Entropy Gradient Maximization for Local-Ergodic Systems

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Abstract

As the further development of the local formalism of entropy gradient maximization (EGM), the construction of weak maximization condition (local ergodicity) is established.

The concerning mathematical techniques for generation and interpretation of dynamic equations for system consisting of two degrees of freedom are consequently elaborated.

An application of the method is discussed and illustrated with standard examples of mechanics.

1 Introduction

The local formalism of entropy gradient maximization (EGM) [1] recently proposed as an alternative way to generate the dynamical equations for physical systems with arbitrary number of degrees of freedom (DoF's)

On the one hand, it eliminates the classical inconsistencies of conventional formalisms, on the other hand it provides no contradiction to their outcomes, since the set of classical solutions of second-order dynamic equations are the restriction of solution space of the first-order ones resulting from the formalism.

An additional feature of the formulation of EGM, amongst others, is a possibility to impose the additinal contition - *ergodicity* as a direct function of infinitesimal variations, since we carry out the maximization on the space of variations of arguments.

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This option allows to distinguish this kind of *weak* or *local* ergodicity from the usual one, called in this context *strong* or *global*, as it was shortly mentioned in the first report [1].

This investigation was focused on the main features of the formalism, considered there in preassumption of *global*-ergodic systems. In the present note we put an attention on the alternative possibility to impose an ergodicity condition as a *local* one.

It will be shown, that this approach works successfully, despite its unusual form using differentials of variables as arguments instead of the variables itself.

Conditions of such kind reveal the principal difference from the usual (global) ergodic equations, and appear in fact as a generalization of them.

In particular this approach leads to, generally speaking, different dynamical equations producing also different classes of solutions respectively, as it will be shown by comparing the outcomes in examples supported below.

Starting with a brief recall of the postulates of the entropy gradient maximization concept, we highlight the case of local ergodicity and illustrate it with simple problems, well known in a framework of the usual approach.

The examples discussed below suggest, that the generalization of ergodicity to a local approach is a promising way for extension of available classes of dynamic systems as well as of the variety of possibilities to construct such systems.

1.1 Formulation of the *entropy-gradient-maximization*

We recall briefly the local formalism of entropy gradient maximization, recently proposed as a way to obtain the dynamical equations, avoding hamiltonian or lagrangian formalism [1].

For a *closed* system with $n+1$ degrees of freedom (DoF's).

$$q_i = \{q_1, q_2, ..., q_{n+1} =: \tau\}$$

which define the state of the system completely, the states $\{q_i\}$ are ordered with respect to increasing values of the scalar function $S(q_i)$- entropy - a scalar field on the space $\{q_i\}$

The main proposition of this formalism is to maximize the entropy variation δS around the given state $\overline{q} = \{q_1, q_2, ..., q_{n+1} = \tau\}$ instead of the conventional maximization of the entropy function itself (usually used to find stationary states):

$$\begin{aligned} \delta S(q_i) &= \sum_{i=1}^{n} S_{q_i} dq_i + S_\tau d\tau \\ &+ \frac{1}{2}\left[\sum_{i,k=1}^{n} S_{q_i q_k} dq_i dq_k + S_{\tau\tau} d\tau^2\right] \\ &+ \sum_{i=1}^{n} S_{q_i \tau} dq_i d\tau + \frac{1}{3!}(higher \;\; orders) \; ... \end{aligned} \quad (1)$$

where the degree of freedom τ is any of the degrees of freedom $\{q_1, ..., q_{n+1}\}$, e.g. the q_{n+1}, which satisfies the *time-eligibility condition*: it means, for a chosen DoF $q_k =: \tau$ with the values $\tau_1, \tau_2, \; ... \; \tau_i \; ...$ ordered so that:

$$S(q_i, \tau_1) \leq S(q_i, \tau_2) \leq ... \leq S(q_i, \tau_n) \leq S(q_i, \tau_{n+1}) \leq \; ...)$$

$$\text{for } \tau_1 < \tau_2 < \; ... \; < \tau_n < \tau_{n+1} < ..., \text{ for discrete set, or}$$

$S(q_i, \tau)$ is a monotonic non-decreasing function of τ:

$$\frac{\partial}{\partial \tau} S := S_\tau \geq 0, \quad (2)$$

if the values $\tau_1 < \tau_2 < \; ... \; < \tau_n < \tau_{n+1}$ form a continuum.

The DoF τ obeying is called "time-eligible" and can be used as a time-reference degree of freedom (or simply "time ") for the system

$$\{q_i\}, i = 1 \; ... \; n$$

The case of strong inequality for entropy " ¡ " instead of " $\geq$" corresponds to the *strong time eligibilty* of τ.

