G8,2 Geometric Algebra, DCGA

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1 Introduction

This paper[1] introduces[2] an application of the $\mathcal{G}_{8,2}$ geometric algebra [3][4], tentatively named in this paper the *Double Conformal Geometric Algebra* (DCGA).

The $\mathcal{G}_{8,2}$ geometric algebra contains two subspaces of the $\mathcal{G}_{4,1}$ conformal geometric algebra (CGA) [1][5][7][8][9]. The first CGA subspace, called CGA1, is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & | \;\; i = j, 1 \leq i \leq 4 \\ -1 & | \;\; i = j = 5 \\ 0 & | \;\; i \neq j. \end{cases}$$

The second CGA subspace, called CGA2, is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & | \;\; i = j, 6 \leq i \leq 9 \\ -1 & | \;\; i = j = 10 \\ 0 & | \;\; i \neq j. \end{cases}$$

The metric we use for $\mathcal{G}_{8,2}$ is $[1,1,1,1,-1,1,1,1,1,-1]$. This metric makes it very simple to use the CGA subalgebras of the CGA1 and CGA2 subspaces in a way fully compatible with CGA. These two CGAs are used as mirror copies or doubles to create *bivector-valued entities* for points and surfaces. As such, the $\mathcal{G}_{8,2}$ geometric algebra of these new bivector-valued point and surface entities could be called *Double-Conformal Geometric Algebra* (DCGA), or even *Bi-conformal Geometric Algebra* (Bi-CGA or 2-CGA).

The $\mathcal{G}_{8,2}$ DCGA surface entities include all of the types of surface entities available in $\mathcal{G}_{4,1}$ CGA and in the $\mathcal{G}_{6,3}$ geometric algebra [10] known as $\mathcal{G}_{6,3}$ Quadric Geometric Algebra (QGA) [2][6]. DCGA also includes a new toroid (torus) surface entity. All DCGA entities (both GIPNS and their GOPNS duals) can be rotated, isometrically dilated, and translated by versor operations on the entities. The DCGA rotor, dilator, and translator are each a new form of bi-versor or double-versor of multivector-grade four. The *DCGA entities* will refer to *all* entities of the DCGA algebra, and the *bi-CGA entities* will refer specifically to the CGA-subset of entities (points, lines, circles, planes, and spheres) as they exist in DCGA. The CGA1 and CGA2 entities also exist within their CGA subalgebras, offering the flexibility to use them separately, or also to double them to form bi-CGA entities when it is desired to bring them into the larger DCGA space.

1. Revision v3, *August 17, 2015*, with some minor corrections and improvements.

2. DCGA is the work of an independent research by this author. No prior published work on DCGA was consulted or known to this author at the time of original research and publication of this paper (v1) on *August 11, 2015*.

Compared to QGA, DCGA extends CGA in a new way and has a new toroid entity. While QGA can only rotate the CGA 6,3D entities, DCGA provides a rotor that can rotate all DCGA entities around any axis by any angle. While QGA has an isotropic dilator and methods for anisotropic dilation, DCGA has only an isotropic dilator. Both QGA and DCGA have a translator for translations of all entities. While QGA supports intersecting all QGA GIPNS entities, DCGA can intersect all DCGA GIPNS entities only with the subset of bi-CGA GIPNS entities.

Depending on the needs of a particular application, DCGA may provide a larger set of operations and entities than QGA. As with QGA, there may be performance issues when working with a high-dimensional Clifford algebra such as DCGA. For applications where anisotropic dilation is not required, but where rotation of all surfaces is required and intersecting them with CGA entities is sufficient, then DCGA provides a powerful geometric algebra with all of the standard operations as versors.

2 CGA1 and CGA2

The CGA1 and CGA2 entities follow ordinary $\mathcal{G}_{4,1}$ CGA and these subalgebras would be easily available to an application that is also using the $\mathcal{G}_{8,2}$ algebra. This subsection gives a quick review of CGA.

2.1 CGA point

In $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA), the embedding of a 3D point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ in $\mathbb{R}^3$ starts with a stereographic embedding of $\mathbf{p}$ onto a hypersphere or 3-sphere $\mathbb{S}^3$ using $\mathbf{e}_4$ as the stereographic 3-sphere pole. As shown in Figure 1, this requires finding the intersection of the line through $\mathbf{e}_4$ and $\mathbf{p}$ with the 3-sphere. The vectors $\mathbf{e}_4$ and $\mathbf{p}$ are perpendicular, and we can treat the embedding of $\mathbf{p}$ similarly to a 1D axis embedding into a stereographic 1-sphere or circle.

The identities

$$\begin{aligned}
|\mathbf{p}| &= \sqrt{x^2 + y^2 + z^2}\\
\hat{\mathbf{p}} &= \frac{\mathbf{p}}{|\mathbf{p}|}\\
\mathbf{p} &= |\mathbf{p}|\hat{\mathbf{p}}\\
\mathbf{p}^2 &= |\mathbf{p}|^2 = x^2 + y^2 + z^2
\end{aligned}$$

are used in the following.

The stereographic embedding of $|\mathbf{p}|\hat{\mathbf{p}}$ is the intersection $\alpha\hat{\mathbf{p}} + \beta\mathbf{e}_4$ of the unit 3-circle on the $\hat{\mathbf{p}}\mathbf{e}_4$-hyperplane with the line through $\mathbf{e}_4$ and $|\mathbf{p}|\hat{\mathbf{p}}$. The Minkowski homogenization is $\alpha\hat{\mathbf{p}} + \beta\mathbf{e}_4 + \mathbf{e}_5$. The point at the origin embeds to $\mathbf{e}_o = -\mathbf{e}_4 + \mathbf{e}_5$ and the point at infinity embeds to $\mathbf{e}_\infty = \mathbf{e}_4 + \mathbf{e}_5$. It is convenient to scale $\mathbf{e}_o$ as $\mathbf{e}_o = \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5)$ such that $\mathbf{e}_o \cdot \mathbf{e}_\infty = -1$. The values for α and β are solved as follows.

The initial relations are the unit circle $\alpha^2+\beta^2=1$ and, by similar triangles, the line $\frac{1-\beta}{\alpha}=\frac{1}{|\mathbf{p}|}$.

$$\begin{aligned}
\alpha^2 &= 1-\beta^2=(1+\beta)(1-\beta)=((1-\beta)|\mathbf{p}|)^2\\
(1+\beta) &= (1-\beta)|\mathbf{p}|^2\\
\beta|\mathbf{p}|^2+\beta &= |\mathbf{p}|^2-1\\
\beta &= \frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}
\end{aligned}$$

$$\begin{aligned}
\alpha &= (1-\beta)|\mathbf{p}|\\
&= \left(1-\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)|\mathbf{p}|\\
&= \left(\frac{|\mathbf{p}|^2+1}{|\mathbf{p}|^2+1}-\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)|\mathbf{p}|\\
&= \frac{2|\mathbf{p}|}{|\mathbf{p}|^2+1}
\end{aligned}$$

The stereographic embedding of $|\mathbf{p}|\hat{\mathbf{p}}$, denoted $\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})$, can now be written as

$$\begin{aligned}
\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}) &= \alpha\hat{\mathbf{p}}+\beta\mathbf{e}_4\\
&= \left(\frac{2|\mathbf{p}|}{|\mathbf{p}|^2+1}\right)\hat{\mathbf{p}}+\left(\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)\mathbf{e}_4.
\end{aligned}$$

The homogenization of $\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})$, denoted $\mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}))$, can be written as

$$\mathbf{P}=\mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}})) = \left(\frac{2|\mathbf{p}|}{|\mathbf{p}|^2+1}\right)\hat{\mathbf{p}}+\left(\frac{|\mathbf{p}|^2-1}{|\mathbf{p}|^2+1}\right)\mathbf{e}_4+\mathbf{e}_5.$$

Since this point entity is homogeneous, and $|\mathbf{p}|^2+1$ is never zero, it is permissible to scale it by $\frac{|\mathbf{p}|^2+1}{2}$ and define

$$\begin{aligned}
\mathbf{P} &= \mathcal{H}_M(\mathcal{S}(|\mathbf{p}|\hat{\mathbf{p}}))\simeq\mathcal{C}_{4,1}(\mathbf{p})\\
&= |\mathbf{p}|\hat{\mathbf{p}}+\frac{|\mathbf{p}|^2-1}{2}\mathbf{e}_4+\frac{|\mathbf{p}|^2+1}{2}\mathbf{e}_5\\
&= |\mathbf{p}|\hat{\mathbf{p}}+\frac{|\mathbf{p}|^2}{2}(\mathbf{e}_4+\mathbf{e}_5)+\frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5)\\
&= \mathbf{p}+\frac{1}{2}\mathbf{p}^2(\mathbf{e}_4+\mathbf{e}_5)+\frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5).
\end{aligned}$$

When $|\mathbf{p}|=0$,

$$\mathbf{P}_{|\mathbf{p}|=0} = \frac{1}{2}(-\mathbf{e}_4+\mathbf{e}_5)=\mathbf{e}_o$$

representing the point at the origin. In the limit as $|\mathbf{p}|\to\pm\infty$, we find that

$$\mathbf{P}_{|\mathbf{p}|\to\infty} = \mathbf{e}_4+\mathbf{e}_5=\mathbf{e}_\infty$$

represents the point at infinity. By taking inner products, it can be shown that these points are all null vectors on a null 4-cone, and the inner product $\mathbf{e}_o\cdot\mathbf{e}_\infty=-1$. The CGA embedding of vector $\mathbf{p}$ as point $\mathbf{P}$ can now be defined as

$$\begin{aligned}
\mathbf{P}=\mathbf{P}_{\mathcal{C}} &= \mathcal{C}(\mathbf{p})=\mathcal{C}_{4,1}(\mathbf{p})\\
&= \mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty+\mathbf{e}_o.
\end{aligned}$$

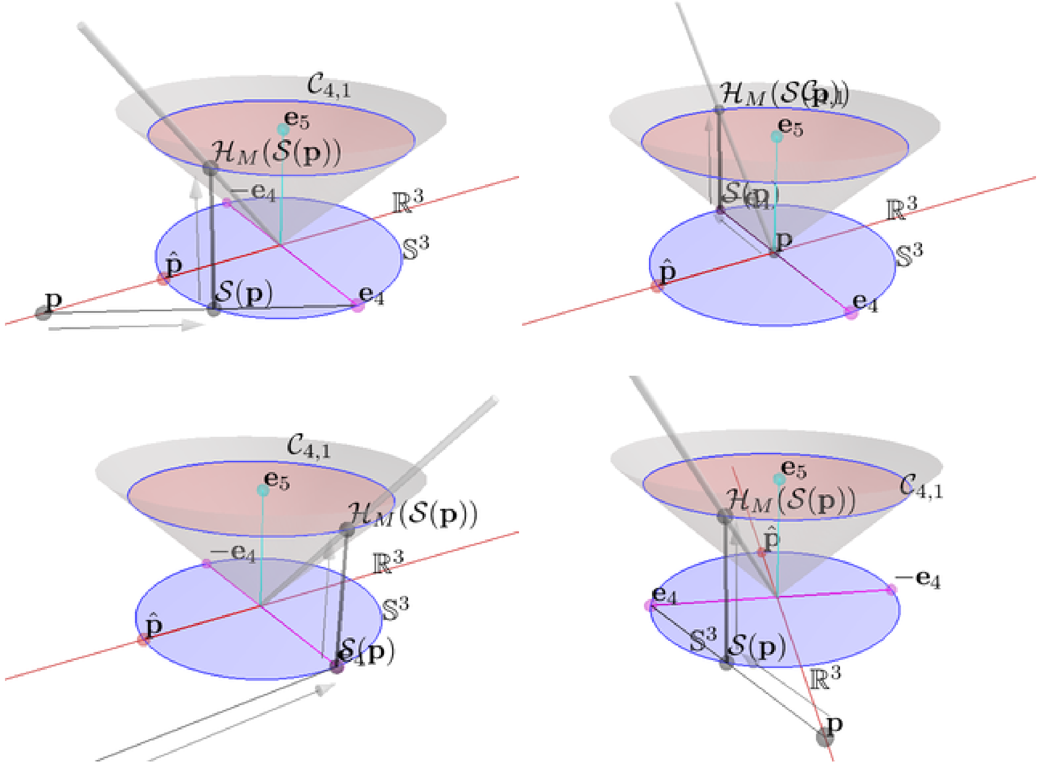

Figure 1. CGA point embedding

Figure 1 shows the CGA embedding procedure. The upper-left image shows a 3D vector $\mathbf{p}$ in $\mathbb{R}^3 = \mathcal{G}_3^1$ that is intersected with the 3-sphere $\mathbb{S}^3$, similar to an ordinary circle, at $\mathcal{S}(\mathbf{p})$ and then raised or homogenized to $\mathcal{H}_M(\mathcal{S}(\mathbf{p})) = \mathcal{C}(\mathbf{p})$ as the CGA embedding of $\mathbf{p}$. The *null cone* is the space of all homogeneous CGA points. A CGA point may be arbitrarily scaled along a line in the null cone without affecting the point being represented. A normalized CGA point has its $\mathbf{e}_o$ component scaled to 1. The upper-right image shows $\mathbf{p}$ at the origin where it is embedded as $2\mathbf{e}_o = -\mathbf{e}_4 + \mathbf{e}_5$; but after scaling this by $\frac{1}{2}$ to our preferred normalization, then it embeds to $\mathcal{C}(\mathbf{p}) = \mathbf{e}_o = \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5)$. The lower-left image shows $\mathbf{p}$ moved very far from the origin off screen, and it approaches the embedding $\mathcal{C}(\mathbf{p}) = \mathbf{e}_\infty = \mathbf{e}_4 + \mathbf{e}_5$ as it moves to an infinite distance from the origin in any direction. The lower-right image shows $\mathbf{p}$ at a relatively negative position where it embeds into a different quadrant of the null cone. The figure was generated using the program *CLUCalc* written by CHRISTIAN PERWASS, and is a modification of Fig 4.14 in [7].

2.2 CGA GIPNS surfaces

A CGA point $\mathbf{T}_\mathcal{C} = \mathcal{C}(\mathbf{t}) = \mathcal{C}_{4,1}(\mathbf{t})$ is on a CGA GIPNS surface $\mathbf{S}$ if $\mathbf{T}_\mathcal{C} \cdot \mathbf{S} = 0$.

2.2.1 CGA GIPNS sphere

The CGA GIPNS 1-vector *sphere* $\mathbf{S}$, centered at CGA point $\mathbf{P}_\mathcal{C}$ with radius r, is defined as

$$\mathbf{S} = \mathbf{P}_\mathcal{C} - \frac{1}{2} r^2 \mathbf{e}_\infty.$$

2.2.2 CGA GIPNS plane

The CGA GIPNS 1-vector *plane* $\mathbf{\Pi}$, normal to unit vector $\mathbf{n}$ at distance d from the origin, is defined as

$$\mathbf{\Pi} = \mathbf{n} + d\mathbf{e}_\infty.$$

2.2.3 CGA GIPNS line

The CGA GIPNS 2-vector *line* $\mathbf{L}$, in the direction of the unit vector $\mathbf{d}$, perpendicular to $\mathbf{D} = \mathbf{d}^{*\mathcal{E}} = \mathbf{d}/\mathbf{I}_\mathcal{E}$, and through 3D point $\mathbf{p}$, is defined as

$$\mathbf{L} = \mathbf{D} - (\mathbf{p} \cdot \mathbf{D})\mathbf{e}_\infty.$$

The Euclidean 3D pseudoscalar is $\mathbf{I}_\mathcal{E} = \mathbf{I}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, and the Euclidean 3D dual of any multivector $\mathbf{d}$ in this space is defined as $\mathbf{d}^{*\mathcal{E}} = \mathbf{d}/\mathbf{I}_\mathcal{E}$.

2.2.4 CGA GIPNS circle

The CGA GIPNS 2-vector *circle* $\mathbf{C}$ is defined as

$$\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi}$$

which is the intersection of a sphere $\mathbf{S}$ and plane $\mathbf{\Pi}$.

2.3 CGA GOPNS surfaces

A CGA point $\mathbf{T}_\mathcal{C} = \mathcal{C}(\mathbf{t}) = \mathcal{C}_{4,1}(\mathbf{t})$ is on a CGA GOPNS surface $\mathbf{S}^{*\mathcal{C}}$ if $\mathbf{T}_\mathcal{C} \wedge \mathbf{S}^{*\mathcal{C}} = 0$.

2.3.1 CGA GOPNS sphere

The CGA GOPNS 4-vector *sphere* $\mathbf{S}^{*\mathcal{C}}$ is the wedge of four CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the sphere

$$\begin{aligned}\mathbf{S}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{P}_{\mathcal{C}_4} \\ &= \mathbf{S}/\mathbf{I}_\mathcal{C}\end{aligned}$$

and is the CGA dual of the CGA GIPNS 1-vector sphere $\mathbf{S}$.

The CGA unit pseudoscalar is $\mathbf{I}_\mathcal{C} = \mathbf{I}_5 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$, and the CGA dual of any multivector $\mathbf{S}$ in this space is defined as $\mathbf{S}^{*\mathcal{C}} = \mathbf{S}/\mathbf{I}_\mathcal{C}$. The CGA GOPNS and CGA GIPNS entities are CGA duals of each other.

2.3.2 CGA GOPNS plane

The CGA GOPNS 4-vector *plane* $\mathbf{\Pi}^{*\mathcal{C}}$ is the wedge of three CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the plane and the point $\mathbf{e}_\infty$

$$\begin{aligned}\mathbf{\Pi}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3} \wedge \mathbf{e}_\infty \\ &= \mathbf{\Pi}/\mathbf{I}_\mathcal{C}\end{aligned}$$

and is the CGA dual of CGA GIPNS 1-vector plane $\mathbf{\Pi}$.

2.3.3 CGA GOPNS line

The CGA GOPNS 3-vector *line* $\mathbf{L}^{*\mathcal{C}}$ is the wedge of two CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the line and the point $\mathbf{e}_\infty$

$$\begin{aligned}\mathbf{L}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{e}_\infty \\ &= \mathbf{L}/\mathbf{I}_\mathcal{C}\end{aligned}$$

and is the CGA dual of the CGA GIPNS 2-vector line $\mathbf{L}$.

2.3.4 CGA GOPNS circle

The CGA GOPNS 3-vector *circle* $\mathbf{C}^{*\mathcal{C}}$ is the wedge of three CGA points $\mathbf{P}_{\mathcal{C}_i}$ on the circle

$$\begin{aligned}\mathbf{C}^{*\mathcal{C}} &= \mathbf{P}_{\mathcal{C}_1} \wedge \mathbf{P}_{\mathcal{C}_2} \wedge \mathbf{P}_{\mathcal{C}_3}\\ &= \mathbf{C}/\mathbf{I}_{\mathcal{C}}\end{aligned}$$

and is the CGA dual of the CGA GIPNS 2-vector circle $\mathbf{C}$.

