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A simple exact solution to the Navier Stokes equation.

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Abstract. In this paper it is demonstrated that the Navier Stokes equation has a smooth nontrivial exact solution. The solution is a heuristic and is the smoothly glueing together of $x_k \geq 0$ with $x_k < 0$ solutions.

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1 Introduction

In the present paper a simple solution to the Navier-Stokes equation is proposed that observes the requirements of vanishing divergence, finite energy and bounded absolute differentials of velocity and force [1]. The claim is that the pair of exact solutions (u,p) exists that observe the requirements. Here, the velocity vector, u, $\{u_i\}_{i=1}^3$, is matched with a simultaneous solution for pressure p. We have for the i-th element $u_i = u_i(x_1,x_2,x_3,t)$, $(i = 1,2,3)$ of the velocity vector and $p = p(x_1,x_2,x_3,t)$ in the NS equation

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i - \nu \nabla^2 u_i + \frac{\partial}{\partial x_i} p = f_i \tag{1.1}$$

Following [1] it is allowed to have $\nu = 1$. The function f_i is external and we may assume to be able to select $f_i, (i=1,2,3)$ such that requirement (5) of [1] also applies. This assumption will be checked. The solution, u_i in (1.1) must have finite energy [1]

$$\int_{\mathbb{R}^3} \sum_{i=1}^{3} u_i^2(x_1,x_2,x_3,t) d^3x \leq C(t) \tag{1.2}$$

and a vanishing divergence $\sum_{i=1}^3 \frac{\partial}{\partial x_i} u_i = 0$. The challenge is to demonstrate that a non-trivial smooth exact solution (type A, [1]) is possible with the zero time initial conditions $u_{0,i}(x_1,x_2,x_3) = u_i(x_1,x_2,x_3,0)$.

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2 Solution heuristics

Let us define a heuristic solution for $u_i = u_i(x_1,x_2,x_3,t)$, with,

$$u_i = \begin{cases} c_i \exp\left[-at - \sum_{k=1}^3 \alpha_k x_k\right], & \forall_{x_k>0} \& k=1,2,3 \\ \\ c_i \exp\left[-at + \sum_{k=1}^3 \alpha_k x_k\right], & \forall_{x_k<0} \& k=1,2,3 \end{cases} \tag{2.1}$$

and, $a > 0$ real and $\alpha_k > 0$ real, with $||\alpha|| = 1$. If $x_k = 0$ for $k = 1,2,3$, then $u_i = c_i \exp[-at]$. We may assume that the constants $\{c_i\}_{i=1}^3$ and $\{\alpha_i\}_{i=1}^3$ are such that $\sum_{j=1}^3 \alpha_j c_j = 0$.

2.1 Finite energy

The requirement of finite energy is given in equation (1.2). Per entry of the sum $||u||^2$ this can be written

$$\int_{-\infty}^{\infty} u_k^2 dx_k = \int_{-\infty}^{0} u_k^2 dx_k + \int_{0}^{\infty} u_k^2 dx_k \tag{2.2}$$

Looking at equation (2.1) we see

$$\int_{-\infty}^{\infty} u_k^2 dx_k = 2c_k^2 e^{-2at} \int_0^\infty e^{-2\alpha_k x_k} dx_k = \frac{c_k^2 e^{-2at}}{\alpha_k} \tag{2.3}$$

For finite $\alpha_k > 0, k = 1,2,3$, the requirement of finite energy in equation (1.2) is observed.

2.2 Solution for $x_k > 0, (k = 1,2,3)$

From (2.1) observe that, if the dot denotes the time differentiation, then, $\dot{u}_i = -a u_i$. Subsequently

$$\frac{\partial u_i}{\partial x_i} = -c_i \alpha_i \exp\left[-at - \sum_{k=1}^3 \alpha_k x_k\right] \tag{2.4}$$

From this equation it follows that

$$\sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = -\left(\sum_{i=1}^3 c_i \alpha_i\right) \exp\left[-at - \sum_{k=1}^3 \alpha_k x_k\right] = 0 \tag{2.5}$$

Hence, the divergence of u, vanishes, i.e. $\nabla \cdot u = 0$, as required. In addition,

$$\frac{\partial u_i}{\partial x_j} = -c_i \alpha_j \exp\left[-at - \sum_{k=1}^3 \alpha_k x_k\right] \tag{2.6}$$