The maximization of the entropy variation (1) with the additional condition - *ergodicity*

$$h(q_i, \tau) = \varepsilon_0 \quad (3)$$

provides:

- the first order differential equation (system of equations) for evolution of the system dq_i with respect to $d\tau$ called - the dynamical or *trajectory* equation, which defines the trajectory of the system in the state space $\{q_i, \tau\}$ and
- the set of second order differential inequalities for $S(q_i, \tau)$ and/or $h(q_i, \tau)$

Thus the ergodicity condition plays the role of a local metrics on the $n+1$-dimensional state space $\overline{q} = \{q_i, \tau\}$.

2 Local-ergodic systems

2.1 Global and local ergodicity

Consider the first order variation of entropy $S(q_i, \tau)$ in the point $\{q_i, \tau\}$.

$$\delta S(q_i, \tau) = S_\tau d\tau + \sum_i S_{q_i} dq_i \tag{4}$$

(values of partial derivatives are taken in the point $\{q_i, \tau\}$)
Here, the **variation of entropy is** considered as a **function of the variation vector** $d\{q_i, \tau\} = \{d\tau, dq_i\}$ about the chosen point q_i, τ in the state space.

$$\delta S = \delta S(q_i, \tau \mid d\tau, dq_i).$$

Thus, the components $\{d\tau, dq_i\}$ are considered as arguments, whereas q_i, τ are parameter.

The evolution of state in $\overline{q} = \{q_i, \tau\}$ occurs in the direction, where the entropy variation $\delta S(q_i, \tau)$ maximizes; with $\{dq_i, d\tau\}$ - the variation vector obeying the additional relation constraining $(dq_i, d\tau)$ - *ergodicity condition*:

$$h(q_i, \tau | dq_i, d\tau) - \varepsilon_0 = 0 \tag{5}$$

At this point, two principally different cases can be distinguished

1. $$\delta h \sim \delta S \text{ - strong or } \textit{global ergodicity} \text{ condition}$$

2. $$\delta h << \delta S \text{ - weak or } \textit{local ergodicity} \text{ condition}$$

The remaining case $\delta h >> \delta S$ provides no ergodicity condition.

The condition in the first case defines the invariant of the state space and corresponds e.g. to a conventional conservation law, as it has been primary introduced and considered in [1].

In the nexti sections we consider in detail the less usual but most interesting second case of the *local ergodicity*

$$h(q_i,\tau|dq_i,d\tau) = h_{q_i,\tau}(dq_i,d\tau) = \varepsilon_0. \tag{6}$$

It corresponds to situations when the entropy changes sufficiently sensitiver to $dq_i,\ d\tau$ as the ergodic function h.

It means, the parameter ε_0 remains approximately conserved. In this sense the condition 2.can be interpreted as an invariant of a local state variation.

For example, for a quantum system in a certain state $Q = Q_i$ with the transition probabilities $P(Q \to q^k) =< q^k|Q >$ in all possible states q^k (inclusive the Q itself),the normalisation of probability

$$h(Q,q) = \sum_k P(Q \to q^k) = 1$$

can be used as a local ergodicty condition.

2.2 Local ergodicity in canonical systems with two DoF's in the 1 order-variation formalism

Here we consider some simplest classical examples with the ergodicity condition replaced by a local one.

We start like [1] with the case of the system possessiing only two degrees of freedom $\{q = x, \tau\}$

Example 1a: *Local-free motion*

In the problem of the entropy gradient maximization we search for a variation vector $\{dx, d\tau\}$ maximizing the first order variation of entropy

$$\delta S(x,\tau) = S_\tau d\tau + S_x dx \tag{7}$$

$$h = h(dx,d\tau) = \frac{m}{2}\left[\frac{dx}{d\tau}\right]^2 = \varepsilon. \tag{8}$$

This condition with the positive solution for

$$\frac{dx}{d\tau} = +\sqrt{2m\varepsilon}$$

is equivalent to the ergodicity condition

$$h(dx, d\tau) = \frac{dx}{d\tau} = p$$

like the 1D local momentum for a unit mass, in opposition to a global conservation law $\frac{x}{\tau} = p$

The detailed application of the formalism, e.g. with the lagrangian multiplier λ like the introducing example in [1], provides

$$\begin{aligned} \frac{\partial}{\partial\ dx}\left[S_x dx + S_\tau d\tau + \lambda\left(\frac{dx}{d\tau} - p\right)\right] &= S_x + \lambda\frac{1}{d\tau} = 0 \\ \frac{\partial}{\partial\ d\tau}\left[S_\tau d\tau + \lambda\left(\frac{dx}{d\tau} - p\right)\right] &= S_\tau - \lambda\frac{dx}{d\tau^2} = 0 \end{aligned} \tag{9}$$

Then, the trajectory equation reads entirely:

$$\frac{dx}{d\tau} \equiv \dot{x} = p(x,\tau) = -\frac{S_\tau}{S_x}$$

and the causality contition is fulfilled identically (see the example 2 below).

