26}$$

with $||\alpha||^2 = 1$. Note $\delta(x_j) = 0$ for $x_j \neq 0$, then $\nabla^2 u_i = u_i$. Hence, the Navier-Stokes equation reduces for $x_k \neq 0$ with $k = 1,2$, to $(\nu = 1)$

$$-(a+1)u_i + \frac{\partial p}{\partial x_i} = f_i \tag{2.27}$$

Suppose we select $p = constant$, for all $x \in \mathbb{R}^2$ and $t \geq 0$, then,

$$f_i^{\iota(x)} = -(a+1) u_i^{\iota(x)}.$$

The requirement of multiple differentiability and finite bounded f_i is observed per quandrant, $(\mathbb{R}_- \backslash \{0\}) \times (\mathbb{R}_- \backslash \{0\})$, $(\mathbb{R}_- \backslash \{0\}) \times (\mathbb{R}_+ \backslash \{0\})$, $(\mathbb{R}_+ \backslash \{0\}) \times (\mathbb{R}_- \backslash \{0\})$and $(\mathbb{R}_+ \backslash \{0\}) \times (\mathbb{R}_+ \backslash \{0\})$ and $\iota(x)$ associated to the quadrants.

$$\left| \frac{\partial^k}{\partial x_1^k} \frac{\partial^m}{\partial x_2^m} \frac{\partial^n}{\partial t^n} f_i^{\iota(x)} \right| \leq \Lambda_{i,k,m,n}^{\iota(x)} \left(1 + ||x||^{C_{i,k,m,n}^{\iota(x)}}\right)^{-1} \tag{2.28}$$

and $\Lambda^{\iota(x)}_{i,k,m,n}\in\mathbb{R}_+\backslash\{0\}$ and $C^{\iota(x)}_{i,k,m,n}\in\mathbb{R}_+\backslash\{0\}$. This is so because in each quadrant we may write u_i as an exponent that for x_k and for $t>0$ vanishes.

2.3 Smoothness of u_i arguments

The following sections will be devoted to the connection between the solutions of the four $\iota(x)\neq 0$ subspaces via x_i in $(-\epsilon,\epsilon)$ intervals. The left and right hand limit of u_i at each (x_1,x_2) must be equal in order to claim a smooth solution. For $x_1=0$ and/or $x_2=0$ this can be accomplished with the λ coefficients in the functions with one or both spatial variables inside $(-\epsilon,\epsilon)$. The λ coefficients disturb the function such that it is no longer a solution of the NS. The $0<\epsilon\to 0$ warrants that this 'not valid' interval is vanishingly small. In physics, a real fluid will drastically change if the ϵ decreases beyond the size of the atoms. Boundary values in $x_k=0$, i.e. along the axes, are initial givens to the Navier Stokes equation.

2.3.1 coefficients

In the first place let us look at the following limit, where $x_2\notin(-\epsilon,\epsilon)$,

$$\lim_{\substack{0>x_1\to 0^-\\ x_2\notin(-\epsilon,\epsilon)}} u_i^{\iota(x)}(x_1,x_2,t)=c_i^{(-,\pm)}\lambda^-_{i1}w(0^-,x_2,t) \tag{2.29}$$

with $w(x_1,x_2,t)=\exp[-at-\alpha_1^{(-,\pm)}|x_1|-\alpha_2^{(-,\pm)}|x_2|]$. Note that, $w(0^-,x_2,t)=w(0,x_2,t)$. The λ^-_{i1} is included because for $0<x_1\to 0$, we have at a certain point $H(\epsilon,x_1)=1$ when $0<x_1\to 0$ 'faster' than $0<\epsilon\to 0$. The second limit we need to look at is

$$\lim_{\substack{0<x_1\to 0^+\\ x_2\notin(-\epsilon,\epsilon)}} u_i^{\iota(x)}(x_1,x_2,t)=c_i^{(+,\pm)}\lambda^+_{i1}w(0^+,x_2,t) \tag{2.30}$$

