JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **x**, No. x (200x), pp. 1-5

An exact radial smooth type A solution to the Navier-Stokes equation.

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Abstract. In this paper it is demonstrated that the Navier Stokes equation has a smooth type A nontrivial exact solution combining two radial solutions inside and outside the unit sphere.

Key Words: exact solution Navier Stokes equation

AMS Subject Classifications: 35Q30, 76D05, 76M35

Chinese Library Classifications: O175.27

1 Introduction

One of the Clay institute millenium problems is the yes or no existence of an exact solution of the Navier-Stokes equation for the velocity vector, with elements $\{u_i\}_{i=1}^3$, matched with the pressure p . We have $u_i = u_i(x_1,x_2,x_3,t),(i=1,2,3)$ and $p(x_1,x_2,x_3,t)$ in the Navier stokes equation

$$\frac{\partial}{\partial t}u_i+\sum_{j=1}^{3}u_j\frac{\partial}{\partial x_j}u_i-\nu\nabla^2u_i+\frac{\partial}{\partial x_i}p=f_i \tag{1.1}$$

The function f_i is considered externally given. Furthermore, the solution, u_i in (1.1) must have finite energy. We have $\nu>0$ and

$$\int_{\mathbf{R}^3}\sum_{i=1}^{3}u_i^2(x_1,x_2,x_3,t)d^3x\leq C(t) \tag{1.2}$$

and a vanishing divergence $\sum_{i=1}^3\frac{\partial}{\partial x_i}u_i=0$. The idea is to demonstrate that an exact solution is possible or not given the requirements and the zero time initial conditions $u_{0,i}(x_1,x_2,x_3)=u_i(x_1,x_2,x_3,0)$

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2 Solution

Let us start to define $x_i = r\beta_i$ for fixed $\beta_i,(i=1,2,3)$ and $\sum_{i=1}^{3}\beta_i^2=1$. Here, $r=\sqrt{x_1^2+x_2^2+x_3^2}$. Subsequently, let us define a heuristic solution for $u_i = u_i(x_1,x_2,x_3,t)$, with,

$$u_i = \{ \begin{array}{ll} c_i \exp[-at-b/r], & 0<r\leq 1 \\ (c_i/r)\exp[-at-br], & r\geq 1 \end{array} \tag{2.1}$$

with, $a>0, b>0$ real and $c_i \in \mathbf{R}$. The initial value function equals $u_{0,i}(x_1,x_2,x_3) = u_i(x_1,x_2,x_3,0)$. The function in equation (2.1) is "sufficiently smooth" for $r>0$ and $t>0$.

2.1 Finite energy

In the inspection of the requirements, given in the introductory section, let us check (2.1) for finite energy. We note that generally the solution must show,

$$\int_{\mathbf{R}^3} \sum_{i=1}^{3} u_i^2(x_1,x_2,x_3,t) d^3x \leq C(t) \tag{2.2}$$

The $C(t)$ is finite. The angular terms of (2.1) give a finite contribution to the energy. Below it will be demonstrated that the velocity in radial terms, including the r^2 from the Jacobian $J=r^2\sin\theta$, gives finite energy too. Firstly,

$$\int_0^\infty r^2 u_i^2(r,t) dr \leq C_i(t) \tag{2.3}$$

From the definition in (2.1) the requirement is

$$\int_0^1 r^2 u_i^2(r,t) dr + \int_1^\infty r^2 u_i^2(r,t) dr \leq C_i(t) \tag{2.4}$$

Inside the unit sphere we see, for $b>0$, $r^2 \leq 1$ together with $-\frac{b}{r} \leq -b$

$$\int_0^1 r^2 u_i^2(r,t) dr \leq c_i^2 \exp[-2(at+b)] \tag{2.5}$$

Secondly, for $r \geq 1$, including the r^2 from the Jacobian

$$\int_1^\infty r^2 u_i^2(r,t) dr = c_i^2 e^{-2at} \int_1^\infty e^{-2br} dr \leq c_i^2 e^{-2at} \frac{e^{-2b}}{2b} \tag{2.6}$$

Here, $b>0$ and finite real. Hence, from the previous equations (2.3)-(2.6) it follows that $C_i(t) \geq \max\{1,\frac{1}{2b}\} c_i^2 \exp[-2(at+b)]$ can be finite. The finite energy requirement is correctly observed for the solution in (2.1).