Example 1b: *Local oscillator*

The next example of the standard template list is the local version of a harmonic oscillator with a generalized (not necessary a space) coordinate q

$$\delta S(q,\tau) = S_\tau d\tau + S_q dq \tag{10}$$

$$h = h(dq_i, d\tau) = dq^2 + \frac{m}{2}\left[\frac{dq}{d\tau}\right]^2 = \varepsilon, \tag{11}$$

instead of a global case, which would be $h = q^2 + \frac{m}{2}\left[\frac{dq}{d\tau}\right]^2$.

The procedure like that outlined above is applied as follows:

$$\begin{aligned} S_q + \lambda \left(2\ dq + m\frac{dq}{d\tau^2} \right) &= 0 \\ S_q + \lambda \left(-m\frac{dq^2}{d\tau^3} \right) &= 0 \end{aligned}$$

providing with $\dot{q} := \frac{dq}{d\tau}$:

$$-2\left(d\tau^2 + \frac{m}{2} \right) = m\dot{q}\frac{S_q}{S_\tau}$$

and with the ergodicity

$$\dot{q}^2 \left(d\tau^2 + \frac{m}{2} \right) = \varepsilon$$

leads finally to the evolution trajectory determined by the nonlinear 1-order ODE

$$\frac{S_q}{S_\tau}\dot{q}^3 = -\frac{2\varepsilon}{m},$$

which is very diffrerent from the canonical 2-order form $\ddot{q} + \omega^2 q = 0$

Example 2: *Local hamiltonian system*

The generalized system with the local ergodicity condition

$$h(dx, d\tau) = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + U(dx) = \varepsilon_0$$

(1D mechanical energy conservation),
and the entropy variation

$$\delta S(x, \tau) = S_\tau d\tau + S_x dx,$$

includes the both of the above examples as particular cases.

The formalism provides the equation for evolution trajectory in the form:

$$U^{-1}\left(\varepsilon_0 - \frac{m\dot{x}^2}{2} \right) = -\frac{m\dot{x}^2}{S_\tau U'(dx)}(S_\tau + \dot{x}S_x)$$

or

$$\varepsilon_0 - \frac{m\dot{x}^2}{2} = U\left[-\frac{m\dot{x}^2(S_\tau + \dot{x}S_x)}{S_\tau U'(dx)}\right] \tag{12}$$

where

$$U'(dx) = U'(\dot{x}d\tau) = U'\left[U^{-1}\left(\varepsilon_0 - \frac{m\dot{x}^2}{2}\right)\right]$$

Especially:

2.1 For $U(dx) = 0$ this case is equivalent to the previous one.

$$\dot{x} = v = -\frac{S_\tau}{S_x}, \quad S_\tau > 0$$

for the trajectory equation.

The causality condition

$$\begin{vmatrix} h_{dxdx} & h_{dxd\tau} \\ h_{d\tau dx} & h_{d\tau d\tau} \end{vmatrix} \leq 0 \Rightarrow \begin{vmatrix} 0 & -\frac{1}{d\tau^2} \\ -\frac{1}{d\tau^2} & \frac{2dx}{d\tau^3} \end{vmatrix} \leq 0$$

is fulfilled automatically.

2.2 For the special case $U(dx) = dx^2$ (local oscillator) the formula (12) provides:

$$\frac{2\varepsilon}{m\dot{x}^2} = \pm\left(\dot{x}\frac{S_x}{S_\tau} + 1\right) + 1$$

which reproduces the result of the **example 1b** in the (also physically meaningful) case of the lower sign "-" . The case of the upper sign "+"

$$\frac{2\varepsilon}{m\dot{x}^2} = \dot{x}\frac{S_x}{S_\tau} + 2$$

can be interpreted as a pointer to the possible presence of a rest energy and belongs together with the analysis of causality, which is postponed here for the next report.

As a short note concerning this thema, we perform the causality condition for the local oscillator, which reads here:

$$\begin{vmatrix} h_{dxdx} & h_{dxd\tau} \\ h_{d\tau dx} & h_{d\tau d\tau} \end{vmatrix} = \begin{vmatrix} \left(2 + \frac{m}{d\tau^2}\right) & -2m\frac{\dot{x}}{d\tau^2} \\ -2m\frac{\dot{x}}{d\tau^2} & 3m\frac{\dot{x}^2}{d\tau^2} \end{vmatrix} \leq 0,$$

resulting finally in

$$\frac{\varepsilon}{T} \leq \frac{4}{3}, \quad T := \frac{m\dot{x}^2}{2} - \text{ generalized kinetic energy,}$$

while the ε can only be intrepreted as a total energy of the oscillator, if the coefficient at dx^2 is the corresponding $\frac{U'}{2}$.