Here, $w(0^+,x_2,t)=w(0,x_2,t)$. For $x_1=0$ we have

$$u_i^{\iota(x)}(0,x_2,t)=c_i^{(0,\pm)}\lambda^0_{i1}w(0,x_2,t) \tag{2.31}$$

For continuous connection of the expressions in (2.29)-(2.31), we need to have $\lambda^-_{i1},\lambda^+_{i1}$ and λ^0_{i1} such that

$$c_i^{(-,\pm)}\lambda^-_{i1}=c_i^{(+,\pm)}\lambda^+_{i1}=c_i^{(0,\pm)}\lambda^0_{i1} \tag{2.32}$$

For $x_1\notin(-\epsilon,\epsilon)$ and $0\neq x_2\to 0$ an equation similar to (2.32) is

$$c_i^{(\pm,-)}\lambda^-_{i2}=c_i^{(\pm,+)}\lambda^+_{i2}=c_i^{(\pm,0)}\lambda^0_{i2} \tag{2.33}$$

The question is whether it is possible to have $c^{\iota(x)}$ vectors of constants. Let us define $\mu^\pm_{ij}=\lambda^\pm_{ij}/\lambda^0_{ij}$. Hence,

$$c_i^{(-,\pm)}\mu^-_{i1}=c_i^{(+,\pm)}\mu^+_{i1}=c_i^{(0,\pm)} \tag{2.34}$$

For $x_1 \notin (-\epsilon,\epsilon)$ and $0 \neq x_2 \to 0$ an equation similar to (2.32) is

$$c_i^{(\pm,-)}\mu_{i2}^{-} = c_i^{(\pm,+)}\mu_{i2}^{+} = c_i^{(\pm,0)} \tag{2.35}$$

Subsequently, for the case where $x_k \in (-\epsilon,\epsilon), k=1,2$, the continuous condition gives in the μ form

$$c_i^{(-,-)}\mu_{i1}^{-}\mu_{i2}^{-} = c_i^{(+,+)}\mu_{i1}^{+}\mu_{i2}^{+} = c_i^{(-,+)}\mu_{i1}^{-}\mu_{i2}^{+} = c_i^{(+,-)}\mu_{i1}^{+}\mu_{i2}^{-} = c_i^{(0,0)} \tag{2.36}$$

The second equation of (2.34) gives $c_i^{(+,+)}\mu_{i1}^{+}\mu_{i2}^{+} = c_i^{(0,+)}\mu_{i2}^{+}$ and $c_i^{(-,-)}\mu_{i1}^{-}\mu_{i2}^{-} = c_i^{(0,-)}\mu_{i2}^{-}$. Looking at (2.36) we see that $c_i^{(0,-)}\mu_{i2}^{-} = c_i^{(0,+)}\mu_{i2}^{+}$ such that

$$\frac{\lambda_{i2}^{+}}{\lambda_{i2}^{-}} = \frac{c_i^{(0,-)}}{c_i^{(0,+)}} \tag{2.37}$$

Using equations (2.35) and(2.36) we see $c_i^{(+,+)}\mu_{i1}^{+}\mu_{i2}^{+} = c_i^{(+,0)}\mu_{i1}^{+} = c_i^{(0,0)}$ together with $c_i^{(-,-)}\mu_{i1}^{-}\mu_{i2}^{-} = c_i^{(-,0)}\mu_{i1}^{-} = c_i^{(0,0)}$. So,

$$\frac{\lambda_{i1}^{+}}{\lambda_{i1}^{-}} = \frac{c_i^{(-,0)}}{c_i^{(+,0)}} \tag{2.38}$$

From (2.36) it can be derived that

$$c_i^{(-,+)}\lambda_{i1}^{-}\lambda_{i2}^{+} = c_i^{(+,-)}\lambda_{i1}^{+}\lambda_{i2}^{-} = c_i^{(0,0)}\lambda_{i1}^{0}\lambda_{i2}^{0}$$

Hence,

$$\left(\frac{\lambda_{i1}^{+}}{\lambda_{i1}^{-}}\right)\left(\frac{\lambda_{i2}^{-}}{\lambda_{i2}^{+}}\right) = \frac{c_i^{(-,+)}}{c_i^{(+,-)}} \tag{2.39}$$