2.4 CGA operations

The rotor R, dilator D, and translator T are called *versors*. Their operation on a CGA entity $\mathbf{X}$ has the form $\mathbf{X}' = O\mathbf{X}O^{-1}$, called a *versor operation*. The versor O of the operation often has an exponential form which can be expanded by Taylor series into circular trigonometric, hyperbolic trigonometric, or dual number form.

2.4.1 CGA rotor

A *rotor* is a *rot*ation operat*or*, or vers*or*. The CGA rotor R, for rotation around unit vector axis $\mathbf{n}$ by θ radians, is defined as

$$\begin{aligned}R &= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)\mathbf{n}^{*\mathcal{E}}\\ &= e^{\frac{1}{2}\theta\mathbf{n}^{*\mathcal{E}}} = e^{\frac{1}{2}\theta\mathbf{N}}\end{aligned}$$

The unit bivector $\mathbf{N} = \mathbf{n}^{*\mathcal{E}}$ represents the plane of rotation. Any multivector in $\mathcal{G}_3$ or any CGA entity $\mathbf{X}$ in $\mathcal{G}_{4,1}$ is rotated as

$$\begin{aligned}\mathbf{X}' &= R\mathbf{X}R^{\sim}\\ &= R\mathbf{X}R^{-1}\end{aligned}$$

where $R^{\sim}$ is the reverse of R, and is also equal to the inverse R^{-1}. Rotation is also defined by reflection in two planes as

$$\mathbf{X}' = \mathbf{\Pi}_2\mathbf{\Pi}_1\mathbf{X}\mathbf{\Pi}_1\mathbf{\Pi}_2$$

which rotates $\mathbf{X}$ by *twice* the angle between the planes from $\mathbf{\Pi}_1$ to $\mathbf{\Pi}_2$.

2.4.2 CGA dilator

A *dilator* is a *dilat*ion operat*or*. The CGA isotropic dilator D by factor d is defined as

$$\begin{aligned}D &= \frac{1}{2}(1+d) + \frac{1}{2}(1-d)\mathbf{e}_\infty \wedge \mathbf{e}_o\\ &\simeq 1 + \frac{(1-d)}{(1+d)}\mathbf{e}_\infty \wedge \mathbf{e}_o\\ &\simeq e^{\operatorname{atanh}\left(\frac{(1-d)}{(1+d)}\right)\mathbf{e}_\infty \wedge \mathbf{e}_o}.\end{aligned}$$

Any CGA entity $\mathbf{X}$ in $\mathcal{G}_{4,1}$ is dilated by the factor d as

$$\mathbf{X}' = D\mathbf{X}D^{\sim}.$$

Dilation is also defined by inversion in two spheres as

$$\mathbf{X}' = \mathbf{S}_2\mathbf{S}_1\mathbf{X}\mathbf{S}_1\mathbf{S}_2$$

which dilates by $d = \frac{r_2^2}{r_1^2}$, with radius r_1 of $\mathbf{S}_1$ and radius r_2 of $\mathbf{S}_2$.

2.4.3 CGA translator

A *translator* is a *translat*ion operat*or*. The CGA translator T by a vector $\mathbf{d} = d_x\mathbf{e}_1 + d_y\mathbf{e}_2 + d_z\mathbf{e}_3$ is defined as

$$\begin{aligned} T &= 1 - \frac{1}{2}\mathbf{d}\mathbf{e}_\infty \\ &= e^{-\frac{1}{2}\mathbf{d}\mathbf{e}_\infty}. \end{aligned}$$

Any CGA entity $\mathbf{X}$ is translated by the vector $\mathbf{d}$ as

$$\mathbf{X} = T\mathbf{X}T^{\sim}.$$

Translation is also defined by reflection in two parallel planes as

$$\mathbf{X}' = \mathbf{\Pi}_2\mathbf{\Pi}_1\mathbf{X}\mathbf{\Pi}_1\mathbf{\Pi}_2$$

which translates by *twice* the vector $\mathbf{d} = (d_2 - d_1)\mathbf{n}$, with common normal unit vector $\mathbf{n}$ of each plane (they are parallel) and plane distances from origin d_1 and d_2 of planes $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$, respectively.

2.4.4 CGA motor

A *motor* is a *mot*ion operat*or*. A rotation around a unit vector axis $\mathbf{n}$, followed by a translation parallel to $\mathbf{n}$ are commutative operations. Either the translation or the rotation can be done first, and the other second, to reach the same final position. This commutative operation, being a screw or helical motion, can be seen physically without mathematics. The motor is a special case where the commutative rotor and translator can be composed into a single versor M with an *exponential form* as

$$\begin{aligned} M &= RT = TR \\ &= e^{\frac{1}{2}\theta\mathbf{n}^{*\mathcal{E}}}e^{-\frac{1}{2}d\mathbf{n}\mathbf{e}_\infty} = e^{-\frac{1}{2}d\mathbf{n}\mathbf{e}_\infty}e^{\frac{1}{2}\theta\mathbf{n}^{*\mathcal{E}}} \\ &= e^{\frac{1}{2}\theta\mathbf{n}^{*\mathcal{E}} - \frac{1}{2}d\mathbf{n}\mathbf{e}_\infty} \\ &= e^{-\frac{1}{2}\mathbf{n}(\theta\mathbf{I}_\mathcal{E} + d\mathbf{e}_\infty)}. \end{aligned}$$

The exponents or *logarithms* of commutative exponentials can be added. A motor can be used to model smoothly-interpolated *screw*, *twistor*, or helical motions, performed in n steps using the nth root of M

$$M^{\frac{1}{n}} = e^{-\frac{1}{2n}\mathbf{n}(\theta\mathbf{I}_\mathcal{E} + d\mathbf{e}_\infty)}$$

applied at each step.

2.4.5 CGA intersection

CGA GIPNS *intersection* entities which represent the surface intersections of two or more CGA GIPNS entities are formed by the wedge of the CGA GIPNS entities. The CGA GIPNS circle is defined as a CGA GIPNS intersection entity $\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi}$.

Almost any combination of CGA GIPNS entities may be wedged to form a CGA GIPNS intersection entity up to grade 4, except that the CGA GIPNS 2-vector line and circle entities that are coplanar cannot be intersected unless their common plane is first contracted out of each of them, then the common plane is wedged back onto their intersection entity.

Like any CGA GIPNS entity, a CGA GIPNS intersection entity $\mathbf{X}$ can be taken dual as $\mathbf{X}^{*\mathcal{C}} = \mathbf{X}/\mathbf{I}_{\mathcal{C}}$ into its CGA GOPNS intersection entity $\mathbf{X}^{*\mathcal{C}}$.

De Morgan's law for the intersection $\mathbf{X}$ of two objects $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{X} = \operatorname{not}((\operatorname{not}\mathbf{A}) \operatorname{and} (\operatorname{not}\mathbf{B}))$$

and translates into the CGA intersection

$$\mathbf{X}^{*\mathcal{C}} = (\mathbf{A}^{*\mathcal{C}} \wedge \mathbf{B}^{*\mathcal{C}})^{*\mathcal{C}}.$$

This is just the creation of the CGA GOPNS intersection entity $\mathbf{X}^{*\mathcal{C}}$ of two *CGA GOPNS entities* $\mathbf{A}$ and $\mathbf{B}$. In this case, $\mathbf{A}^{*\mathcal{C}}$ and $\mathbf{B}^{*\mathcal{C}}$ are the *undual* CGA GIPNS entities, which can then be intersected by wedge product. The CGA GIPNS intersection $\mathbf{X}$ is then dualized as the CGA GOPNS entity $\mathbf{X}^{*\mathcal{C}}$. The classical view of intersections is by working with spanning objects, which are the CGA GOPNS entities.

2.4.6 CGA dualization

The Euclidean 3D or $\mathcal{G}_3$ unit pseudoscalar $\mathbf{I}_{\mathcal{E}}$ is defined as

$$\begin{aligned}
\mathbf{I}_{\mathcal{E}} = \mathbf{I}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\
\mathbf{I}_{\mathcal{E}}^{\sim} &= (-1)^{3(3-1)/2}\mathbf{I}_{\mathcal{E}} = -\mathbf{I}_{\mathcal{E}}\\
\mathbf{I}_{\mathcal{E}}^2 &= -\mathbf{I}_{\mathcal{E}}\mathbf{I}_{\mathcal{E}}^{\sim} = -1\\
\mathbf{I}_{\mathcal{E}}^{-1} &= \mathbf{I}_{\mathcal{E}}^{\sim} = -\mathbf{I}_{\mathcal{E}}
\end{aligned}$$

and is the dualization operator on multivectors in $\mathcal{G}_3$. A blade $\mathbf{B} \in \mathcal{G}_3^k$ of grade k is taken to its Euclidean 3D or $\mathcal{G}_3$ dual $\mathbf{B}^{*\mathcal{E}} \in \mathcal{G}_3^{3-k}$ of grade $3-k$ as

$$\mathbf{B}^{*\mathcal{E}} = \mathbf{B}/\mathbf{I}_{\mathcal{E}} = -\mathbf{B} \cdot \mathbf{I}_{\mathcal{E}}.$$

Duals represent the same objects from two converse spatial spans, and the duals have different behavior as operators or algebraic factors on other multivectors. The dual of a unit vector is a unit bivector that can act as the unit of a rotor around the vector, but a unit vector can only operate as a reflector through the vector.

The CGA or $\mathcal{G}_{4,1}$ unit pseudoscalar $\mathbf{I}_{\mathcal{C}}$ defined as

$$\begin{aligned}
\mathbf{I}_{\mathcal{C}} = \mathbf{I}_5 &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\\
\mathbf{I}_{\mathcal{C}}^{\sim} &= (-1)^{5(5-1)/2}\mathbf{I}_{\mathcal{C}} = \mathbf{I}_{\mathcal{C}}\\
\mathbf{I}_{\mathcal{C}}^2 &= \mathbf{I}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{\sim} = -1\\
\mathbf{I}_{\mathcal{C}}^{-1} &= -\mathbf{I}_{\mathcal{C}}^{\sim} = -\mathbf{I}_{\mathcal{C}}
\end{aligned}$$

and is the dualization operator on CGA entities that takes CGA GIPNS entities to or from CGA GOPNS entities. A CGA entity $\mathbf{X}\in\mathcal{G}^{k}_{4,1}$ of grade k is taken to its CGA dual entity $\mathbf{X}^{*\mathcal{C}}\in\mathcal{G}^{5-k}_{4,1}$ of grade $5-k$ as

$$\mathbf{X}^{*\mathcal{C}} = \mathbf{X}/\mathbf{I}_{\mathcal{C}} = -\mathbf{X}\cdot\mathbf{I}_{\mathcal{C}}.$$

The pseudoscalar $\mathbf{I}_{\mathcal{C}}$ does not represent any CGA entity, so no CGA GOPNS entity nor CGA GIPNS intersection entity can have grade 5. The max grade of a CGA entity is grade 4.

2.5 CGA1 and CGA2 notations

The CGA1 and CGA2 spaces are used as exact copies of CGA. All that is needed is a little notation to separate the two spaces.

Multivectors in the $\mathcal{G}_3$ subspace of the CGA1 space will use the subscript $\mathcal{E}^1$. For example, a Euclidean 3D vector $\mathbf{p}$ in the CGA1 space is denoted in the form

$$\mathbf{p}_{\mathcal{E}^1} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3.$$

A CGA entity in the CGA1 space will use the subscript $\mathcal{C}^1$. For example, the embedding of $\mathbf{p}_{\mathcal{E}^1}$ as a CGA1 point $\mathbf{P}_{\mathcal{C}^1}$ is denoted

$$\begin{aligned}\mathbf{P}_{\mathcal{C}^1} &= \mathcal{C}^1(\mathbf{p}_{\mathcal{E}^1})\\ &= \mathbf{p}_{\mathcal{E}^1} + \frac{1}{2}\mathbf{p}^2_{\mathcal{E}^1}\mathbf{e}_{\infty 1} + \mathbf{e}_{o1}\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}_{\infty 1} &= (\mathbf{e}_4 + \mathbf{e}_5)\\ \mathbf{e}_{o1} &= \frac{1}{2}(-\mathbf{e}_4 + \mathbf{e}_5).\end{aligned}$$

The CGA1 point embedding function has been named $\mathcal{C}^1$. Likewise, a CGA1 surface entity is named $\mathbf{X}_{\mathcal{C}^1}$. The CGA1 point at the origin $\mathbf{e}_{o1}$ and point at infinity $\mathbf{e}_{\infty 1}$ are named with suffix 1 to indicate their version as being the CGA1 versions.

Multivectors in the $\mathcal{G}_3$ subspace of the CGA2 space will use the subscript $\mathcal{E}^2$ (e.g., $\mathbf{p}_{\mathcal{E}^2}$). A CGA entity in the CGA2 space will use the subscript $\mathcal{C}^2$ (e.g., $\mathbf{X}_{\mathcal{C}^2}$).

With this notation, the CGA1 unit pseudoscalars are named as

$$\begin{aligned}\mathbf{I}_{\mathcal{E}^1} &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\ \mathbf{I}_{\mathcal{C}^1} &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\end{aligned}$$

and the CGA2 unit pseudoscalars are named as

$$\begin{aligned}\mathbf{I}_{\mathcal{E}^2} &= \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8\\ \mathbf{I}_{\mathcal{C}^2} &= \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}.\end{aligned}$$

A Euclidean 3D vector $\mathbf{p}$ in the CGA2 space is denoted in the form

$$\mathbf{p}_{\mathcal{E}^2} = p_x\mathbf{e}_6 + p_y\mathbf{e}_7 + p_z\mathbf{e}_8.$$

The CGA2 point embedding is

$$\begin{aligned}\mathbf{P}_{\mathcal{C}^2} &= \mathcal{C}^2(\mathbf{p}_{\mathcal{E}^2})\\ &= \mathbf{p}_{\mathcal{E}^2}+\frac{1}{2}\mathbf{p}_{\mathcal{E}^2}^2\mathbf{e}_{\infty 2}+\mathbf{e}_{o2}\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}_{\infty 2} &= (\mathbf{e}_9+\mathbf{e}_{10})\\ \mathbf{e}_{o2} &= \frac{1}{2}(-\mathbf{e}_9+\mathbf{e}_{10}).\end{aligned}$$

The CGA2 point at the origin $\mathbf{e}_{o2}$ and point at infinity $\mathbf{e}_{\infty 2}$ are named with suffix 2 to indicate their version as being the CGA2 versions.

A versor O (rotor, dilator, translator, or motor) that is in the CGA1 space is denoted $O_{\mathcal{C}^1}$, and if it is in the CGA2 space it is denoted $O_{\mathcal{C}^2}$.

3 DCGA point

The standard DCGA *null* 2-vector *point* entity $\mathbf{P}_{\mathcal{D}}$ is the embedding of a vector

$$\mathbf{p}_{\mathcal{E}^1} = \mathbf{p}=p_x\mathbf{e}_1+p_y\mathbf{e}_2+p_z\mathbf{e}_3$$

as

$$\begin{aligned}\mathbf{P}_{\mathcal{D}} &= \mathcal{D}(\mathbf{p})\\ &= \mathcal{C}^1(\mathbf{p}_{\mathcal{E}^1})\wedge\mathcal{C}^2(\mathbf{p}_{\mathcal{E}^2})\\ &= \mathbf{P}_{\mathcal{C}^1}\wedge\mathbf{P}_{\mathcal{C}^2}\end{aligned}$$

where

$$\begin{aligned}\mathbf{p}_{\mathcal{E}^2} &= (\mathbf{p}_{\mathcal{E}^1}\cdot\mathbf{e}_1)\mathbf{e}_6+(\mathbf{p}_{\mathcal{E}^1}\cdot\mathbf{e}_2)\mathbf{e}_7+(\mathbf{p}_{\mathcal{E}^1}\cdot\mathbf{e}_3)\mathbf{e}_8\\ &= p_x\mathbf{e}_6+p_y\mathbf{e}_7+p_z\mathbf{e}_8.\end{aligned}$$

The DCGA point $\mathbf{P}_{\mathcal{D}}$, which could be called a *double point*, is the wedge of a CGA1 point $\mathbf{P}_{\mathcal{C}^1}$ with a CGA2 point $\mathbf{P}_{\mathcal{C}^2}$, which are the CGA embeddings of the same Euclidean 3D vector $\mathbf{p}$ into each CGA.

CGA1 and CGA2 points and surface entities can be rotated, translated, and dilated using CGA1 and CGA2 versors for these operations. The wedge of a CGA1 versor with its copy CGA2 versor (rotor, translator, dilator, or motor) creates the DCGA versor on DCGA points and surface entities. The DCGA versors could be called *double versors* or bi-CGA versors.

The DCGA point at the origin $\mathbf{e}_o$ is defined as

$$\mathbf{e}_o = \mathbf{e}_{o1}\wedge\mathbf{e}_{o2}.$$

The DCGA point at infinity $\mathbf{e}_\infty$ is defined as

$$\mathbf{e}_\infty = \mathbf{e}_{\infty 1}\wedge\mathbf{e}_{\infty 2}.$$

As in CGA, these DCGA points also have the inner product

$$\mathbf{e}_\infty\cdot\mathbf{e}_o = -1.$$

All DCGA points are null 2-vectors, $\mathbf{P}_{\mathcal{D}}^2 = 0$. However, compared to CGA, all values are squared and this changes the formulas for the metrical results known in CGA. For example, the squared-squared distance d^4 between two DCGA points $\mathbf{P}_{\mathcal{D}_1}$ and $\mathbf{P}_{\mathcal{D}_2}$ is

$$\begin{aligned}
d^4 &= -4\mathbf{P}_{\mathcal{D}_1} \cdot \mathbf{P}_{\mathcal{D}_2} \\
&= -4(\mathbf{P}_{\mathcal{C}_1^1} \wedge \mathbf{P}_{\mathcal{C}_1^2}) \cdot (\mathbf{P}_{\mathcal{C}_2^1} \wedge \mathbf{P}_{\mathcal{C}_2^2}) \\
&= -4(\mathbf{P}_{\mathcal{C}_1^1} \cdot ((\mathbf{P}_{\mathcal{C}_1^2} \cdot \mathbf{P}_{\mathcal{C}_2^1})\mathbf{P}_{\mathcal{C}_2^2} - \mathbf{P}_{\mathcal{C}_2^1}(\mathbf{P}_{\mathcal{C}_1^2} \cdot \mathbf{P}_{\mathcal{C}_2^2}))) \\
&= -4((\mathbf{P}_{\mathcal{C}_1^2} \cdot \mathbf{P}_{\mathcal{C}_2^1})(\mathbf{P}_{\mathcal{C}_1^1} \cdot \mathbf{P}_{\mathcal{C}_2^2}) - (\mathbf{P}_{\mathcal{C}_1^1} \cdot \mathbf{P}_{\mathcal{C}_2^1})(\mathbf{P}_{\mathcal{C}_1^2} \cdot \mathbf{P}_{\mathcal{C}_2^2})) \\
&= -4\left((0)(0) - \left(\frac{-d^2}{2}\right)\left(\frac{-d^2}{2}\right)\right).
\end{aligned}$$

The squared distance d^2 between points is also

$$\begin{aligned}
d^2 &= -2\frac{-\mathbf{P}_{\mathcal{D}_1} \cdot \mathbf{e}_{\infty 2}}{(\mathbf{P}_{\mathcal{D}_1} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} \cdot \frac{-\mathbf{P}_{\mathcal{D}_2} \cdot \mathbf{e}_{\infty 2}}{(\mathbf{P}_{\mathcal{D}_2} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} \\
&= -2\mathbf{P}_{\mathcal{C}_1^1} \cdot \mathbf{P}_{\mathcal{C}_2^1}
\end{aligned}$$

where each DCGA point is contracted and renormalized into CGA1 points.