3 Comparison and intermediate forms

By means of the local form of ergodicity outlined above, we can remark, that the global ergodicity can be viewed entirely as a particular form of the local one:

$$h(q,\tau) = \varepsilon; \quad \delta h := h_{loc}(q,\tau;\ dq, d\tau) = h_q dq + h_\tau d\tau = 0 \tag{13}$$

For example, the strictly global hamiltonian function of x, τ

$$h(x,\tau) = U(x) + \frac{m}{2}\frac{x^2}{\tau^2} = \varepsilon; \quad \left(U'(x) + \frac{mx}{\tau^2}\right) dx - \frac{mx^2}{\tau^3} d\tau = 0 \tag{14}$$

provides the trajectory equation together with relation to the entropy structure [1] as:

$$-\frac{S_\tau}{S_x} = \dot{x} = \frac{dx}{d\tau} = \frac{mx^2}{U'(x)\tau^3 + mx\tau}$$

In particular for a harmonic potential $U(x) = \frac{kx^2}{2}$ we obtain

$$\dot{x} = x\frac{1}{\tau\left(\frac{k}{m}\tau^2 + 1\right)} = x\frac{1}{\tau\left((\omega\tau)^2 + 1\right)}$$

with the solution

$$x(\tau) = \frac{C}{\sqrt{\omega^2 + 1/\tau^2}},$$

quite different from the known harmonic oscillations $C_+ e^{i\omega t} + C_- e^{-i\omega t}$.

Historically, the dynamical equations and ergodicity conditions respectively, are formulated in terms of coordinates x and local velocities $v = \frac{dx}{dt}$ rather than x/t.

In this way we try to use the mixed approach, a kind of an intermediate form between global and local ergodicity

$$h(x,\tau;\ dx, d\tau) = U(x) + U'(x)dx + \frac{m}{2}\frac{dx^2}{d\tau^2} = \varepsilon; \tag{15}$$

which corresponds to the conventional Hamilton function and can be called therefore *hamiltonian ergodicity*.

For this condition the formalism provides:

$$\frac{S_x}{S_\tau} m\dot{x}^2 = -U'(x)d\tau - m\dot{x}$$

$$U'(x)d\tau = \frac{\varepsilon - U(x) - \frac{m\dot{x}^2}{2}}{\dot{x}}$$

what results in the evolution equation.

$$\frac{m\dot{x}^2}{2} + \frac{S_x}{S_\tau} m\dot{x}^3 + \varepsilon - U(x) = 0 \tag{16}$$

Here two important remarks compared to the global ergodicity case, are in order:

- The dynamical equation contains now the entropy function, i.e. the entropy and ergodic function are in general not discoupled anymore, as in the global case. It means, for any conventional dynamical equation (for example newtonian)

  $$m\ddot{x} = -U'(x)$$

  a such restriction on the special class of entropy functions $S(x, \tau)$ exists, that the entropy gradient maximization (EGM) reproduces the conventional results. In terms of dynamical equations, it comes about if the first order equation of EGM is the first integral of the second order conventional one.

  For example, for a harmonic potential (example 1b) considered above, the equation with a unit mass (16) is then the first integral of the conventional

  $$\ddot{x} + x = 0 \tag{17}$$

  if the entropy $S(x, \tau)$ satisfies

  $$\frac{\frac{d}{d\tau}\left[\frac{S_x}{S_\tau}\right]\dot{x}^2}{3\frac{S_x}{S_\tau}\dot{x} + 1} = \frac{x}{m}$$

for any solution of 17, fro example with the solution $x(\tau) = \cos\tau$ the entropy obeys

$$\tan\tau \frac{d}{d\tau}\frac{S_x}{S_\tau} = 3\frac{S_x}{S_\tau} + \frac{1}{\sin\tau}$$

(a kind of a "step-structure")

- The **entropy** construction in form of the function $\frac{S_x}{S_\tau}\dot{x}$ **contributes to** the inertial *mass*. It gives rise for the suggestion, it is a formulation possible, where the ergodicity condition does not contain any mass parameter at all and the issue of the inertial mass is displaced to the entropy structure.

4 Conclusions

A generalization of the additional EGM condition to a local form supplies a sufficient extension of dynamic systems available for this formalism as well as the variety of possible solutions, as it has been shown in this note

A comparison with the existent result in the framework of EGM formalism leads entirely to the restriction of this variety for special classes of entropy functions.

In this way there is a possibility to relate the issue of the mass to the entropy structure, i.e. the statistical structure of the state space

In simple cases an explicit form of these functions is available; in particular, for the case of *discoupled time* (or any other, which admits transformations to such classes)

Since this discussion requires an introduction and explanation of further related concepts, it would be preferable to postpone it for the next report [2], together with the related causality analysis, which has not been performed in the present note.

References

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