Approximation of exp(x) deduced from the implicit Euler numerical solution of first order linear differential equations

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Abstract

In this paper it is shown a simple approximation of the function exp(x) for positive values of x, deduced from the implicit Euler numerical solution of first order lineal differential equations (ODE). The results show that the approximation has an error of less than 10% for exp(x) when x < 0.35 and for exp(-x) when x < 0.5, which is acceptable for many engineering applications, and helps facilitate the analysis of some systems without the use of computers.

Keywords: approximation, Euler implicit method, exponential function, first order ODE.

1. Introduction

This work was done with two objectives in mind: 1) to find a simple approximation for the exponential function which could be used to solve several problems without the use of a calculator which has more operations than addition, subtraction, multiplication and division; 2) to show the usefulness of numerical methods in finding approximations for functions without having to enter into the realm of mathematical analysis, which is an area not many people can handle easily.

2. Deduction of the approximation

Consider one of the simplest first order linear ODE:

$$\frac{du}{dx} = u \quad (1)$$

With initial value $u(0) = u_0$ and for $x \geq 0$. The analytical solution for (1) is found by direct integration:

$$u = e^{x}u_0 \quad (2)$$

Now, consider the Euler implicit method [1] for the numerical (and thus approximate) solution of (1), where we consider the previous step value of u as the initial value of the problem and the independent variable step as x, then:

$$\frac{u - u_0}{x} = u \quad (3)$$

This gives the approximate solution of (1):

$$u = \frac{1}{1 - x} u_0 \quad (4)$$

Comparing (2) and (4) we see that:

$$\boxed{e^x \approx \frac{1}{1 - x}} \quad (5)$$

Now consider the differential equation that describes heat exchange from a body of uniform temperature u to a surrounding medium at temperature u_∞:

$$\frac{du}{dt} = k(u_\infty - u) \quad (6)$$

With initial value $u(0) = u_0$ and for $t \geq 0$. The analytical solution of (6) by variable separation is:

$$u = (1 - e^{-kt})u_\infty + e^{-kt}u_0 \quad (7)$$

The implicit Euler method for (6) is:

$$\frac{u - u_0}{t} = k(u_\infty - u) \quad (8)$$

And thus the approximate solution for (6) is:

$$u = \frac{kt}{1 + kt} u_\infty + \frac{1}{1 + kt} u_0 \quad (9)$$

Comparing each term of (7) and (9) –remembering that (9) is an approximate solution- and making $x = -kt$ we arrive again at:

$$\boxed{e^x \approx \frac{1}{1 - x}} \quad (10)$$

3. Validity of the approximation

The curves of the exponential functions with different arguments and their corresponding approximations and the errors of those are shown in Figures 1 to 8.

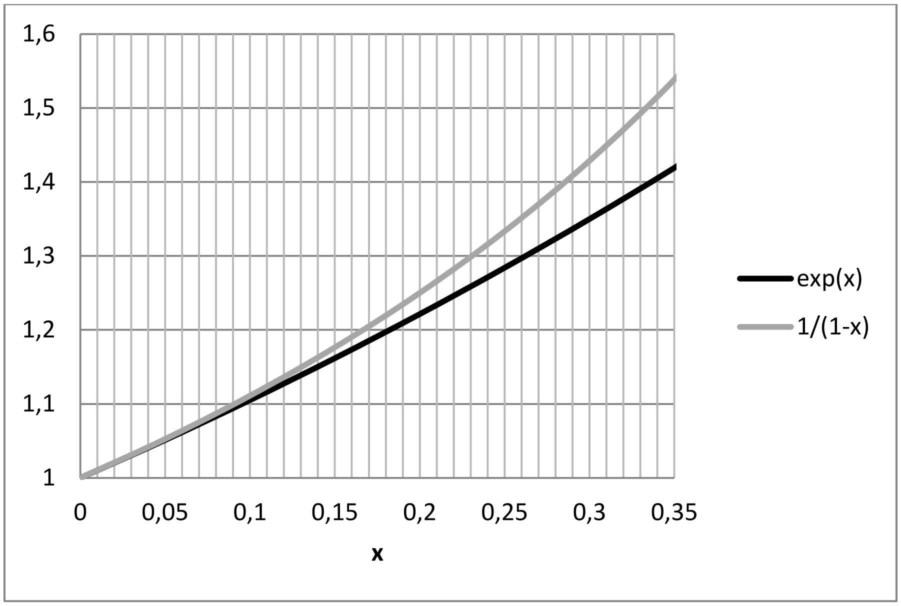

Figure 1. Curves of $f(x) = e^x$ and the approximation.