The DCGA 2-vector point $\mathbf{P}_{\mathcal{D}}$ allows for the extraction of more polynomial terms than only the x, y, z, x^2, y^2, z^2 terms that CGA or QGA 1-vector points allow. The terms that can be extracted from a point determine what polynomial equations or entities that can be represented as GIPNS entities that test against the point.

When expanded, the DCGA point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is

$$\begin{aligned}
\mathbf{T}_{\mathcal{D}} = {} & \frac{x}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_1 \wedge \mathbf{e}_9 + \frac{x}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_1 \wedge \mathbf{e}_{10} + \\
& \frac{x}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_4 \wedge \mathbf{e}_6 + \frac{x}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_5 \wedge \mathbf{e}_6 + \\
& \frac{y}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_2 \wedge \mathbf{e}_9 + \frac{y}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_2 \wedge \mathbf{e}_{10} + \\
& \frac{y}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_4 \wedge \mathbf{e}_7 + \frac{y}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_5 \wedge \mathbf{e}_7 + \\
& \frac{z}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_3 \wedge \mathbf{e}_9 + \frac{z}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_3 \wedge \mathbf{e}_{10} + \\
& \frac{z}{2}(x^2+y^2+z^2-1)\,\mathbf{e}_4 \wedge \mathbf{e}_8 + \frac{z}{2}(x^2+y^2+z^2+1)\,\mathbf{e}_5 \wedge \mathbf{e}_8 + \\
& xy\,\mathbf{e}_1 \wedge \mathbf{e}_7 + xy\,\mathbf{e}_2 \wedge \mathbf{e}_6 + \\
& yz\,\mathbf{e}_2 \wedge \mathbf{e}_8 + yz\,\mathbf{e}_3 \wedge \mathbf{e}_7 + \\
& xz\mathbf{e}_1 \wedge \mathbf{e}_8 + xz\,\mathbf{e}_3 \wedge \mathbf{e}_6 + \\
& x^2\mathbf{e}_1 \wedge \mathbf{e}_6 + y^2\,\mathbf{e}_2 \wedge \mathbf{e}_7 + z^2\,\mathbf{e}_3 \wedge \mathbf{e}_8 + \\
& \left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} + \frac{y^4}{4} + \frac{y^2z^2}{2} + \frac{z^4}{4} - \frac{1}{4}\right)\mathbf{e}_4 \wedge \mathbf{e}_{10} + \\
& \left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} + \frac{y^4}{4} + \frac{y^2z^2}{2} + \frac{z^4}{4} - \frac{1}{4}\right)\mathbf{e}_5 \wedge \mathbf{e}_9 + \\
& \left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} - \frac{x^2}{2} + \frac{y^4}{4} + \frac{y^2z^2}{2} - \frac{y^2}{2} + \frac{z^4}{4} - \frac{z^2}{2} + \frac{1}{4}\right)\mathbf{e}_4 \wedge \mathbf{e}_9 + \\
& \left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} + \frac{x^2}{2} + \frac{y^4}{4} + \frac{y^2z^2}{2} + \frac{y^2}{2} + \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4}\right)\mathbf{e}_5 \wedge \mathbf{e}_{10}
\end{aligned}$$

where

$$\begin{aligned}\mathbf{t} &= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3\\ \mathbf{t}^2 &= x^2 + y^2 + z^2\\ \mathbf{t}^4 &= x^4 + y^4 + z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2.\end{aligned}$$

The vector $\mathbf{t}$, and its DCGA point embedding $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$, will be used as a *test point* for position on surfaces. If we define the following value-extraction elements or operators on DCGA points,

$$\begin{aligned}T_x &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_6)\\ T_y &= \frac{1}{2}(\mathbf{e}_2 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7)\\ T_z &= \frac{1}{2}(\mathbf{e}_3 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_8)\end{aligned}$$

$$\begin{aligned}T_{xy} &= \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_2)\\ T_{yz} &= \frac{1}{2}(\mathbf{e}_7 \wedge \mathbf{e}_3 + \mathbf{e}_8 \wedge \mathbf{e}_2)\\ T_{zx} &= \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_1 + \mathbf{e}_6 \wedge \mathbf{e}_3)\end{aligned}$$

$$\begin{aligned}T_{x^2} &= \mathbf{e}_6 \wedge \mathbf{e}_1\\ T_{y^2} &= \mathbf{e}_7 \wedge \mathbf{e}_2\\ T_{z^2} &= \mathbf{e}_8 \wedge \mathbf{e}_3\end{aligned}$$

$$\begin{aligned}T_1 &= -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}) = -\mathbf{e}_\infty\\ T_{\mathbf{t}^2} &= -(\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_{\infty 2})\\ T_{\mathbf{t}^4} &= -4(\mathbf{e}_{o1} \wedge \mathbf{e}_{o2}) = -4\mathbf{e}_o\end{aligned}$$

then we can extract values from a DCGA point $\mathbf{T}_{\mathcal{D}}$ as $s = T_s \cdot \mathbf{T}_{\mathcal{D}}$. These extraction operators are used to define the DCGA GIPNS entities.

4 DCGA GIPNS surfaces

The DCGA *geometric inner product null space* (GIPNS) surface entities are constructed using the value extractions $T_s \cdot \mathbf{T}_{\mathcal{D}}$ from the DCGA point entity. The DCGA GIPNS surface entities are the standard surface entities in DCGA since the direct construction of DCGA *geometric outer product null space* (GOPNS) surface entities is limited to the wedge of up to four DCGA points which cannot construct all of the DCGA GOPNS surface entities. The DCGA GIPNS surface entities can be rotated, dilated, and translated by DCGA versors, and they can be intersected with the bi-CGA GIPNS surface entities.

A DCGA test point $\mathbf{T}_{\mathcal{D}}$ that is on a DCGA GIPNS surface entity $\mathbf{S}$ must satisfy the GIPNS condition

$$\mathbf{T}_{\mathcal{D}}\cdot\mathbf{S} \;=\; 0.$$

The DCGA GIPNS k-vector surface entity $\mathbf{S}$ represents the set $\mathbb{NI}_G(\mathbf{S}\in\mathcal{G}^k_{8,2})$ of all 3D vector test points $\mathbf{t}$ that are surface points

$$\mathbb{NI}_G(\mathbf{S}\in\mathcal{G}^k_{8,2}) \;=\; \left\{\ \mathbf{t}\in\mathcal{G}^1_3\ :\ (\mathcal{D}(\mathbf{t})=\mathbf{T}_{\mathcal{D}})\cdot\mathbf{S}=0\ \right\}.$$

4.1 DCGA GIPNS toroid

The homogeneous quartic equation for a circular toroid (torus), which is positioned at the origin and surrounds the z-axis, is

$$\mathbf{t}^4+2\mathbf{t}^2(R^2-r^2)+(R^2-r^2)-4R^2(x^2+y^2) \;=\; 0$$

where

$$\mathbf{t} \;=\; x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$$

is a test point, R is the major radius, and r is the minor radius. The equation is true if the test point $\mathbf{t}$ is on the toroid. The radius R is that of a circle around the origin in the xy-plane. The radius r is that of circles centered on the circle of R and which span the z-axis dimension for $z=\pm r$. The toroid spans $x,y=\pm(R+r)$.

The DCGA GIPNS 2-vector *toroid* surface entity $\mathbf{O}$ is defined as

$$\mathbf{O} \;=\; T_{\mathbf{t}^4}+2(R^2-r^2)T_{\mathbf{t}^2}+(R^2-r^2)T_1-4R^2(T_{x^2}+T_{y^2}).$$

A test DCGA point $\mathbf{T}_{\mathcal{D}}=\mathcal{D}(\mathbf{t})$ is on the toroid surface represented by $\mathbf{O}$ if $\mathbf{T}_{\mathcal{D}}\cdot\mathbf{O}=0$. Using symbolic mathematics software, such as the *Geometric Algebra Module* for *Sympy* by Alan Bromborsky et al., the inner product $\mathbf{T}_{\mathcal{D}}\cdot\mathbf{O}$ generates the homogeneous *scalar* expression of the toroid when $\mathbf{t}$ is a variable symbolic vector. When $\mathbf{t}$ is a specific vector, $\mathbf{T}_{\mathcal{D}}\cdot\mathbf{O}$ is a test operation on the toroid for the specific point.

We can denote the *DCGA-dual* of $\mathbf{O}$ as $\mathbf{O}^{*\mathcal{D}}$, and define it as

$$\mathbf{O}^{*\mathcal{D}} \;=\; \mathbf{O}/\mathbf{I}_{\mathcal{D}}=\mathbf{O}\mathbf{I}_{\mathcal{D}}^{-1}=-\mathbf{O}\cdot\mathbf{I}_{\mathcal{D}}.$$

The DCGA GOPNS 8-vector *toroid* surface entity is $\mathbf{O}^{*\mathcal{D}}$, where a test point $\mathbf{t}$ on the surface must satisfy the GOPNS condition $\mathbf{T}_{\mathcal{D}}\wedge\mathbf{O}^{*\mathcal{D}}=0$. Since $\mathbf{T}_{\mathcal{D}}$ is a 2-vector and $\mathbf{O}^{*\mathcal{D}}$ is an 8-vector, then $\mathbf{T}_{\mathcal{D}}\wedge\mathbf{O}^{*\mathcal{D}}$ is a homogeneous DCGA 10-vector *pseudoscalar* expression of the toroid when $\mathbf{t}$ is a variable symbolic vector. The *undual* operation returns the DCGA GIPNS surface $\mathbf{O}=\mathbf{O}^{*\mathcal{D}}\cdot\mathbf{I}_{\mathcal{D}}$. The other DCGA GOPNS surface entities will be discussed later in this paper.

Although the toroid **O** is created at the origin and aligned around the z-axis, it can then be rotated, dilated, and translated away from the origin using DCGA versor operations. Like all DCGA GIPNS surface entities, the DCGA GIPNS toroid can be intersected with any bi-CGA GIPNS (2, 4, or 6)-vector surface, which are 2-vector spheres and planes, 4-vector circles and lines, and 6-vector point-pairs.

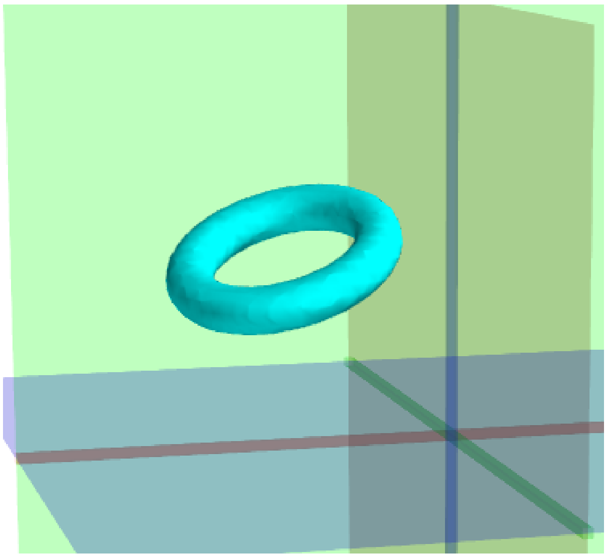

Figure 2. DCGA toroid rotated and translated

4.2 DCGA GIPNS ellipsoid

The homogeneous quadric equation for a principal axes-aligned ellipsoid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_x)^2}{r_z^2}-1 \;=\; 0$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c). Expanding the squares, the equation can be written as

$$\frac{-2p_x x}{r_x^2}+\frac{-2p_y y}{r_y^2}+\frac{-2p_z z}{r_z^2}+\left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}\right)+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right) \;=\; 0.$$

Using the DCGA point value-extraction elements, an ellipsoid equation can be constructed. This construction will be similar for the remaining surface entities that follow.

The DCGA GIPNS 2-vector *ellipsoid* surface entity **E** is defined as

$$\begin{aligned}\mathbf{E} \;=\; & \frac{-2p_xT_x}{r_x^2}+\frac{-2p_yT_y}{r_y^2}+\frac{-2p_zT_z}{r_z^2}+\\ & \frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}+\frac{T_{z^2}}{r_z^2}+\\ & \left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right)T_1.\end{aligned}$$

A DCGA 2-vector point $\mathbf{T}_{\mathcal{D}} = \mathcal{D}(\mathbf{t})$ is tested against the DCGA 2-vector ellipsoid $\mathbf{E}$ as

$$\mathbf{T}_{\mathcal{D}} \cdot \mathbf{E} \begin{cases} <0 & : \ \mathbf{t} \text{ is } \textit{inside} \text{ ellipsoid} \\ =0 & : \ \mathbf{t} \text{ is } \textit{on} \text{ ellipsoid} \\ >0 & : \ \mathbf{t} \text{ is } \textit{outside} \text{ ellipsoid.} \end{cases}$$

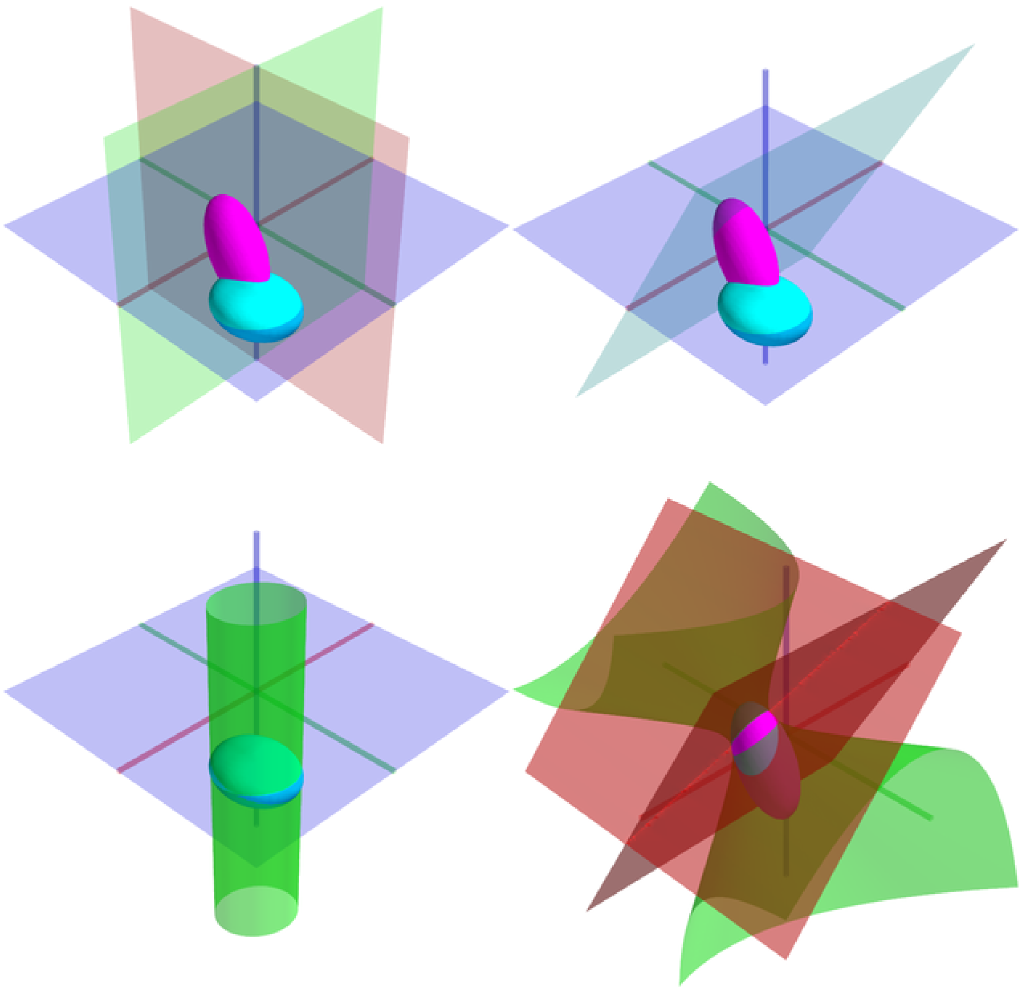

Figure 3. DCGA ellipsoids rotated, translated, and intersected with planes

Figure 3 Shows two ellipsoids that have been rotated and translated into their intersecting positions using DCGA versor operations. The view is from the first quadrant, the x-axis is red, the y-axis is green, and the z-axis is blue. The cyan DCGA GIPNS ellipsoid $\mathbf{E}_1$ ($r_x = 4$, $r_y = 5$, $r_z = 3$) is rotated 25° around the line $\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, then rotated 45° around the z-axis, then translated by $\mathbf{d} = 10\mathbf{e}_1 + 10\mathbf{e}_2$. The magenta DCGA GIPNS ellipsoid $\mathbf{E}_2$ ($p_z = 6$, $r_x = 2$, $r_y = 3$, $r_z = 6$) is rotated -35° around the line $\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, then rotated 35° around the z-axis, then translated by $\mathbf{d} = 10\mathbf{e}_1 + 10\mathbf{e}_2$. The ellipsoids intersect in a curved ellipse which, unfortunately, could *not* be represented as an intersection entity.

Although not rigorously proved here, in tests performed by this author, the initial assessment appears as though the ellipsoids and other DCGA entities can be intersected with DCGA planes and other bi-CGA entities, but the DCGA entities cannot be intersected in full generality.