2.2 Vanishing divergence of the solution

If we suppose $0<r\leq 1$ then

$$\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=\sum_{i=1}^{3}e^{-at}c_i\frac{\partial}{\partial x_i}e^{-b/r}=\frac{be^{-at}}{r^2}\sum_{i=1}^{3}\beta_i c_i \tag{2.7}$$

Hence, from the assumption

$$\sum_{i=1}^{3}\beta_i c_i=0 \tag{2.8}$$

it follows that $\nabla\cdot u=0$. Suppose then that, $r\geq 1$. The requirement for $r\geq 1$, is to have, $\sum_{i=1}^{3}\frac{\partial}{\partial x_i}u_i=0$ so

$$\frac{\partial u_i}{\partial x_i}=c_i e^{-at}\left\{-\frac{x_i}{r^3}e^{-br}-\frac{bx_i}{r^2}e^{-br}\right\} \tag{2.9}$$

In this equation the product $c_i\beta_i$ is identified and note, $\sum_{i=1}^{3}c_i\beta_i=0$. Hence, the required vanishing divergence also applies to the $r\geq 1$ case.

2.3 Navier-Stokes for $0<r\leq 1$

In the first part of the solution we have $\frac{\partial}{\partial t}u_i=-au_i$. Subsequently, from $\frac{\partial}{\partial x_j}u_i=\frac{bx_j}{r^3}u_i$

$$\sum_{j=1}^{3}u_j\frac{\partial}{\partial x_j}u_i=\sum_{j=1}^{3}u_j\frac{bx_j}{r^3}u_i \tag{2.10}$$

In (2.10) we may note the co-occurrence of c_j and $x_j=\beta_j r$, so from (2.8) it follows that for $0<r\leq 1$ we have $\sum_{j=1}^{3}u_j\frac{\partial}{\partial x_j}u_i=0$. In addition, the algebraic consequence of (2.1) for the Navier - Stokes is

$$\frac{\partial^2}{\partial x_j^2}u_i=b\left\{\frac{1}{r^3}-\frac{3r^2x_j^2}{r^7}\right\}u_i+\frac{b^2x_j^2}{r^6}u_i \tag{2.11}$$

The previous algebraic excercise gives the following

$$\nabla^2 u_i=\frac{b^2}{r^4}u_i \tag{2.12}$$

Looking back at equation (1.1) gives for $\frac{\partial}{\partial x_j}p$

$$-au_i-\nu\frac{b^2}{r^4}u_i+\frac{\partial}{\partial x_i}p=f_i \tag{2.13}$$

When $p=p(r,t)$ it is $\frac{\partial}{\partial x_i}p=\beta_i p'(r,t)$ with the prime indicating the r derivation. Hence,

$$\sum_{i=1}^{3}\left(-a\beta_i u_i-\nu\frac{b^2}{r^4}\beta_i u_i+\beta_i^2 p'(r,t)\right)=\sum_{i=1}^{3}\beta_i f_i \tag{2.14}$$

From this equation the $\beta_i u_i$ in the sum warrants the vanishing of the first two terms in (2.14) based on the vanishing divergence (2.8). Hence, because $\sum_{i=1}^{3}\beta_i^2=1$, we see

$$p(r,t)=p(0,t)+\sum_{i=1}^{3}\beta_i\int_0^r f_i(r_1,t)dr_1 \tag{2.15}$$

Given $0<r\leq 1$ it then follows that (2.1) contains the $(u_1,u_2,u_3)^T$ solution associated with $p=p(r,t)$ in (2.15). The choice of f_i in (2.15) is still "free".