In this previous equation (2.39), the λ's can be eliminated with the results in (2.37) and (2.38) to give

$$\frac{c_i^{(-,+)}}{c_i^{(+,-)}} = \frac{c_i^{(-,0)}c_i^{(0,+)}}{c_i^{(+,0)}c_i^{(0,-)}} \tag{2.40}$$

Moreover, from(2.36)

$$\frac{c_i^{(-,-)}}{c_i^{(+,+)}} = \frac{c_i^{(-,0)}c_i^{(0,-)}}{c_i^{(+,0)}c_i^{(0,+)}} \tag{2.41}$$

2.3.2 example

Suppose, $\alpha_1 = \frac{1}{2} > 0, \alpha_2 = \frac{1}{2}\sqrt{3} > 0$. Hence, $||\alpha||^2 = 1$. The next step is to see if c vectors are possible such that (2.40) and (2.41) hold. Let us define the c coefficients with unequal signs in the superscript. Hence,

$$\begin{array}{ll} c_1^{(+,-)} = \sqrt{3}, & c_2^{(+,-)} = 1 \\ c_1^{(-,+)} = -\sqrt{3}, & c_2^{(-,+)} = -1 \end{array} \tag{2.42}$$

So,

$$\alpha_1 c_1^{(+,-)} - \alpha_2 c_2^{(+,-)} = \left(\frac{1}{2} * \sqrt{3}\right) - \left(\frac{1}{2}\sqrt{3} * 1\right) = 0$$

and,

$$-\alpha_1 c_1^{(-,+)} + \alpha_2 c_2^{(-,+)} = -\left(-\frac{1}{2} * \sqrt{3}\right) + \left(-\frac{1}{2}\sqrt{3} * 1\right) = 0.$$

For equal sign coefficients

$$\begin{array}{ll} c_1^{(+,+)} = \sqrt{3}, & c_2^{(+,+)} = -1 \\ c_1^{(-,-)} = -\sqrt{3}, & c_2^{(-,-)} = 1 \end{array} \tag{2.43}$$

Subsequently, we have to check consistency

$$\alpha_1 c_1^{(+,+)} + \alpha_2 c_2^{(+,+)} = \left(\frac{1}{2} * \sqrt{3}\right) + \left(-\frac{1}{2}\sqrt{3} * 1\right) = 0$$

together with

$$-\alpha_1 c_1^{(-,-)} - \alpha_2 c_2^{(-,-)} = -\left(-\frac{1}{2} * \sqrt{3}\right) - \left(\frac{1}{2}\sqrt{3} * 1\right) = 0$$

In order to match with (2.40) and (2.41), we could have $c_i^{(-,0)} = c_i^{(+,0)} = c_i^{(0,+)} = 1$ and $c_i^{(0,-)} = -1$. Hence, $\frac{\lambda_{i1}^+}{\lambda_{i1}^-} = 1$ and $\frac{\lambda_{i2}^+}{\lambda_{i2}^-} = -1$. We can conclude that the algebra for continuous connection is possible. So the process $0 < \epsilon \rightarrow 0$ holds consistent selections of coefficients.

3 Conclusion and discussion

In the previous section it was demonstrated that the Navier Stokes equation has a non-trivial exact solution for the 4 quadrants of $\mathbb{R}^2$ and $x_k \neq 0$.