The upper-left image in Figure 3 shows the ellipsoids with standard planes drawn. The upper-right image shows the ellipsoids drawn with DCGA GIPNS plane $\mathbf{\Pi}_1$ representing the plane $z=0$, and with the DCGA GIPNS plane $\mathbf{\Pi}_2$ representing the plane $z=0$ rotated 60° around the x-axis. The lower-left image shows the DCGA GIPNS intersection entity $\mathbf{E}_1 \wedge \mathbf{\Pi}_1$; the green elliptic cylinder is an intersection entity component and represents the ellipse in which they intersect. The lower-right image shows the DCGA GIPNS intersection entity $\mathbf{E}_2 \wedge \mathbf{\Pi}_2$; the green *hyperboloid of one sheet* and the red *non-parallel planes pair* are intersection entity components which are also coincident and represent the intersection.

4.3 DCGA GIPNS sphere

The standard DCGA GIPNS 2-vector *sphere* will be defined as a bi-CGA sphere, *not* the DCGA GIPNS ellipsoid with equal semi-diameters. If $r=r_x=r_y=r_z$, then the DCGA GIPNS ellipsoid $\mathbf{E}$ is a kind of sphere that can be written as

$$\begin{aligned}\mathbf{E} &= -2(p_xT_x+p_yT_y+p_zT_z)+T_{x^2}+T_{y^2}+T_{z^2}+(p_x^2+p_y^2+p_z^2-r^2)T_1\\ &\simeq (p_xT_x+p_yT_y+p_zT_z)-\frac{1}{2}(p_x^2+p_y^2+p_z^2)T_1-\frac{1}{2}(T_{x^2}+T_{y^2}+T_{z^2})+\frac{1}{2}r^2T_1.\end{aligned}$$

Taking $r=0$ suggests that this ellipsoid-based sphere degenerates into some type of point entity. With $T_1=-\mathbf{e}_\infty$, the middle term has a familiar CGA point form. However, if this were a CGA point, the last term should reduce to $\mathbf{e}_o$, but it does not. The result here is that, the DCGA GIPNS ellipsoid with $r=0$ degenerates into a DCGA GIPNS *non-null* 2-vector *point* entity, which is not the standard DCGA *null* 2-vector *point* that we might expect. The DCGA ellipsoid can form a kind of sphere entity and it degenerates into a kind of non-null point entity, but we can form a sphere in another way which *does* degenerate into a standard DCGA point.

The standard DCGA GIPNS 2-vector *sphere* surface entity $\mathbf{S}$, also being called a bi-CGA GIPNS 2-vector sphere, is defined as

$$\mathbf{S} = \mathbf{S}_{\mathcal{C}^1}\wedge\mathbf{S}_{\mathcal{C}^2}$$

where

$$\begin{aligned}\mathbf{S}_{\mathcal{C}^1} &= \mathbf{P}_{\mathcal{C}^1}-\frac{1}{2}r^2\mathbf{e}_{\infty1}\\ \mathbf{S}_{\mathcal{C}^2} &= \mathbf{P}_{\mathcal{C}^2}-\frac{1}{2}r^2\mathbf{e}_{\infty2}.\end{aligned}$$

The CGA1 GIPNS sphere $\mathbf{S}_{\mathcal{C}^1}$ and the CGA2 GIPNS sphere $\mathbf{S}_{\mathcal{C}^2}$, both representing the same sphere, with radius r at center position $\mathbf{p}$ in our main 3D Euclidean space $\mathcal{E}^1$, are wedged to form the DCGA or bi-CGA GIPNS sphere $\mathbf{S}$. If $r=0$, the sphere is degenerated into a DCGA point

$$\mathbf{P}_{\mathcal{D}} = \mathbf{P}_{\mathcal{C}^1}\wedge\mathbf{P}_{\mathcal{C}^2}$$

that would represent the center position of the sphere. This form a sphere allows greater consistency, and it can also be intersected with any DCGA GIPNS entity. A sphere that is formed using the DCGA GIPNS ellipsoid can only be intersected with bi-CGA GIPNS entities. In general, the other bi-CGA GIPNS entities for lines, circles, and planes follow this same pattern, that they are the wedge of the CGA1 and CGA2 copies of the entity.

A DCGA 2-vector point $\mathbf{T}_{\mathcal{D}}=\mathcal{D}(\mathbf{t})$ is tested against the DCGA 2-vector sphere $\mathbf{S}$ as

$$-2\left(\frac{-\mathbf{T}_{\mathcal{D}}\cdot\mathbf{e}_{\infty 2}}{(\mathbf{T}_{\mathcal{D}}\cdot\mathbf{e}_{o2})\cdot\mathbf{e}_{\infty 1}}\right)\cdot\left(\frac{-\mathbf{S}\cdot\mathbf{e}_{\infty 2}}{(\mathbf{S}\cdot\mathbf{e}_{o2})\cdot\mathbf{e}_{\infty 1}}\right)\begin{cases} <0 & : \ \mathbf{t} \text{ is } \textit{inside} \text{ sphere}\\ =0 & : \ \mathbf{t} \text{ is } \textit{on} \text{ sphere}\\ >0 & : \ \mathbf{t} \text{ is } \textit{outside} \text{ sphere}\\ >0 & : \ =d^2, \text{ squared tangent}\end{cases}$$

To determine inside or outside, this incidence test requires the bi-CGA point $\mathbf{T}_{\mathcal{D}}$ to be contracted into a CGA1 point, and the bi-CGA sphere $\mathbf{S}$ to be contracted into a CGA1 sphere, and both are renormalized. The entity $\mathbf{e}_{\infty 2}$ is both a CGA2 point and a CGA2 sphere of infinite radius, and it serves as the contraction operator on both the point and sphere into CGA1 entities, up to scale. The result is reduced to a CGA1 incidence test. When the test is positive, it is the squared distance d^2 from the point to the sphere along any line tangent to the sphere surface. Similarly for other bi-CGA entities, they can be contracted into CGA1 entities and then all the usual CGA tests are available on them.

4.4 DCGA GIPNS line

The DCGA GIPNS 4-vector *line* 1D surface entity $\mathbf{L}$ is defined as

$$\mathbf{L} \ = \ \mathbf{L}_{\mathcal{C}^1}\wedge\mathbf{L}_{\mathcal{C}^2}$$

where

$$\begin{aligned}\mathbf{L}_{\mathcal{C}^1} &= \mathbf{D}_{\mathcal{E}^1}-(\mathbf{p}_{\mathcal{E}^1}\cdot\mathbf{D}_{\mathcal{E}^1})\mathbf{e}_{\infty 1}\\ \mathbf{L}_{\mathcal{C}^2} &= \mathbf{D}_{\mathcal{E}^2}-(\mathbf{p}_{\mathcal{E}^2}\cdot\mathbf{D}_{\mathcal{E}^2})\mathbf{e}_{\infty 2}.\end{aligned}$$

This is the wedge of the line as represented in CGA1 with the same line as represented in CGA2. It could also be called a bi-CGA GIPNS line entity. The $\mathbf{D}$ are unit bivectors perpendicular to the line, and $\mathbf{p}$ is any sample point on the line. The undual unit vector $\mathbf{d}=\mathbf{D}\mathbf{I}_3$, or $\mathbf{d}_{\mathcal{E}^1}=\mathbf{D}_{\mathcal{E}^1}\mathbf{I}_{\mathcal{E}^1}$ and $\mathbf{d}_{\mathcal{E}^2}=\mathbf{D}_{\mathcal{E}^2}\mathbf{I}_{\mathcal{E}^2}$, is in the direction of the line.

4.5 DCGA GIPNS plane

The DCGA GIPNS 2-vector *plane* surface entity $\mathbf{\Pi}$ is defined as

$$\mathbf{\Pi} \ = \ \mathbf{\Pi}_{\mathcal{C}^1}\wedge\mathbf{\Pi}_{\mathcal{C}^2}$$

where

$$\begin{aligned}\mathbf{\Pi}_{\mathcal{C}^1} &= \mathbf{n}_{\mathcal{E}^1}+d\mathbf{e}_{\infty 1}\\ \mathbf{\Pi}_{\mathcal{C}^2} &= \mathbf{n}_{\mathcal{E}^2}+d\mathbf{e}_{\infty 2}.\end{aligned}$$

This is the wedge of the plane as represented in CGA1 with the same plane as represented in CGA2. It could also be called a bi-CGA GIPNS plane entity. The vector $\mathbf{n}$ is a unit vector perpendicular (normal) to the plane, and the scalar d is the distance of the plane from the origin.

The DCGA GIPNS 4-vector line $\mathbf{L}$ can also be defined as the intersection of two DCGA GIPNS planes as

$$\begin{aligned}
\mathbf{L} &= \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2 \\
&= (\mathbf{n}_{1\mathcal{E}^1} + d_1 \mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{1\mathcal{E}^2} + d_1 \mathbf{e}_{\infty 2}) \wedge (\mathbf{n}_{2\mathcal{E}^1} + d_2 \mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{2\mathcal{E}^2} + d_2 \mathbf{e}_{\infty 2}) \\
&= -((\mathbf{n}_{1\mathcal{E}^1} + d_1 \mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{2\mathcal{E}^1} + d_2 \mathbf{e}_{\infty 1})) \wedge ((\mathbf{n}_{1\mathcal{E}^2} + d_1 \mathbf{e}_{\infty 2}) \wedge (\mathbf{n}_{2\mathcal{E}^2} + d_2 \mathbf{e}_{\infty 2})) \\
&\simeq (\mathbf{n}_{1\mathcal{E}^1} \wedge \mathbf{n}_{2\mathcal{E}^1} - (d_1 \mathbf{n}_{2\mathcal{E}^1} - d_2 \mathbf{n}_{1\mathcal{E}^1}) \mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{1\mathcal{E}^2} \wedge \mathbf{n}_{2\mathcal{E}^2} - (d_1 \mathbf{n}_{2\mathcal{E}^2} - d_2 \mathbf{n}_{1\mathcal{E}^2}) \mathbf{e}_{\infty 2}) \\
&= (\mathbf{D}_{\mathcal{E}^1} - (\mathbf{p}_{\mathcal{E}^1} \cdot \mathbf{D}_{\mathcal{E}^1}) \mathbf{e}_{\infty 1}) \wedge (\mathbf{D}_{\mathcal{E}^2} - (\mathbf{p}_{\mathcal{E}^2} \cdot \mathbf{D}_{\mathcal{E}^2}) \mathbf{e}_{\infty 2}) \\
&= \mathbf{L}_{\mathcal{C}^1} \wedge \mathbf{L}_{\mathcal{C}^2}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_{\mathcal{E}^1} &= \mathbf{n}_{1\mathcal{E}^1} \wedge \mathbf{n}_{2\mathcal{E}^1} \\
\mathbf{D}_{\mathcal{E}^2} &= \mathbf{n}_{1\mathcal{E}^2} \wedge \mathbf{n}_{2\mathcal{E}^2} \\
\mathbf{p}_{\mathcal{E}^1} \cdot \mathbf{D}_{\mathcal{E}^1} &= (\mathbf{p}_{\mathcal{E}^1} \cdot \mathbf{n}_{1\mathcal{E}^1}) \mathbf{n}_{2\mathcal{E}^1} - (\mathbf{p}_{\mathcal{E}^1} \cdot \mathbf{n}_{2\mathcal{E}^1}) \mathbf{n}_{1\mathcal{E}^1} \\
&= d_1 \mathbf{n}_{2\mathcal{E}^1} - d_2 \mathbf{n}_{1\mathcal{E}^1} \\
\mathbf{p}_{\mathcal{E}^2} \cdot \mathbf{D}_{\mathcal{E}^2} &= (\mathbf{p}_{\mathcal{E}^2} \cdot \mathbf{n}_{1\mathcal{E}^2}) \mathbf{n}_{2\mathcal{E}^2} - (\mathbf{p}_{\mathcal{E}^2} \cdot \mathbf{n}_{2\mathcal{E}^2}) \mathbf{n}_{1\mathcal{E}^2} \\
&= d_1 \mathbf{n}_{2\mathcal{E}^2} - d_2 \mathbf{n}_{1\mathcal{E}^2}
\end{aligned}$$

such that $\mathbf{p}$ is any point on both planes (the line), and $\mathbf{D} = \mathbf{d}^{*\mathcal{E}} = \mathbf{d} / \mathbf{I}_{\mathcal{E}}$ is the unit bivector perpendicular to the line. The unit vector $\mathbf{d} = \mathbf{D}\mathbf{I}_{\mathcal{E}}$ points in the direction of the line. Other bi-CGA GIPNS entities are formed similarly as the wedge of the entity in CGA1 with the same entity in CGA2.

Some of the subscripting notation may seem confusing. For example, $\mathbf{n}_{1\mathcal{E}^1}$ is the first of the two Euclidean 3D unit vectors in the CGA1 space, and this could also be denoted as $\mathbf{n}_{\mathcal{E}_1^1}$. Recall that $\mathcal{E}^1$ is the space of the unit pseudoscalar $\mathbf{I}_{\mathcal{E}^1} = \mathbf{I}_{3^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and it is a subspace of the $\mathcal{C}^1$ CGA1 space $\mathbf{I}_{\mathcal{C}^1} = \mathbf{I}_{5^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$. The CGA2 space uses notations $\mathbf{n}_{1\mathcal{E}^2}$ or $\mathbf{n}_{\mathcal{E}_1^2}$, where $\mathbf{I}_{\mathcal{E}^2} = \mathbf{I}_{3^2} = \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8$ and $\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{5^2} = \mathbf{e}_6\mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}$. The subscripting indicates the index number for multiple entities sharing the same name, and also the space in which the entity exists. Finally, $\mathbf{n}_{1\mathcal{E}^1}$ and $\mathbf{n}_{1\mathcal{E}^2}$ have the same index number 1, so they represent the same 3D unit vector $\mathbf{n}$ copied or doubled into the $\mathcal{E}^1$ and $\mathcal{E}^2$ Euclidean subspace of the $\mathcal{C}^1$ CGA1 and $\mathcal{C}^2$ CGA2 space, respectively.

4.6 DCGA GIPNS circle

A circle is the intersection of a sphere and plane. We can intersect a bi-CGA GIPNS 2-vector plane $\mathbf{\Pi}$ with either a bi-CGA GIPNS 2-vector sphere $\mathbf{S}$ or with a spherical DCGA GIPNS 2-vector ellipsoid $\mathbf{E}$ and get two different GIPNS 4-vector circle entities. The first can be intersected again with any other entity, but the latter can only be intersected again with another bi-CGA GIPNS entity.

Intersections are limited to an GIPNS intersection entity of maximum grade 8, so up to four 2-vector entities, two 4-vector entities, or a 4-vector entity and two 2-vector GIPNS entities can be intersected, but only one of the intersecting entities can be a non-biCGA quadric surface or toroid GIPNS entity.

As the standard DCGA GIPNS 4-vector *circle* 1D surface entity $\mathbf{C}$, we will define it as the bi-CGA GIPNS circle

$$\begin{aligned}\mathbf{C} &= \mathbf{S}\wedge\mathbf{\Pi}\\ &= \mathbf{S}_{\mathcal{C}^1}\wedge\mathbf{S}_{\mathcal{C}^2}\wedge\mathbf{\Pi}_{\mathcal{C}^1}\wedge\mathbf{\Pi}_{\mathcal{C}^2}\\ &= -(\mathbf{S}_{\mathcal{C}^1}\wedge\mathbf{\Pi}_{\mathcal{C}^1})\wedge(\mathbf{S}_{\mathcal{C}^2}\wedge\mathbf{\Pi}_{\mathcal{C}^2})\\ &\simeq \mathbf{C}_{\mathcal{C}^1}\wedge\mathbf{C}_{\mathcal{C}^2}.\end{aligned}$$

4.7 DCGA GIPNS elliptic cylinder

An axes-aligned elliptic cylinder is the limit of an ellipsoid as one of the semi-diameters approaches ∞. This limit eliminates the terms of the cylinder axis from the homogeneous ellipsoid equation.

The x-axis aligned cylinder takes $r_x\to\infty$, reducing the ellipsoid equation to

$$\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_z)^2}{r_z^2}-1 = 0.$$

Similarly, the y-axis and z-axis aligned cylinders are

$$\begin{aligned}\frac{(x-p_x)^2}{r_x^2}+\frac{(z-p_z)^2}{r_z^2}-1 &= 0\\ \frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-1 &= 0\end{aligned}$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c).

The DCGA GIPNS 2-vector *x,y,z-axis aligned cylinder* surface entities $\mathbf{H}^{||\{x,y,z\}}$ are defined as

$$\begin{aligned}\mathbf{H}^{||x} &= \frac{-2p_yT_y}{r_y^2}+\frac{-2p_zT_z}{r_z^2}+\frac{T_{y^2}}{r_y^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right)T_1\\ \mathbf{H}^{||y} &= \frac{-2p_xT_x}{r_x^2}+\frac{-2p_zT_z}{r_z^2}+\frac{T_{x^2}}{r_x^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_z^2}{r_z^2}-1\right)T_1\\ \mathbf{H}^{||z} &= \frac{-2p_xT_x}{r_x^2}+\frac{-2p_yT_y}{r_y^2}+\frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}-1\right)T_1.\end{aligned}$$

These elliptic cylinders are created as axes-aligned, but like all DCGA entities, they can be rotated, dilated, and translated using DCGA versor operations.

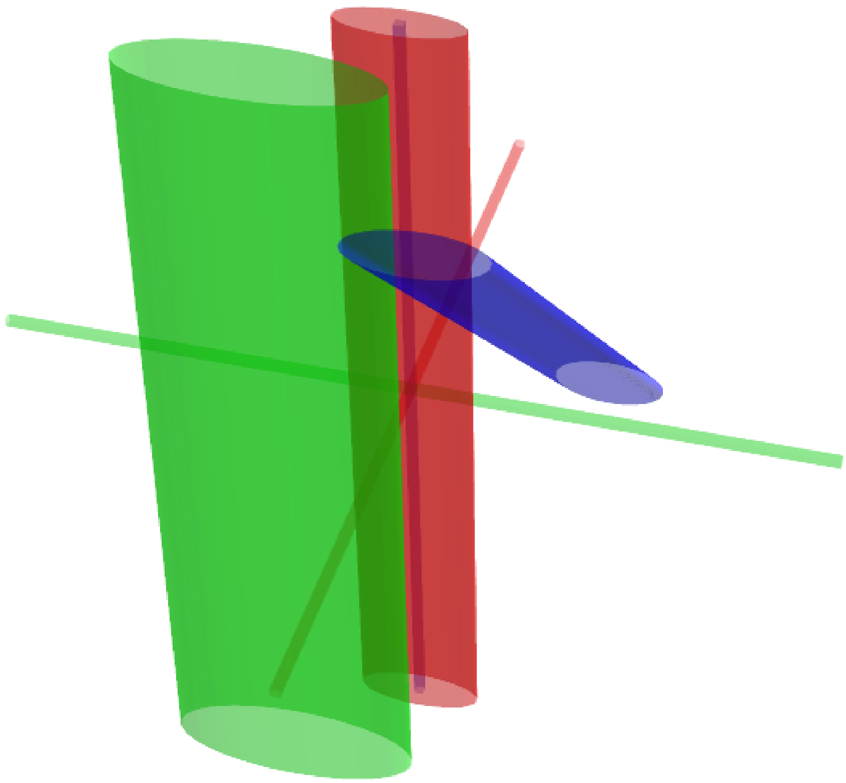

Figure 4. DCGA elliptic cylinders

Figure 4 shows a red DCGA GIPNS z-axis aligned elliptic cylinder at the origin with semi-diameters $r_x = 1$ and $r_y = 3$. The green cylinder is the red cylinder dilated by factor 2 and translated $5\mathbf{e}_1 - 5\mathbf{e}_2$ using DCGA versors. The blue cylinder is the red cylinder rotated 45° around the y-axis and translated $-5\mathbf{e}_1 + 5\mathbf{e}_2$.