Hence,

$$u_j \frac{\partial u_i}{\partial x_j} = -c_i c_j \alpha_j \exp\left[-2at - 2\sum_{k=1}^3 \alpha_k x_k\right] \tag{2.7}$$

Because, $\sum_{j=1}^{3}\alpha_j c_j=0$,we see that

$$\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}=0 \tag{2.8}$$

From equation (2.6) it also follows that $\nabla^2 u_i=u_i$, when it is noted that $||\alpha||=1$. Hence, the Navier-Stokes equation reduces for $x_k>0$ with $k=1,2,3$, to $(\nu=1)$

$$-(a+1)u_i+\frac{\partial p}{\partial x_i}=f_i \tag{2.9}$$

Suppose we select $p=\sum_{k=1}^{3}\exp[-x_k]d_k$, with $\{d_k\}_{k=1}^{3}$ constants, then,

$$f_i=-d_i\exp[-x_i]-(a+1)u_i$$

and the requirement of multiple differentiability and finite bounded forms $|\frac{\partial^n f_i}{\partial x_j^n}|$ is observed for $x_k>0, k=1,2,3$.

2.3 Solution for $x_k<0, (k=1,2,3)$

The time differentiation does not change, $\dot{u}_i=-au_i$. Furthermore,

$$\frac{\partial u_i}{\partial x_i}=c_i\alpha_i\exp\left[-at+\sum_{k=1}^{3}\alpha_k x_k\right] \tag{2.10}$$

similarly to the previous case this leads us to vanishing divergence for $x_k<0\,(k=1,2,3)$ because $\sum_{j=1}^{3}\alpha_j c_j=0$ remains unaffected for a change in the sign of the x_k. With a similar argument one can also arrive at

$$\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}=0 \tag{2.11}$$

for $x_k<0\,(k=1,2,3)$. In this domain of x we also have for u_i

$$\frac{\partial^2 u_i}{\partial x_j^2}=c_i\alpha_j^2\exp\left[-at+\sum_{k=1}^{3}\alpha_k x_k\right] \tag{2.12}$$

Because, $||\alpha||=1$, it also follows that $\nabla^2 u_i=u_i$ for $x_k<0\,(k=1,2,3)$. Hence,

$$-(a+1)u_i+\frac{\partial p}{\partial x_i}=f_i \tag{2.13}$$

in the case that $x_k<0\,(k=1,2,3)$. Similarly we can have $p=\sum_{k=1}^{3}\exp[x_k]d_k$ and

$$f_i=d_i\exp[x_i]-(a+1)u_i$$

in $x_k<0\,(k=1,2,3)$.

3 Conclusion

In the previous section it was demonstrated that the Navier Stokes equation has a smooth type A , cite1, solution. Basically $\mathbb{R}^3$ is dissected in $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$ and, via x=(0,0,0) the two parts can be "glued" together resulting in a smooth solution. The algebraic construction of a vanishing sum $\sum_{j=1}^{3} c_i \alpha_i$ with c_i from the entries u_i and α_i from the exponent in the entries, stands at the foundation of resolving the problem. Zero time initial conditions can be found at the $t=0$ point of the solution and obey the requirements as well.

References

[1] C.L. Fefferman, Existence and smoothness of the Navier Stokes equation, (2000), Clay Institute.

[2] A. Kozachok, Navier-Stokes Millennium Prize Problem., preprint and XII International Scientific Kravchuk Conference,(2008) 197-198.

[3] C. Sun, and H. Gao, *Hausdorff dimension of random attractor for stochastic Navier Stokes Voight equations*, Dynamics of PDE, (2010) 7(4), 307-326.

[4] C. Foias, R.M.S. Rosa, and R. Temam, *Properties of the time-dependent statistical solutions of the three dimensional Navier Stokes equation*, (2012), arXiv math.AP:1111.6257.

[5] J.L. Guermond, *Faedo-Galerkin weak solutions of the Navier Stokes equations with Dirichlet boundary conditions are suitable*, J. Math. Pures. Appl. (2007), 88, 87-106.

[6] N.S. Bakhvalov, Ya.M. Zhileikin, E.A. Zabolotskaya, *Nonlinear theory of sound beams*, AIP press (1987), 67-68.