The solution refers to 4 quadrants. We have $u^{(-,-)}$, for, $(\mathbb{R}_-\backslash\{0\}) \times (\mathbb{R}_-\backslash\{0\})$, $u^{(-,+)}$ for $(\mathbb{R}_-\backslash\{0\}) \times (\mathbb{R}_+\backslash\{0\})$, $u^{(+,-)}$ for $(\mathbb{R}_+\backslash\{0\}) \times (\mathbb{R}_-\backslash\{0\})$, and $u^{(+,+)}$ for $(\mathbb{R}_+\backslash\{0\}) \times (\mathbb{R}_+\backslash\{0\})$. The algebraic construction of $\sum_{i=1}^{2} c_i^{\iota(x)} \alpha_i \mathrm{sgn}(x_i) = 0$ is basic to the solution. Use is made of $\frac{\partial}{\partial x_i}|x_i| = \mathrm{sgn}(x_i)$ thereby ignoring the $x_i \delta(x_i)$ term. This is allowed for $|X_i| > 0$. The required zero time initial conditions can be found at the $t = 0$ point of the solution and obey the requirements as well. For $x_1 = 0, x_2 \neq 0$ we have the vector $u^{(0,\mathrm{sgn}(x_2))}(0, x_2, t)$ with $c^{(0,\mathrm{sgn}(x_2))}$. Similarly for $x_2 = 0$, $x_1 \neq 0$. In $(x_1, x_2) = (0,0)$ we have $u^{(0,0)}(0,0,t)$.

The argumentation for smoothness related to $x_1 = 0$ and/or $x_2 = 0$ connecting the separate solutions associated to the quadrants $(\mathbb{R}_-\backslash\{0\}) \times (\mathbb{R}_-\backslash\{0\})$, $(\mathbb{R}_-\backslash\{0\}) \times (\mathbb{R}_+\backslash\{0\})$, $(\mathbb{R}_+\backslash\{0\}) \times (\mathbb{R}_-\backslash\{0\})$and $(\mathbb{R}_+\backslash\{0\}) \times (\mathbb{R}_+\backslash\{0\})$, is given in the paper. For sufficiently small $0 < \epsilon \rightarrow 0$, the 4 quadrant NS solutions are connected and the algebra of coefficients is demonstrated to be consistent. It is noted that for $x_k \in (-\epsilon, \epsilon)_{0 < \epsilon \rightarrow 0}$, the NS equation does not apply. In a physical sense we hence may claim to have obtained an exact modified type A solution. The NS breaks down physically in $x_k \in (-\epsilon, \epsilon)_{0 < \epsilon \rightarrow 0}$ because the absence of continuum mechanics beyond a certain length limit in a real fluid. Note in passing that the origin of $\mathbb{R}^2$ is arbitrary. Moreover, select a pair (x_1,x_2), with, $(\exists_{\epsilon>0}) x_1 \in (-\epsilon,\epsilon)$, $|x_1|>0$, then there always will be an $0<\epsilon'<\epsilon$ such that $x_1 \notin (-\epsilon',\epsilon')$ and (x_1,x_2) is included in the solution space. In fact for all $\{(x_1,x_2)\in\mathbb{R}^2 : \iota(x_i)\neq 0, i=1,2\}$ an exact solution is found. With the $c_i^{(0,\pm)}, c_i^{(\pm,0)}, c_i^{(0,0)}$ we can identify initial boundary values of the problem. Hence, we claim a type A solution with connection to boundary (initial) values in $x_1=0$ and/or $x_2=0$.

Finally let us inspect the modifications to the type A of [1] we employed here. In the first place, we did not select $f_i=0$ for $i=1,2$ such as was officially stated in type A of [1]. In the second place the paper shows the (2+1) version of the NS equation. Of course one can correctly put in $u_3=0$ and come with a formal $n=3$, (3+1) from the previous considerations in the paper. Finite energy remains valid, and $u_3=0$ fits $p=\mathit{constant}$ and $f_3=0$. However, it is most likely that this is not what is intended in [1] when $n=3$ is insisted upon. In the third place the boundary value on the x_1 and x_2 axes and the irreducible breakdown of NS in a vanishingly small section of $\mathbb{R}^2$ is also not present in the type A of [1]. We could argue that, considering physical validity such as for finite energy, cannot avoid the existence of atoms. In other words, if one claims finite energy, should one also not allow irreducible lengths where the atoms prevent continuum mechanics? Despite the modifications of the type A solution of [1], the paper is a step closer to this type A.

References

[1] C.L. Fefferman, Existence and smoothness of the Navier Stokes equation, (2000), Clay Institute.