4.8 DCGA GIPNS elliptic cone

An axis-aligned elliptic cone is an axis-aligned cylinder that is linearly scaled along the axis.

The homogeneous quadric equation for an x-axis aligned cone is

$$\frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(x-p_x)^2}{r_x^2} = 0.$$

where $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is the position (or shifted origin, or center) of the cone apex, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c) of the ellipsoid upon which the cone is based. When

$$\frac{(x-p_x)^2}{r_x^2} = 1$$

the cross section of the cone is the size of the similar cylinder. When $x = p_x$ the cross section of the cone is degenerated into the cone apex point.

Similarly, the homogeneous equations for y-axis and z-axis aligned cones are

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(y-p_y)^2}{r_y^2} = 0$$

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} = 0.$$

The GIPNS cone entities are constructed similarly to the ellipsoid and cylinder entities.

The DCGA GIPNS 2-vector *{x,y,z}-axis aligned elliptic cone* surface entities $\mathbf{K}^{||\{x,y,z\}}$ are defined as

$$\begin{aligned}
\mathbf{K}^{||x} &= \frac{2p_xT_x}{r_x^2} + \frac{-2p_yT_y}{r_y^2} + \frac{-2p_zT_z}{r_z^2} + \frac{-T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{-p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2}\right)T_1 \\
\mathbf{K}^{||y} &= \frac{-2p_xT_x}{r_x^2} + \frac{2p_yT_y}{r_y^2} + \frac{-2p_zT_z}{r_z^2} + \frac{T_{x^2}}{r_x^2} + \frac{-T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{-p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2}\right)T_1 \\
\mathbf{K}^{||z} &= \frac{-2p_xT_x}{r_x^2} + \frac{-2p_yT_y}{r_y^2} + \frac{2p_zT_z}{r_z^2} + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \frac{-T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} + \frac{-p_z^2}{r_z^2}\right)T_1.
\end{aligned}$$

These elliptic cones are created as axes-aligned, but they can be rotated, dilated, and translated using DCGA versor operations. All the DCGA surfaces can have general position, but we initially define them in axes-aligned position for simplicity. Defining the surfaces in general position may be possible if the value-extraction operations T_{xy}, T_{yz}, and T_{zx} are employed.

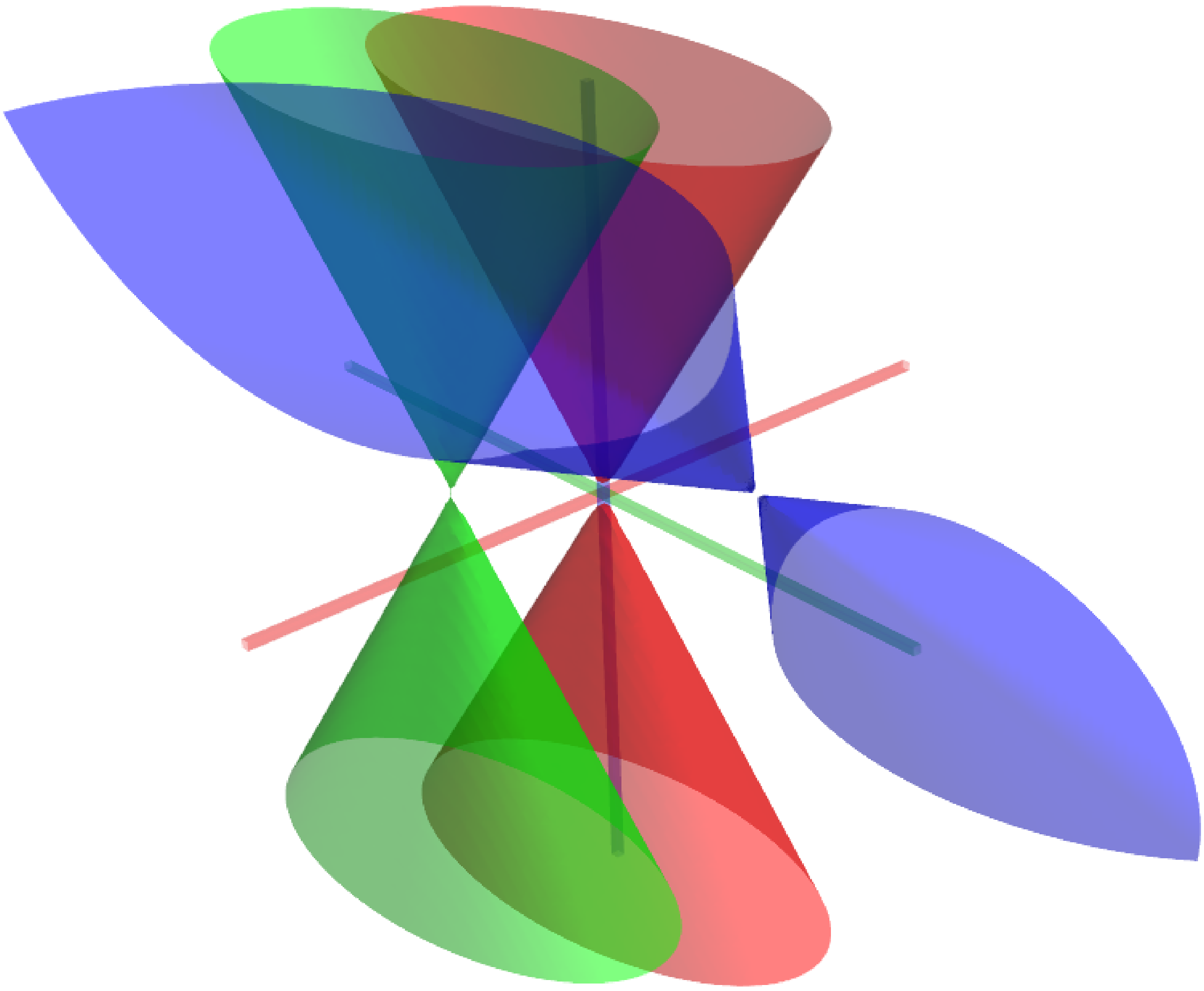

Figure 5. DCGA elliptic cones

Figure 5 shows some DCGA GIPNS cones positioned and transformed similar to the elliptic cylinders of Figure 4. The dilation of a cone does not change the cone shape, but it does dilate the cone center position to effectively translate a cone that is not initially at the origin to be further from the origin by the dilation factor.

4.9 DCGA GIPNS elliptic paraboloid

The elliptic paraboloid has a cone-like shape that opens up or down. The other paraboloid that would open the other way is imaginary with no real solution points.

The homogeneous quadric equation of a z-axis aligned elliptic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(z-p_z)}{r_z} = 0.$$

The surface opens up the z-axis for $r_z > 0$, and opens down the z-axis for $r_z < 0$. Similar equations for x-axis and y-axis aligned elliptic paraboloids are

$$\begin{aligned}\frac{(z-p_z)^2}{r_z^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(x-p_x)}{r_x} &= 0\\ \frac{(x-p_x)^2}{r_x^2}+\frac{(z-p_z)^2}{r_z^2}-\frac{(y-p_y)}{r_y} &= 0.\end{aligned}$$

Expanding the squares, the z-axis aligned equation is

$$\frac{-2p_xx}{r_x^2}+\frac{-2p_yy}{r_y^2}+\frac{-z}{r_z}+\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z}{r_z}\right) = 0$$

and the x-axis and y-axis aligned equations are

$$\begin{aligned}\frac{-2p_zz}{r_z^2}+\frac{-2p_yy}{r_y^2}+\frac{-x}{r_x}+\frac{z^2}{r_z^2}+\frac{y^2}{r_y^2}+\left(\frac{p_z^2}{r_z^2}+\frac{p_y^2}{r_y^2}+\frac{p_x}{r_x}\right) &= 0\\ \frac{-2p_xx}{r_x^2}+\frac{-2p_zz}{r_z^2}+\frac{-y}{r_y}+\frac{x^2}{r_x^2}+\frac{z^2}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_z^2}{r_z^2}+\frac{p_y}{r_y}\right) &= 0.\end{aligned}$$

The DCGA GIPNS 2-vector *{x,y,z}-axis aligned elliptic paraboloid* surface entities $\mathbf{V}^{||\{x,y,z\}}$ are defined as

$$\begin{aligned}\mathbf{V}^{||x} &= \frac{-2p_zT_z}{r_z^2}+\frac{-2p_yT_y}{r_y^2}+\frac{-T_x}{r_x}+\frac{T_{z^2}}{r_z^2}+\frac{T_{y^2}}{r_y^2}+\left(\frac{p_z^2}{r_z^2}+\frac{p_y^2}{r_y^2}+\frac{p_x}{r_x}\right)T_1\\ \mathbf{V}^{||y} &= \frac{-2p_xT_x}{r_x^2}+\frac{-2p_zT_z}{r_z^2}+\frac{-T_y}{r_y}+\frac{T_{x^2}}{r_x^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_z^2}{r_z^2}+\frac{p_y}{r_y}\right)T_1\end{aligned}$$

$$\mathbf{V}^{||z} \qquad \frac{-2p_xT_x}{r_x^2}+\frac{-2p_yT_y}{r_y^2}+\frac{-T_z}{r_z}+\frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z}{r_z}\right)T_1.$$

A DCGA 2-vector point $\mathbf{T}_{\mathcal{D}}=\mathcal{D}(\mathbf{t})$ is tested against the DCGA 2-vector paraboloid $\mathbf{V}$ as

$$\mathbf{T}_{\mathcal{D}}\cdot\mathbf{V}\begin{cases} <0 & : \ \mathbf{t} \text{ is } \textit{inside} \text{ paraboloid} \\ =0 & : \ \mathbf{t} \text{ is } \textit{on} \text{ paraboloid} \\ >0 & : \ \mathbf{t} \text{ is } \textit{outside} \text{ paraboloid.} \end{cases}$$

This is similar to the ellipsoid incidence test, and this test is similar for many of the surfaces.

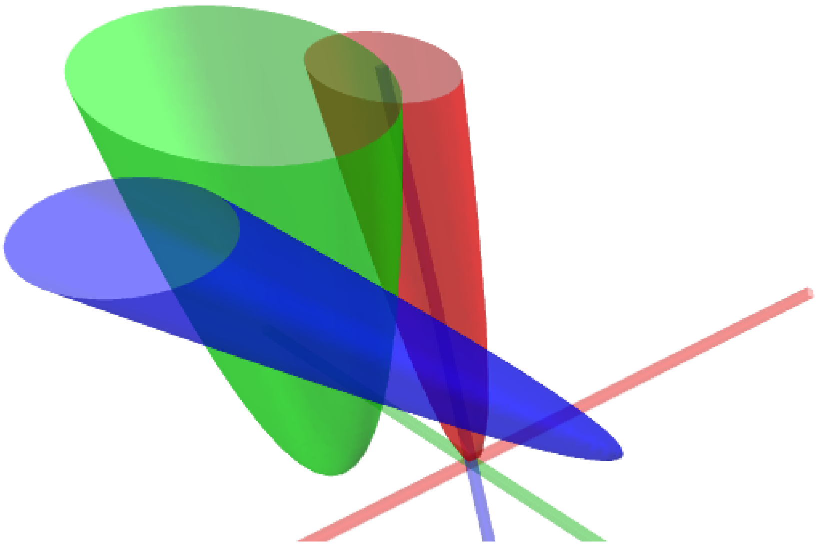

Figure 6. DCGA elliptic paraboloids

4.10 DCGA GIPNS hyperbolic paraboloid

The hyperbolic paraboloid has a saddle shape. The saddle can be mounted or aligned on a *saddle* axis with another axis chosen as the *up* axis. The third axis may be called the *straddle* axis.

The homogeneous quadric equation of a hyperbolic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2}-\frac{(z-p_z)}{r_z} \;=\; 0.$$

This particular form of the equation has saddle x-axis, straddle y-axis, and up z-axis for $r_z>0$ or up negative z-axis for $r_z<0$. By its similarity to the z-axis aligned elliptic paraboloid with the elliptic y-axis inverted, this particular form can be seen as z-axis aligned. Other forms can be made by transposing axes, or by rotation around diagonal lines using DCGA rotor operations.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{-z}{r_z} + \frac{x^2}{r_x^2} + \frac{-y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) = 0.$$

The DCGA GIPNS 2-vector *z-axis aligned hyperbolic paraboloid* surface entity $\mathbf{M}$ is defined as

$$\mathbf{M} = \frac{-2p_x T_x}{r_x^2} + \frac{2p_y T_y}{r_y^2} + \frac{-T_z}{r_z} + \frac{T_{x^2}}{r_x^2} + \frac{-T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) T_1.$$

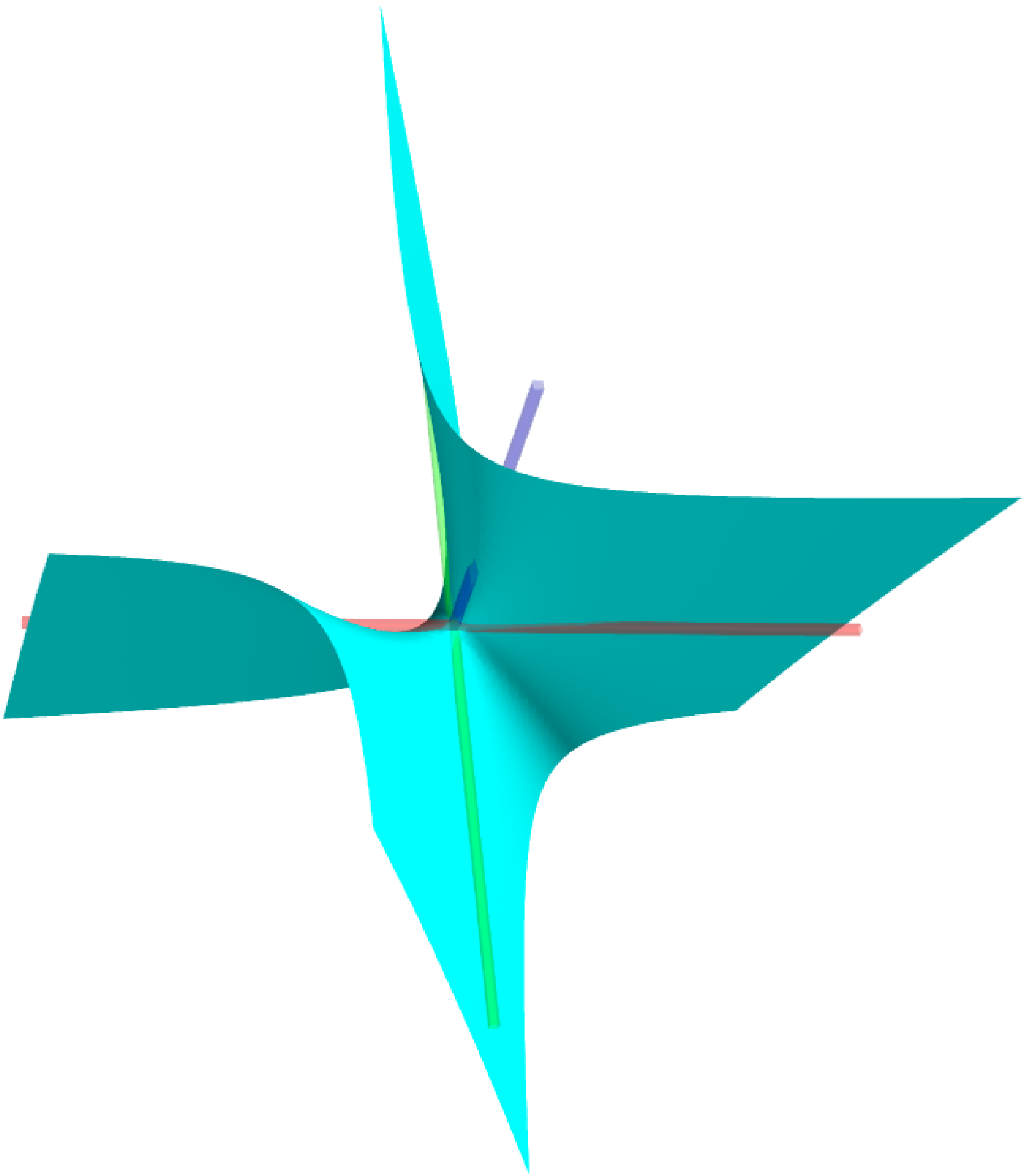

Figure 7. DCGA hyperbolic paraboloid rotated twice

Figure 7 shows the hyperbolic paraboloid entity, which is centered on the origin with parameters $r_x = r_y = r_z = 1$, and which was initially z-axis aligned. It was then rotated twice. The first rotation was 45° around the blue z-axis, pointing nearly out of the page. The second rotation was 25° around the line $\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, not shown but pointing toward the lower-right of the page. The rotations follow the right-hand rule on a right-handed axes model. The red x-axis points to the left, and the green y-axis points down.