2.4 Navier-Stokes for $r\geq 1$

Similarly to the previous algebraic construction we may observe that $\frac{\partial}{\partial t}u_i=-au_i$. We note that

$$\frac{\partial u_i}{\partial x_j}=c_i e^{-at}\left\{-\frac{x_j}{r^3}e^{-br}-\frac{bx_j}{r^2}e^{-br}\right\} \tag{2.16}$$

In the previous equation we see that $\beta_j=x_j/r$ occurs. Together with c_j from the pre-multiplication with u_j the product $c_j\beta_j$ occurs. We have $\sum_{j=1}^{3}c_j\beta_j=0$. Hence the term $\sum_{j=1}^{3}u_j\frac{\partial}{\partial x_j}u_i=0$. Subsequently we note that in the radial terms of u_i,

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

This leads us to $\nabla^2 u_i=b^2u_i$. Hence,

$$-(a+\nu b^2)u_i+\frac{\partial}{\partial x_i}p=f_i \tag{2.17}$$

If f_i conveniently can be selected for $r\geq 1$ such that

$$f_i=g_i-(a+\nu b^2)u_i \tag{2.18}$$

then $p(x_1,x_2,x_3,t)=(x\cdot g)$ for g a real constant vector in $r\geq 1$.

3 Requirements for f_i

In the previous two sections two reduced forms for $p(x_1,x_2,x_3,t)$ were obtained. In (2.15) the selected f_i is "free". So, regarding the requirement that f_i must be multiply differentiable, let us take

$$f_i=g_i-(a+\nu b^2)u_i \tag{3.1}$$

for $r>0$ and the u_i come from (2.1). Suppose, for $r\geq 1$ we have $\varphi(r)=\frac{1}{r}e^{-br}$. Then

$$\frac{\partial\varphi(r)}{\partial r}=-\left(\frac{1}{r}+b\right)\varphi(r) \tag{3.2}$$

for $b>0$ finite. Then noting radial dependence only in $r \geq 1$, we may repeatedly apply $\frac{\partial}{\partial r}$ to (3.2) and be convinced that $|\frac{\partial^n}{\partial x_j^n} f_i|$, with, $n=0,1,2,...$ and $i,j=1,2,3$, will remain finite for $\mathbb{R}^3$ where $r \geq 1$. For $0<r \leq 1$, we have for $\psi(r)=e^{-b/r}$ the limit behavior $\lim_{r\to 0}\psi(r)=0$. The multiple application of $\frac{\partial}{\partial r}$ to $\psi(r)$ provides powers of $1/r$. Note that, $\frac{\partial}{\partial r}\psi(r)=\frac{b}{r^2}\psi(r)$. Hence, for $\frac{\partial^n}{\partial r^n}\psi(r)$, with n finite but perhaps large, we will have $(1/r)^m\psi(r)$ forms and for $r \to 0$ see a vanishing of differentials. Hence, for $n=1,2,.....N$ with N finite integer possibly large, $|\frac{\partial^n}{\partial x_j^n} f_i|$ will be finite for $\mathbb{R}^3$. If $\mathbb{R}^3\backslash(0,0,0)$ may be taken for physical space then $|\frac{\partial^n}{\partial x_j^n} f_i|$ will be finit for $n=1,2,3....$ It appears that the $|\frac{\partial^n}{\partial x_j^n} f_i|$ requirement is also fullfilled by the heuristic in (2.1). Because, $\sum_{j=1}^3 c_j\beta_j=0$, from (3.1) and (2.15) it follows for $0<r\leq 1$ that $p(r,t)=p(0,t)+r\sum_{j=1}^3 \beta_j g_j$. Note $x_j=r\beta_j$, while, we already established, for $r\geq 1$, $p(x_1,x_2,x_3,t)=\sum_{j=1}^3 x_j g_j=(x\cdot g)$.

4 Conclusion

The claim is that in the previous sections an exact smooth nontrivial type A solution to the Navier-Stokes equation is presented. Perhaps that the exclusively radial dependence will prove to be an unphysical form for solution. However, as far as the author can see this is not a reason to reject the mathematics. The author would also like to refer to another approach of getting exact nontrivial solutions of the Navier Stokes equation in [2].

References

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