4.11 DCGA GIPNS hyperboloid of one sheet

The hyperboloid of one sheet has a shape that is similar to an hourglass which continues to open both upward and downward. The homogeneous quadric equation is

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} - 1 \;=\; 0.$$

This particular form opens up and down the z-axis. Planes parallel to the z-axis cut hyperbola sections. Planes perpendicular to the z-axis cut ellipse sections. At $z = p_z$, the ellipse section has a minimum size of the similar cylinder. Other forms can be made by transposing axes, or by rotation around diagonal lines using DCGA rotor operations.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{-2p_y y}{r_y^2} + \frac{2p_z z}{r_z^2} + \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{-z^2}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right) \;=\; 0.$$

The DCGA GIPNS 2-vector *z-axis aligned hyperboloid of one sheet* surface entity $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} \;=\; -2\left(\frac{p_x T_x}{r_x^2} + \frac{p_y T_y}{r_y^2} - \frac{p_z T_z}{r_z^2}\right) + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} - \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right) T_1.$$

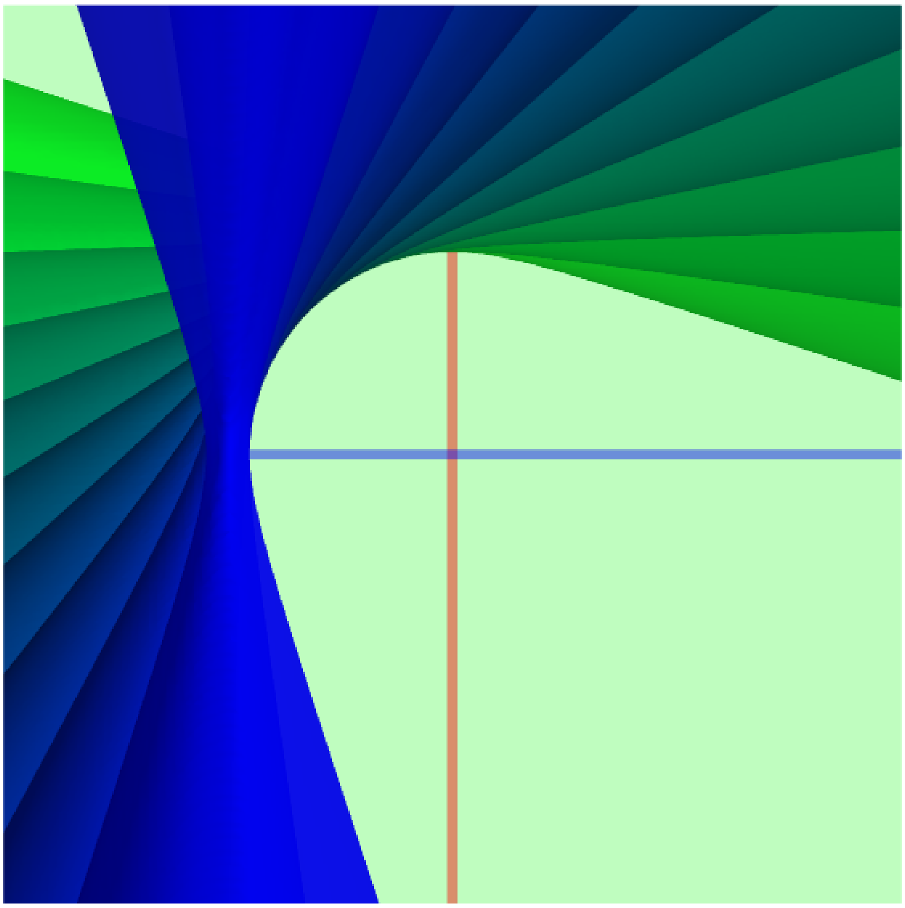

Figure 8. Rotation of DCGA hyperboloid of one sheet

Figure 8 is an orthographic (parallel projection) view from above the zx-plane that shows this hyperboloid with $r_x = 1$, $r_y = 2$, $r_z = 3$, initially with green color, positioned at $p_x = 10$, and aligned up and down the z-axis. It is then rotated using a DCGA rotor by 90° in 10° steps as its color fades to blue, with final position at $p_z = -10$ and aligned up and down the x-axis. The rotation is counter-clockwise around the y-axis coming out of the page on a right-handed system of axes. The x-axis is red and positive up, the y-axis is green (not visible), and the z-axis is blue and positive to the right. The axes are drawn by rendering thin elliptic cylinder entities. The right-hand rule, holding the y-axis, provides orientation for this rotation. The hyperboloid is rotated about the origin, around the y-axis, as a rigid body of points. In the symbolic computer algebra system (CAS) *Sympy*, the hyperboloid equation itself, as a DCGA entity, was rotated symbolically and graphed at each step using the *MayaVi* data visualization software.

4.12 DCGA GIPNS hyperboloid of two sheets

The hyperboloid of two sheets has the shapes of two separate hyperbolic dishes; one opens upward, and the other one opens downward. The shape is like an hourglass that is pinched closed and the two halves are also separated by some distance. The homogeneous quadric equation is

$$-\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_z)^2}{r_z^2}-1 \;=\; 0.$$

This particular form has the two dishes opening up and down the z-axis. The dishes are separated by distance $2r_z$ centered at p_z. At $|z-p_z|=\sqrt{2}r_z$, the sections perpendicular to the z-axis are the size of the similar cylinder.

Expanding the squares, the equation is

$$\frac{2p_x x}{r_x^2}+\frac{2p_y y}{r_y^2}-\frac{2p_z z}{r_z^2}-\frac{x^2}{r_x^2}-\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}+\left(\frac{-p_x^2}{r_x^2}+\frac{-p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right) \;=\; 0.$$

The DCGA GIPNS 2-vector *z-axis aligned hyperboloid of two sheets* surface entity $\mathbf{\Xi}$ is defined as

$$\mathbf{\Xi} \;=\; 2\left(\frac{p_x T_x}{r_x^2}+\frac{p_y T_y}{r_y^2}-\frac{p_z T_z}{r_z^2}\right)-\frac{T_{x^2}}{r_x^2}-\frac{T_{y^2}}{r_y^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_z^2}{r_z^2}-\frac{p_x^2}{r_x^2}-\frac{p_y^2}{r_y^2}-1\right)T_1.$$

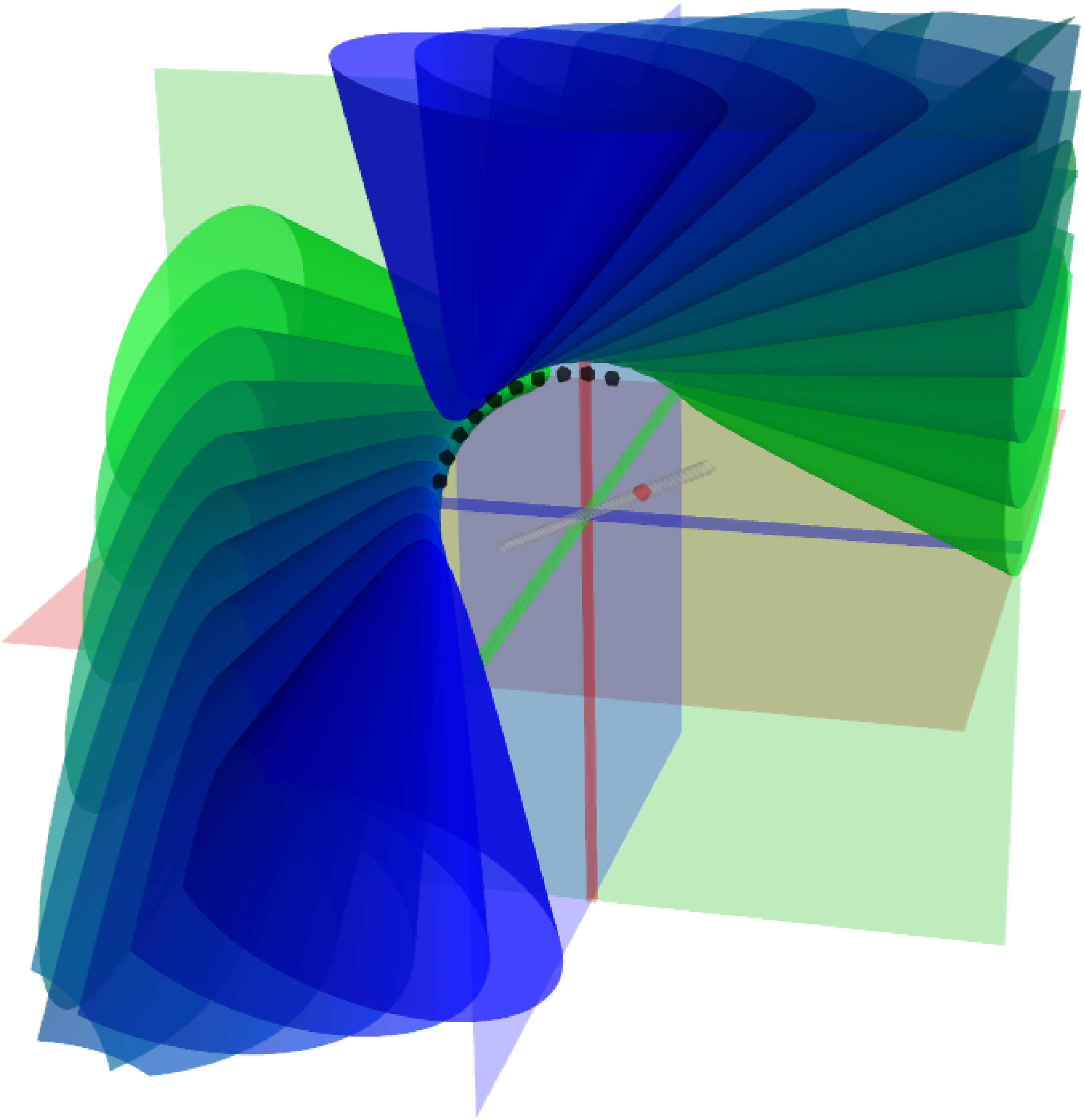

Figure 9. Rotation of DCGA hyperboloid of two sheets

Figure 9 shows a perspective view of the hyperboloid of two sheets initially with green color, centered at $p_x = 5$, $p_y = -5$, and with semi-diameters $r_x = 1$, $r_y = 2$, $r_z = 3$. The black dots (small sphere entities) are the center positions as the surface is rotated around the white line through the origin and the red point $5\mathbf{e}_1 + 10\mathbf{e}_2 + 5\mathbf{e}_3$. The rotation is by 90° in 10° steps until it reaches the position of the blue surface. The first black dot is on the xy-plane (blue plane), and then the black dots go under the blue plane along an arc directly around the axis of rotation. The surface is carried along as a rigid body by the rotation using a DCGA rotor operation. The symbolic CAS *Sympy* was used for each rotation step, where an exact symbolic equation of the hyperboloid was generated by the rotated entity and graphed using *MayaVi* data visualization software.

4.13 DCGA GIPNS parabolic cylinder

The homogeneous quadric equation for the z-axis aligned parabolic cylinder is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)}{r_y} = 0.$$

The z coordinate is free, which creates a type of 2-surface or cylinder with parabolic sections that open up the y-axis for $r_y > 0$, and open down the y-axis for $r_y < 0$. The similar equations for x-axis and y-axis aligned parabolic cylinders are

$$\begin{aligned} \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} &= 0 \\ \frac{(x-p_x)^2}{r_x^2} - \frac{(z-p_z)}{r_z} &= 0 \end{aligned}$$

with parabolas that open up or down the z-axis. Other forms can be made by transpositions using the transposition operations.

Expanding the squares, the equations are

$$\begin{aligned} \frac{-2p_x x}{r_x^2} - \frac{y}{r_y} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right) &= 0 \\ \frac{-2p_y y}{r_y^2} - \frac{z}{r_z} + \frac{y^2}{r_y^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) &= 0 \\ \frac{-2p_x x}{r_x^2} - \frac{z}{r_z} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right) &= 0. \end{aligned}$$

The DCGA GIPNS 2-vector *{x,y,z}-axis aligned parabolic cylinder* surface entities $\mathbf{B}^{||\{x,y,z\}}$ are defined as

$$\mathbf{B}^{||x} = \frac{-2p_y T_y}{r_y^2} - \frac{T_z}{r_z} + \frac{T_{y^2}}{r_y^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) T_1$$

$$\begin{aligned}\mathbf{B}^{||y} &= \frac{-2p_xT_x}{r_x^2} - \frac{T_z}{r_z} + \frac{T_{x^2}}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right)T_1 \\ \mathbf{B}^{||z} &= \frac{-2p_xT_x}{r_x^2} - \frac{T_y}{r_y} + \frac{T_{x^2}}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right)T_1.\end{aligned}$$

These are created as axes-aligned surfaces, but can be rotated, dilated, and translated using DCGA versor operations.

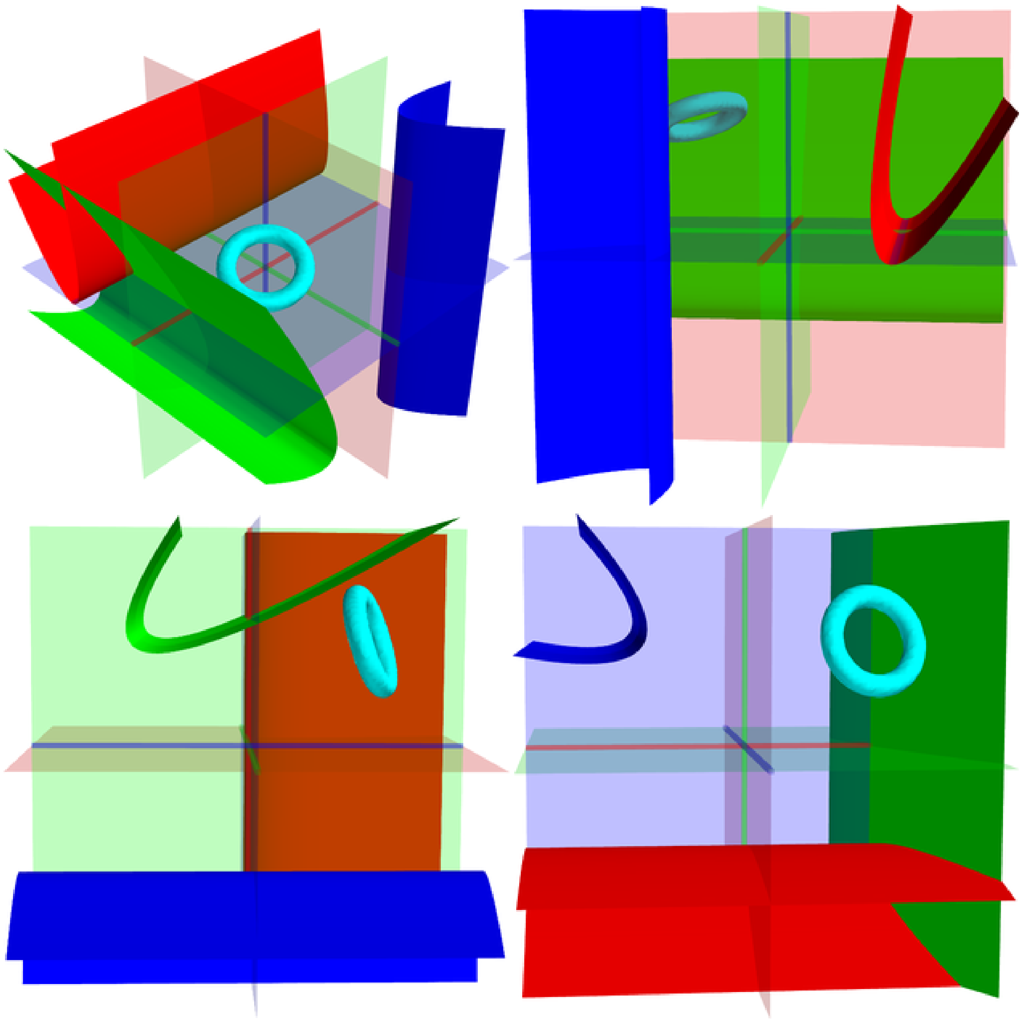

Figure 10. DCGA parabolic cylinders and toroid rotated and translated

Figure 10 shows multiple perspective views of the DCGA GIPNS 2-vector parabolic cylinders and toroid surface entities rendered together in one scene. The red cylinder is x-axis aligned, $r_y = 1$, $r_z = 2$, rotated 20° around the x-axis, and then translated by $\mathbf{d} = -10\mathbf{e}_2$ from the origin. The green cylinder is y-axis aligned, $r_x = 1$, $r_z = 3$, rotated 40° around the y-axis, and then translated by $\mathbf{d} = 10\mathbf{e}_1 - 10\mathbf{e}_3$ from the origin. The blue cylinder is z-axis aligned, $r_x = 1$, $r_y = 4$, rotated 60° around the z-axis, and then translated by $\mathbf{d} = -10\mathbf{e}_1 + 10\mathbf{e}_2$ from the origin. The toroid, with

$R = 4$ and $r = 1$, is rotated $25°$ around the axis $\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$, and then translated by $\mathbf{d} = 10\mathbf{e}_1 + 10\mathbf{e}_2 + 10\mathbf{e}_3$ from the origin. The rotations follow the right-hand rule on right-handed axes. The rotation-translations were performed as compositions of DCGA rotors and translators. Symbolic CAS *Sympy* was used to generate exact equations of the transformed entities, which were then graphed using the *MayaVi* data visualization software.

4.14 DCGA GIPNS hyperbolic cylinder

The homogeneous quadric equation for the z-axis aligned hyperbolic cylinder is

$$\frac{(x - p_x)^2}{r_x^2} - \frac{(y - p_y)^2}{r_y^2} - 1 = 0.$$

The z coordinate is free, which creates a type of cylinder with hyperbolic sections that open up and down the x-axis. The hyperbola branches are separated by distance $2r_x$ centered at $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + z\mathbf{e}_3$. The asymptotes are the lines

$$(y - p_y) = \pm\frac{r_y}{r_x}(x - p_x)$$

through (p_x, p_y), where in the limit as $x \to \pm\infty$ the -1 becomes insignificant.

The similar equations for x-axis and y-axis aligned hyperbolic cylinders are

$$\begin{aligned} \frac{(y - p_y)^2}{r_y^2} - \frac{(z - p_z)^2}{r_z^2} - 1 &= 0. \\ \frac{(z - p_z)^2}{r_z^2} - \frac{(x - p_x)^2}{r_x^2} - 1 &= 0. \end{aligned}$$

with parabolas that open up or down the z-axis and x-axis. Other hyperbolic cylinders can be made by using DCGA versor operations to transform a cylinder with fixed semi-diameters into general positions.

Expanding the squares, the equations for x, y, z-aligned hyperbolic cylinders are

$$\begin{aligned} \frac{-2p_y y}{r_y^2} + \frac{2p_z z}{r_z^2} + \frac{y^2}{r_y^2} - \frac{z^2}{r_z^2} + \left(\frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right) &= 0 \\ \frac{-2p_z z}{r_z^2} + \frac{2p_x x}{r_x^2} + \frac{z^2}{r_z^2} - \frac{x^2}{r_x^2} + \left(\frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2} - 1\right) &= 0 \\ \frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} - 1\right) &= 0. \end{aligned}$$

The DCGA GIPNS 2-vector *{x,y,z}-axis aligned hyperbolic cylinder* surface entities

$\mathbf{J}^{||\{x,y,z\}}$ are defined as

$$\begin{aligned}
\mathbf{J}^{||x} &= \frac{-2p_yT_y}{r_y^2} + \frac{2p_zT_z}{r_z^2} + \frac{T_{y^2}}{r_y^2} - \frac{T_{z^2}}{r_z^2} + \left(\frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right)T_1 \\
\mathbf{J}^{||y} &= \frac{-2p_zT_z}{r_z^2} + \frac{2p_xT_x}{r_x^2} + \frac{T_{z^2}}{r_z^2} - \frac{T_{x^2}}{r_x^2} + \left(\frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2} - 1\right)T_1 \\
\mathbf{J}^{||z} &= \frac{-2p_xT_x}{r_x^2} + \frac{2p_yT_y}{r_y^2} + \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} - 1\right)T_1.
\end{aligned}$$

Figure 11. DCGA hyperbolic cylinder rotated and translated

Figure 11 shows the z-axis aligned hyperbolic cylinder, with initial parameters $p_x = 0$, $p_y = 0$, $r_x = 1$, and $r_y = 2$. The second rendering of it is rotated 60° around the z-axis and then translated by $\mathbf{d} = -10\mathbf{e}_1 + 10\mathbf{e}_2$ using a composition of DCGA rotor and translator operations.

4.15 DCGA GIPNS parallel planes pair

Parallel pairs of axes-aligned planes are represented by the simple quadratic equations in one variable

$$\begin{aligned}(x-p_{x1})(x-p_{x2}) &= 0\\ (y-p_{y1})(y-p_{y2}) &= 0\\ (z-p_{z1})(z-p_{z2}) &= 0.\end{aligned}$$

Each solution is a plane. Expanding the equations gives

$$\begin{aligned}x^2-(p_{x1}+p_{x2})x+p_{x1}p_{x2} &= 0\\ y^2-(p_{y1}+p_{y2})y+p_{y1}p_{y2} &= 0\\ z^2-(p_{z1}+p_{z2})z+p_{z1}p_{z2} &= 0.\end{aligned}$$

The DCGA GIPNS 2-vector *parallel {x,y,z}-planes pair* entities $\mathbf{\Pi}^{\perp\{x,y,z\}}$ are defined as

$$\begin{aligned}\mathbf{\Pi}^{\perp x} &= T_{x^2}-(p_{x1}+p_{x2})T_x+p_{x1}p_{x2}T_1\\ \mathbf{\Pi}^{\perp y} &= T_{y^2}-(p_{y1}+p_{y2})T_y+p_{y1}p_{y2}T_1\\ \mathbf{\Pi}^{\perp z} &= T_{z^2}-(p_{z1}+p_{z2})T_z+p_{z1}p_{z2}T_1\end{aligned}$$

These surfaces can also be described as being types of cylinders with cross sections being two parallel lines.

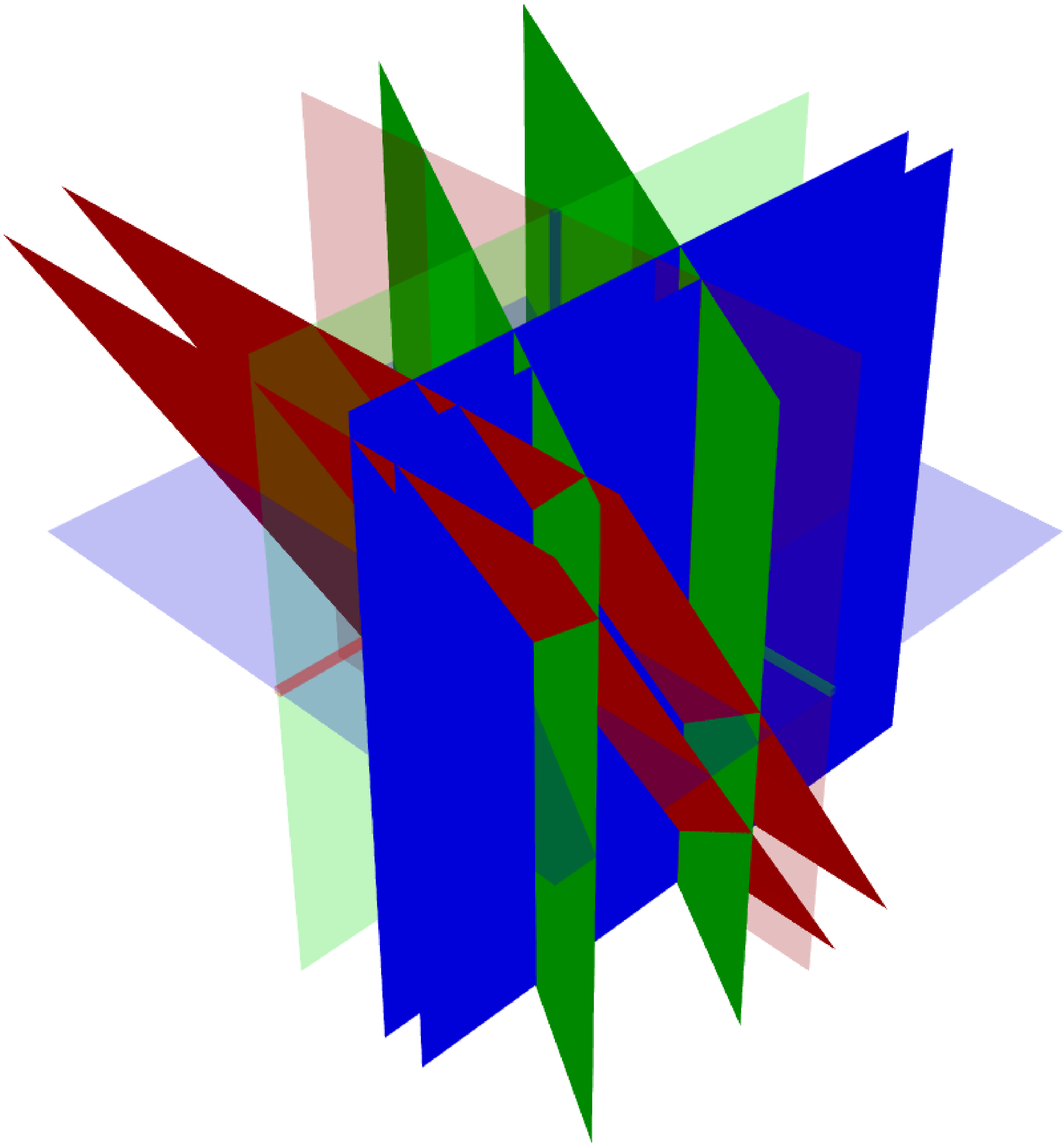

Figure 12. DCGA parallel planes pairs rotated

Figure 12 shows the DCGA GIPNS parallel planes pair entities rotated using DCGA rotor operations. The red planes pair is initially perpendicular to the x-axis through points $p_{x1} = 4$ and $p_{x2} = 8$, then it is rotated 30° around the y-axis. The green planes pair is initially perpendicular to the y-axis through points $p_{y1} = -5$ and $p_{y2} = 5$, then it is rotated 60° around the z-axis. The blue planes pair is initially perpendicular to the z-axis through points $p_{z1} = -10$ and $p_{z2} = -7$, then it is rotated 90° around the x-axis until it is perpendicular to the y-axis through the points

$p_{y1} = 10$ and $p_{y2} = 7$.

4.16 DCGA GIPNS non-parallel planes pair

The homogeneous quadric equation for a pair of intersecting, non-parallel planes that are parallel to the z-axis is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} = 0.$$

This equation can be written as

$$(y-p_y) = \pm\frac{r_y}{r_x}(x-p_x)$$

with the z coordinate free to range. This surface can also be described as a kind of cylinder with a cross section in plane z that is two lines with slopes $\pm\frac{r_y}{r_x}$ intersecting at $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + z\mathbf{e}_3$.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2}\right) = 0.$$

The DCGA GIPNS 2-vector *{x,y,z}-axis aligned non-parallel planes pair* entities $\mathbf{X}^{||\{x,y,z\}}$ are defined as

$$\begin{aligned}
\mathbf{X}^{||x} &= \frac{-2p_yT_y}{r_y^2} + \frac{2p_zT_z}{r_z^2} + \frac{T_{y^2}}{r_y^2} - \frac{T_{z^2}}{r_z^2} + \left(\frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2}\right)T_1\\
\mathbf{X}^{||y} &= \frac{-2p_zT_z}{r_z^2} + \frac{2p_xT_x}{r_x^2} + \frac{T_{z^2}}{r_z^2} - \frac{T_{x^2}}{r_x^2} + \left(\frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2}\right)T_1\\
\mathbf{X}^{||z} &= \frac{-2p_xT_x}{r_x^2} + \frac{2p_yT_y}{r_y^2} + \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2}\right)T_1
\end{aligned}$$

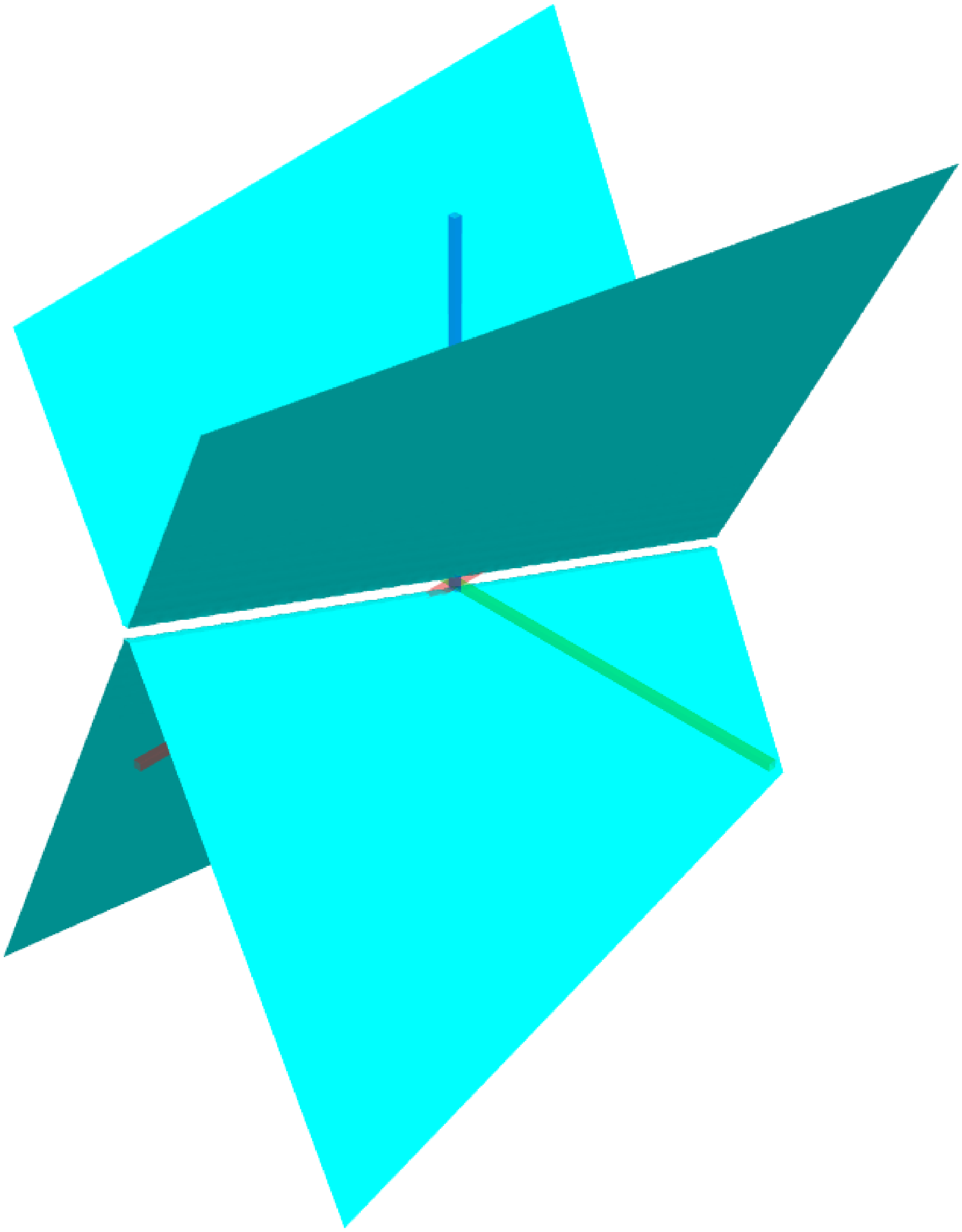

Figure 13. DCGA non-parallel planes pair rotated

Figure 13 shows the entity $\mathbf{X}^{||z}$, initially having planes with slopes $\pm\frac{r_y}{r_x} = \pm\frac{1}{2}$ that cross at the origin point $p_x = 0$, $p_y = 0$ in the xy-plane. It is then rotated using a DCGA rotor operation around the y-axis (green) by 70°. The line of crossing points was initially the z-axis, but after rotation the crossing line is at 70° off the z-axis, around the y-axis. Like the other DCGA entities, the non-parallel planes pair entities can be transformed into general positions using DCGA versor operations.

4.17 DCGA GIPNS ellipse

The ellipse is a conic section, and like all conic sections it can be made as the intersection of a plane and cone, but we are not limited to intersecting with cones.

A simple ellipse representation is made as the intersection of a plane and elliptic cylinder. The parabola and hyperbola are also conic sections, and their simple representations are as planes intersecting parabolic and hyperbolic cylinders. We can just define these conic sections as these plane and cylinder intersections, but these conic sections could be formed by a wide variety of other possible intersections.

The DCGA GIPNS 4-vector *xy-plane ellipse* 1D surface entity $\boldsymbol{\epsilon}^{||xy}$ is defined as

$$\boldsymbol{\epsilon}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{H}^{||z}$$

where the DCGA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ is the entity for the plane $z=0$, and the DCGA GIPNS 2-vector *elliptic cylinder* $\mathbf{H}^{||z}$ is as previously defined and directly represents an ellipse in the xy-plane. Other similar ellipse entities are the wedges of other planes with other elliptic cylinders that are aligned differently.

A DCGA GIPNS ellipse entity $\boldsymbol{\epsilon}$, or its dual DCGA GOPNS ellipse entity $\boldsymbol{\epsilon}^{*\mathcal{D}} = \boldsymbol{\epsilon}/\mathbf{I}_{\mathcal{D}}$, can be rotated, dilated, and translated using DCGA versor operations, where *versor outermorphism* is applied to the wedge of plane and cylinder that form the ellipse entity. In versor operations on the ellipse entity, the plane and cylinder are each transformed by the versor operations, and then the transformed plane and cylinder are intersected.

4.18 DCGA GIPNS parabola

The DCGA GIPNS 4-vector *xy-plane parabola* 1D surface entity $\boldsymbol{\rho}^{||xy}$ is defined as

$$\boldsymbol{\rho}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{B}^{||z}$$

where the DCGA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ is the entity for the plane $z = 0$, and the DCGA GIPNS 2-vector *parabolic cylinder* $\mathbf{B}^{||z}$ is as previously defined and directly represents a parabola in the xy-plane. Other similar parabola entities are the wedges of other planes with other parabolic cylinders that are aligned differently.

4.19 DCGA GIPNS hyperbola

The DCGA GIPNS 4-vector *xy-plane hyperbola* 1D surface entity $\boldsymbol{\eta}^{||xy}$ is defined as

$$\boldsymbol{\eta}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{J}^{||z}$$

where the DCGA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ is the entity for the plane $z=0$, and the DCGA GIPNS 2-vector *hyperbolic cylinder* $\mathbf{J}^{||z}$ is as previously defined and directly represents a hyperbola in the xy-plane. Other similar hyperbola entities are the wedges of other planes with other hyperbolic cylinders that are aligned differently.

5 DCGA GOPNS surfaces

Up to four DCGA points can be wedged to form DCGA *geometric outer product null space* (GOPNS) 4,6,8-vector surface entities of the surface types available in CGA. Unfortunately, the wedge of more than four points, as required for the quadric surfaces, does not work with DCGA points.

The DCGA GOPNS surface entities for quadric surfaces and the toroid would require more than four points to define them. For quadric surfaces in general position, it takes 5 points in 2D, 9 points in 3D, and $\binom{n+2}{2}-1$ points in nD to define a quadric surface. If limited to principal axes-aligned surfaces it still requires 6 points in 3D to define quadric surfaces, as in QGA. Therefore, it seems that it is not possible in DCGA to directly represent the DCGA GOPNS quadric surfaces as the wedge of DCGA surface points. When more than four DCGA surface points are required to define a surface, then more complicated formulas are still possible but they resolve back to the GIPNS entities.

In general, we can always obtain a DCGA GOPNS surface entity $\mathbf{S}^{*\mathcal{D}}$ by taking the DCGA dual of a DCGA GIPNS surface entity $\mathbf{S}$ as $\mathbf{S}^{*\mathcal{D}} = \mathbf{S}/\mathbf{I}_{\mathcal{D}}$. All DCGA versor operations are valid on both the DCGA GIPNS entities and their dual DCGA GOPNS entities.

The following four subsections define the four DCGA GOPNS surface entities which can be constructed synthetically as wedges of up four DCGA surface points. These four DCGA GOPNS surface entities are just the DCGA analogs of the CGA GOPNS surface entities.

A DCGA test point $\mathbf{T}_{\mathcal{D}}$ that is on a DCGA GOPNS surface entity $\mathbf{S}^{*\mathcal{D}}$ must satisfy the GOPNS condition

$$\mathbf{T}_{\mathcal{D}} \wedge \mathbf{S}^{*\mathcal{D}} = 0.$$

The DCGA GOPNS k-vector surface entity $\mathbf{S}^{*\mathcal{D}}$ represents the set $\mathbb{NO}_G(\mathbf{S}^{*\mathcal{D}} \in \mathcal{G}^k_{8,2})$ of all 3D vector test points $\mathbf{t}$ that are surface points

$$\mathbb{NO}_G(\mathbf{S}^{*\mathcal{D}} \in \mathcal{G}^k_{8,2}) = \left\{ \mathbf{t} \in \mathcal{G}^1_3 \ : \ (\mathcal{D}(\mathbf{t}) = \mathbf{T}_{\mathcal{D}}) \wedge \mathbf{S}^{*\mathcal{D}} = 0 \right\}.$$

5.1 DCGA GOPNS sphere

The DCGA GOPNS 8-vector *sphere* $\mathbf{S}^{*\mathcal{D}}$ is defined as the wedge of four DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the sphere as

$$\begin{aligned} \mathbf{S}^{*\mathcal{D}} &= \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} \wedge \mathbf{P}_{\mathcal{D}_4} \\ &= \mathbf{S}/\mathbf{I}_{\mathcal{D}} \end{aligned}$$

and is the DCGA dual of the DCGA GIPNS 2-vector sphere $\mathbf{S}$.

5.2 DCGA GOPNS plane

The DCGA GOPNS 8-vector *plane* $\mathbf{\Pi}^{*\mathcal{D}}$ is defined as the wedge of three DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the plane and the DCGA point at infinity $\mathbf{e}_\infty$ as

$$\begin{aligned} \mathbf{\Pi}^{*\mathcal{D}} &= \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} \wedge \mathbf{e}_\infty \\ &= \mathbf{\Pi}/\mathbf{I}_{\mathcal{D}} \end{aligned}$$

and is the DCGA dual of the DCGA GIPNS 2-vector plane $\mathbf{\Pi}$.

5.3 DCGA GOPNS line

The DCGA GOPNS 6-vector *line* $\mathbf{L}^{*\mathcal{D}}$ is defined as the wedge of two DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the line and the DCGA point at infinity $\mathbf{e}_\infty$ as

$$\begin{aligned}\mathbf{L}^{*\mathcal{D}} &= \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{e}_\infty \\ &= \mathbf{L}/\mathbf{I}_{\mathcal{D}}\end{aligned}$$

and is the DCGA dual of the DCGA GIPNS 4-vector line $\mathbf{L}$.

5.4 DCGA GOPNS circle

The DCGA GOPNS 6-vector *circle* $\mathbf{C}^{*\mathcal{D}}$ is defined as the wedge of three DCGA points $\mathbf{P}_{\mathcal{D}_i}$ on the circle as

$$\begin{aligned}\mathbf{C}^{*\mathcal{D}} &= \mathbf{P}_{\mathcal{D}_1} \wedge \mathbf{P}_{\mathcal{D}_2} \wedge \mathbf{P}_{\mathcal{D}_3} \\ &= \mathbf{C}/\mathbf{I}_{\mathcal{D}}\end{aligned}$$

and is the DCGA dual of the DCGA GIPNS 4-vector circle $\mathbf{C}$.

6 DCGA operations

The DCGA operations are very similar to the CGA operations, but the DCGA versors are the wedges of the two likewise CGA versors in both CGA1 and CGA2.

6.1 DCGA rotor

The DCGA rotor R is defined as

$$R = R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2}.$$

The CGA rotors for the same rotation operation in CGA1 and CGA2 are wedged as the DCGA rotor R. All DCGA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be generally rotated around any axis by any angle by the DCGA rotor operation

$$\mathbf{X}' = R\mathbf{X}R^{\sim}.$$

6.2 DCGA dilator

The DCGA dilator D is defined as

$$D = D_{\mathcal{C}^1} \wedge D_{\mathcal{C}^2}.$$

The CGA dilators for the same dilation operation in CGA1 and CGA2 are wedged as the DCGA dilator D. All DCGA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be dilated by the DCGA dilator operation

$$\mathbf{X}' = D\mathbf{X}D^{\sim}.$$

Keep in mind that dilation also dilates the position of an entity, which may cause an unexpected translational movement. To *scale* an entity, it should be translated to be centered on the origin, dilated around the origin, and then translated back.

6.3 DCGA translator

The DCGA translator T is defined as

$$T = T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2}.$$

The CGA translators for the same translation operation in CGA1 and CGA2 are wedged as the DCGA translator T. All DCGA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be translated by the DCGA translator operation

$$\mathbf{X}' = T\mathbf{X}T^{\sim}.$$

6.4 DCGA motor

The DCGA motor M is defined as

$$M = M_{\mathcal{C}^1} \wedge M_{\mathcal{C}^2}.$$

The CGA motors for the same motion operation in CGA1 and CGA2 are wedged as the DCGA motor M. All DCGA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be moved by the DCGA motor operation

$$\mathbf{X}' = M\mathbf{X}M^{\sim}.$$

6.5 DCGA intersection

Although not rigorously proved here, the intersection tests performed by this author supported the following claims given in this subsection about DCGA intersection. Detailed examinations of ellipsoid-plane and ellipsoid-sphere intersections are shown in Figures 14 and 15. These claims should be considered preliminary, and require additional research to prove for certain what intersections are valid or invalid.

The *bi-CGA GIPNS* entities are the following:

- DCGA GIPNS 2-vector sphere
- DCGA GIPNS 2-vector plane
- DCGA GIPNS 4-vector line
- DCGA GIPNS 4-vector circle.

The DCGA GIPNS *intersection* entity $\mathbf{X}$ is the wedge of multiple bi-CGA GIPNS entities and (optionally) up to *one* more DCGA GIPNS entity that is not a bi-CGA GIPNS entity. Only one quadric surface DCGA GIPNS entity can be included in a wedge that forms an intersection entity. Unfortunately, the quadric surface entities cannot be intersected directly with each other by wedge products. These claims are summarized by the following definition.

The DCGA GIPNS *intersection* entity $\mathbf{X}$ of grade $k \leq 8$ is defined as

$$\mathbf{X}_{\langle k \leq 8 \rangle} = \begin{cases} \text{DCGA GIPNS entity} \wedge (\bigwedge_{i=1}^{n} \text{bi-CGA GIPNS entity}_i) \\ \bigwedge_{i=1}^{n} \text{bi-CGA GIPNS entity}_i. \end{cases}$$

The maximum grade for a valid intersection entity is grade 8. The grade of the wedges is divisible by 2, making the next grade above 8 to be 10, proportional to the DCGA unit pseudoscalar $\mathbf{I}_{\mathcal{D}}$. No valid entity is a pseudoscalar.

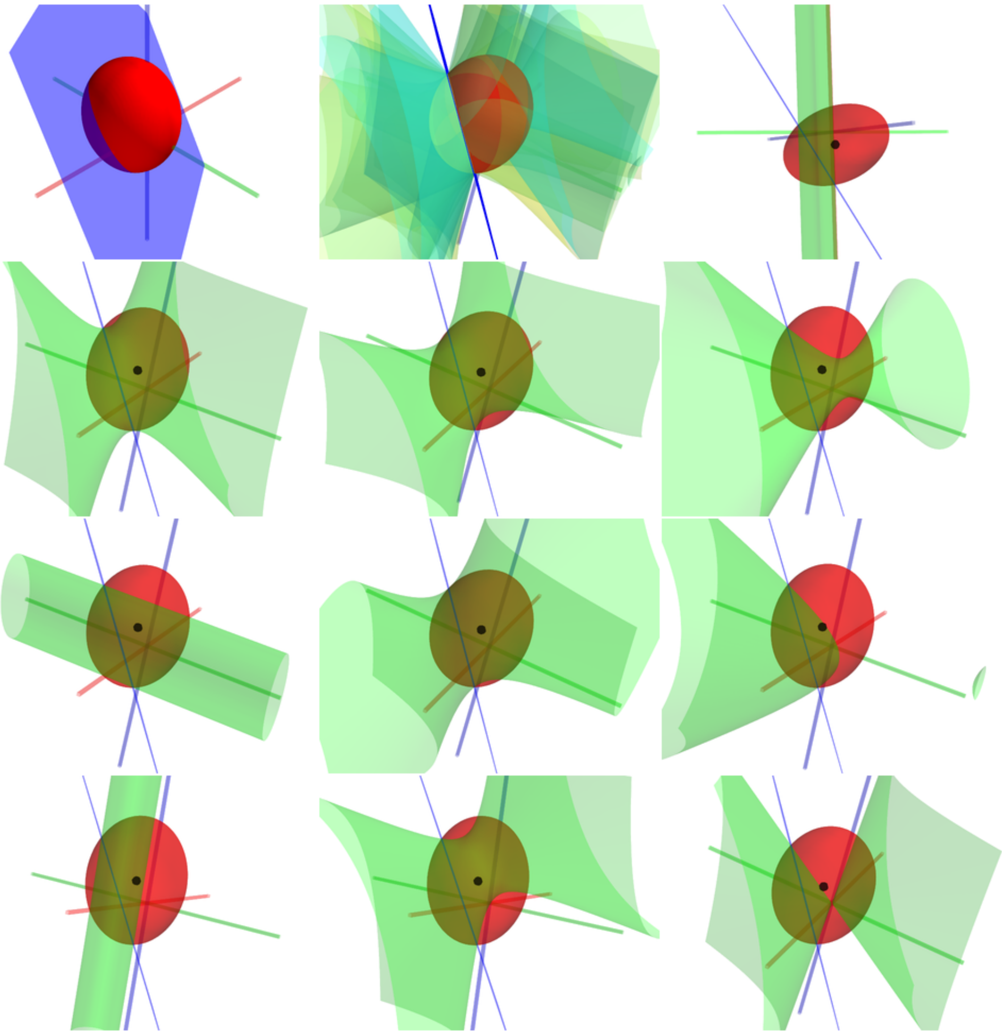

Figure 14. Intersection of ellipsoid and plane in general positions

Figure 14 shows the details of a DCGA GIPNS intersection entity representing the intersection of a DCGA GIPNS 2-vector ellipsoid and DCGA GIPNS 2-vector plane, both rotated and translated differently into general positions that have an intersection. The red ellipsoid $\mathbf{E}$ has initial parameters $r_x = 5$, $r_y = 7$, $r_z = 9$, $p_x = 1$, $p_y = -2$, $p_z = 3$, and is then rotated 30° around the blue z-axis. The *Sympy* test code for the ellipsoid was:

```
Rotor(e3,30*pi*Pow(180,-1))*
GIPNS_Ellipsoid(1,-2,3,5,7,9)*
Rotor(e3,30*pi*Pow(180,-1)).rev()
```

The black dot is the ellipsoid center position. The blue plane $\mathbf{\Pi}$ is initially perpendicular to the x-axis through the origin, then transformed according the following code:

```
Rotor(e1,30*pi*Pow(180,-1))*
Translator(-4*e2)*
Rotor(e3,-60*pi*Pow(180,-1))*
GIPNS_Plane(e1,0)*
Rotor(e3,-60*pi*Pow(180,-1)).rev()*
Translator(-4*e2).rev()*
Rotor(e1,30*pi*Pow(180,-1)).rev()
```

Their DCGA GIPNS intersection is $\mathbf{X} = \mathbf{E} \wedge \mathbf{\Pi}$. The various images in Figure 14 show components of $\mathbf{X}$ that represent other surfaces that are all coincident with the intersection of the ellipsoid and plane. There were ten unique components in $\mathbf{X}$. These components are cylinders, hyperboloids, and a cone. The intersection entity $\mathbf{X}$ represents the locus of points that are simultaneously located on all ten of these surfaces, which is an ellipse-shaped intersection of the ellipsoid and plane.

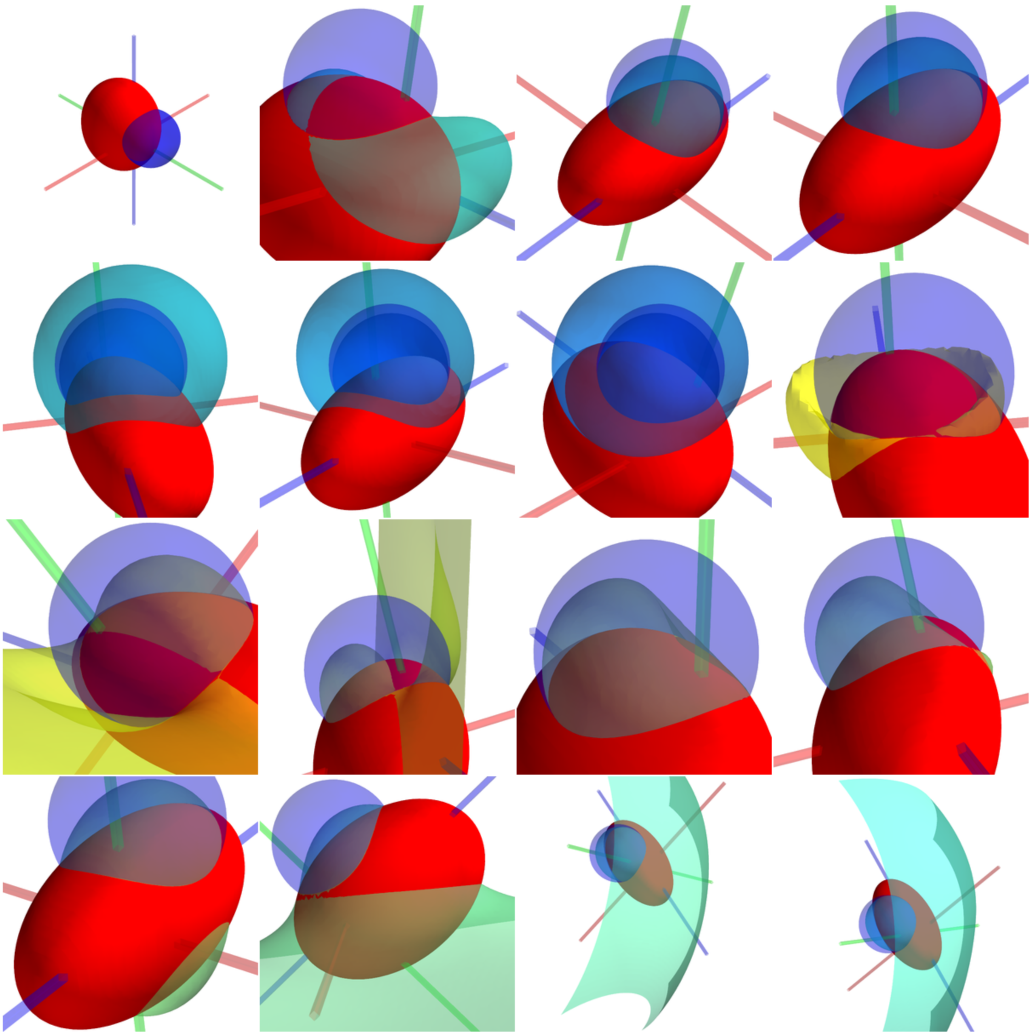

Figure 15. Intersection of ellipsoid and sphere in general positions

Figure 15 shows the same red DCGA GIPNS ellipsoid $\mathbf{E}$ as in Figure 14, but now intersected with a blue DCGA GIPNS sphere $\mathbf{S}$ of radius $r=5$ at position $\mathbf{e}_1+5\mathbf{e}_2+3\mathbf{e}_3$. The DCGA GIPNS intersection entity is now $\mathbf{X}=\mathbf{E}\wedge\mathbf{S}$. The shape of the intersection appears like a curved ellipse or curved circle. The components of the entity $\mathbf{X}$ represent 15 other unique surfaces that are also coincident with the intersection of $\mathbf{E}$ and $\mathbf{S}$. The images of Figure 15 show how each of these 15 surfaces intersect with the intersection of $\mathbf{E}$ and $\mathbf{S}$. Some of these surfaces are unusually shaped, and some have two sheets. The DCGA GIPNS intersection entity $\mathbf{X}$ represents the simultaneous locus or intersection of all of the involved surfaces and appears to be a valid intersection entity for the ellipsoid and sphere.

The DCGA GIPNS quadric surface entities, of the types not available in CGA, could *not* be wedged with each other to form valid intersection entities - *incorrect* or *invalid* intersection entities resulted from their wedge. As a curiosity, it was noticed that the sum and the difference of two intersecting DCGA GIPNS quadric surface entities represent two more coincident intersecting surfaces.

6.6 DCGA dualization

The DCGA unit pseudoscalar $\mathbf{I}_{\mathcal{D}}$ is defined as

$$\begin{aligned}\mathbf{I}_{\mathcal{D}} &= \mathbf{I}_{\mathcal{C}^1}\wedge\mathbf{I}_{\mathcal{C}^2}\\ &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6\mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\end{aligned}$$

and is the DCGA dualization operator on all DCGA entities.

Properties of $\mathbf{I}_{\mathcal{D}}$ include

$$\begin{aligned}\mathbf{I}_{\mathcal{D}}^{\sim} &= (-1)^{10(10-1)/2}\mathbf{I}_{\mathcal{D}}=-\mathbf{I}_{\mathcal{D}}\\ \mathbf{I}_{\mathcal{D}}^2 &= -\mathbf{I}_{\mathcal{D}}\mathbf{I}_{\mathcal{D}}^{\sim}=-1\\ \mathbf{I}_{\mathcal{D}}^{-1} &= \mathbf{I}_{\mathcal{D}}^{\sim}=-\mathbf{I}_{\mathcal{D}}.\end{aligned}$$

According to the sign rule for the commutation of the inner product of two blades $(-1)^{r(10-1)}$, the DCGA pseudoscalar $\mathbf{I}_{\mathcal{D}}$ commutes with blades of even grade r, such as the DCGA 2-vector points, DCGA GIPNS 2,4,6,8-vector surfaces, and their dual DCGA GOPNS surfaces.

A DCGA GIPNS k-vector surface entity $\mathbf{X}$ is *dualized* into its *dual* DCGA GOPNS $(10-k)$-vector surface entity $\mathbf{X}^{*\mathcal{D}}$ as

$$\mathbf{X}^{*\mathcal{D}} = \mathbf{X}/\mathbf{I}_{\mathcal{D}}=-\mathbf{X}\cdot\mathbf{I}_{\mathcal{D}}$$

A DCGA GOPNS k-vector surface entity $\mathbf{X}^{*\mathcal{D}}$ is *undualized* into its *undual* DCGA GIPNS $(10-k)$-vector surface entity $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{X}^{*\mathcal{D}}\mathbf{I}_{\mathcal{D}}=\mathbf{X}^{*\mathcal{D}}\cdot\mathbf{I}_{\mathcal{D}}.$$

This definition of dual and undual preserves the sign on the entities, otherwise the dual applied twice changes signs.

It is understandable that many authors may call the GIPNS entities *dual* and the GOPNS entities *standard*, but since in DCGA we cannot wedge DCGA points into all of the GOPNS entities, the GIPNS entities are considered the *standard* or *undual* entities and the GOPNS entities are the *dual* entities. Most of the DCGA GOPNS entities can only be obtained by the dualization operation as duals.

7 Conclusion

The $\mathcal{G}_{8,2}$ *Double Conformal Geometric Algebra* (DCGA) that has been presented, and possibly introduced for the first time, in this paper may be an interesting algebra for future research or for some applications now.

DCGA provides entities for all the surfaces available in CGA, and DCGA also has entities for all quadric surfaces. DCGA also provides a toroid entity which may be a new entity not previously available in $\mathcal{G}_{4,1}$ CGA nor in $\mathcal{G}_{6,3}$ QGA. DCGA has a complete set of entity transformation operations as versor operations that can transform both the GIPNS and GOPNS forms of all entities, including the toroid. The available versors are rotor, dilator, translator, and motor. DCGA supports the creation of GIPNS intersection entities as the wedge of intersecting GIPNS entities, but this support is limited to intersecting up to a single quadric surface entity or toroid with some combination of planes, spheres, circles, and lines not exceeding a combined grade of 8.

Although not yet tested by this author, the possible extension of $\mathcal{G}_{8,2}$ DCGA to a $\mathcal{G}_{12,3}$ *Triple* or $\mathcal{G}_{16,4}$ *Quadruple Conformal Geometric Algebra* may be theoretically feasible, and may allow for general cubic and quartic surface entities.

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