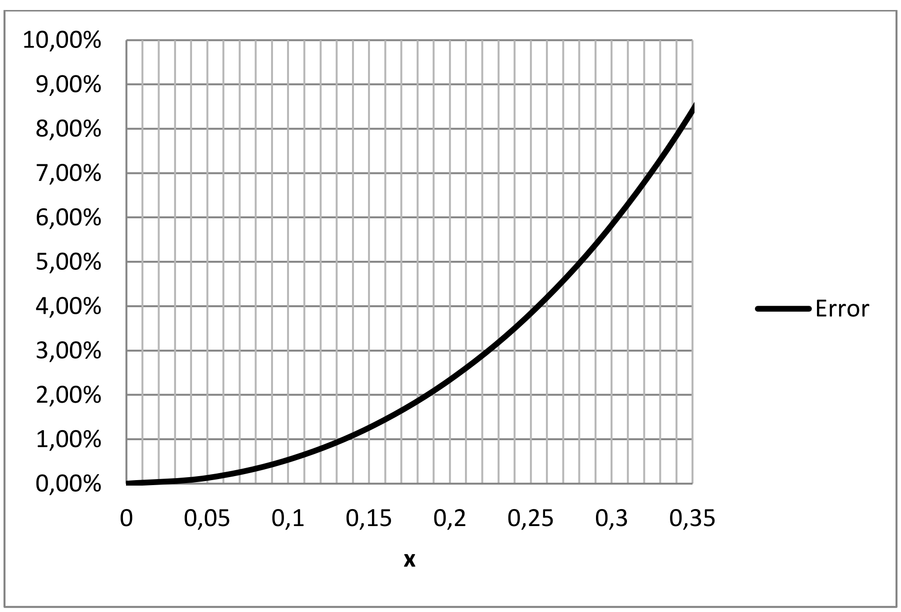

Figure 2. Error of the approximation of $f(x) = e^x$.

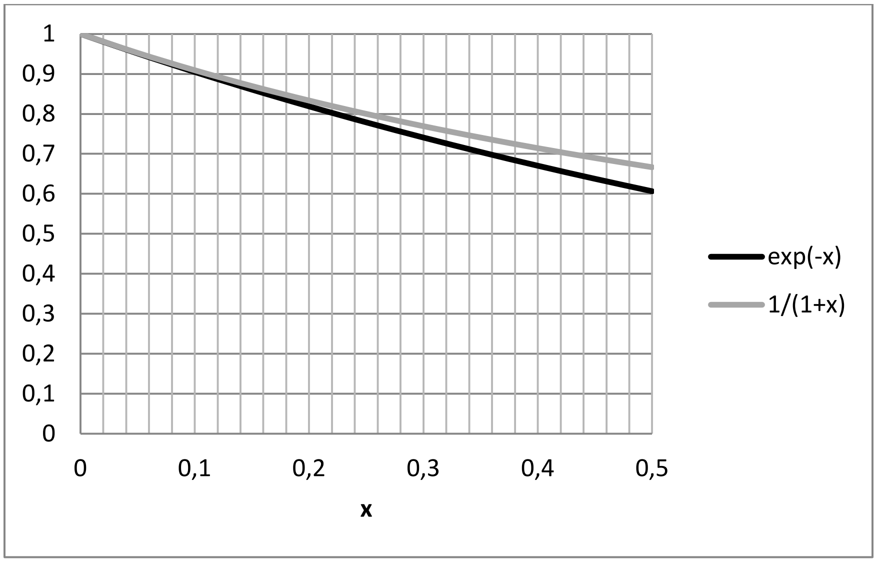

Figure 3. Curves of $f(x) = e^{-x}$ and the approximation.

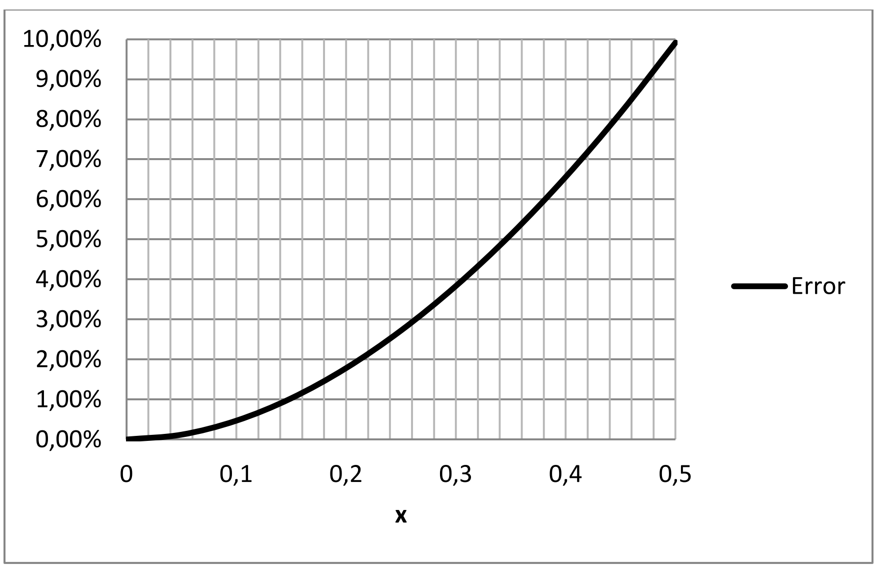

Figure 4. Error of the approximation of $f(x) = e^{-x}$.

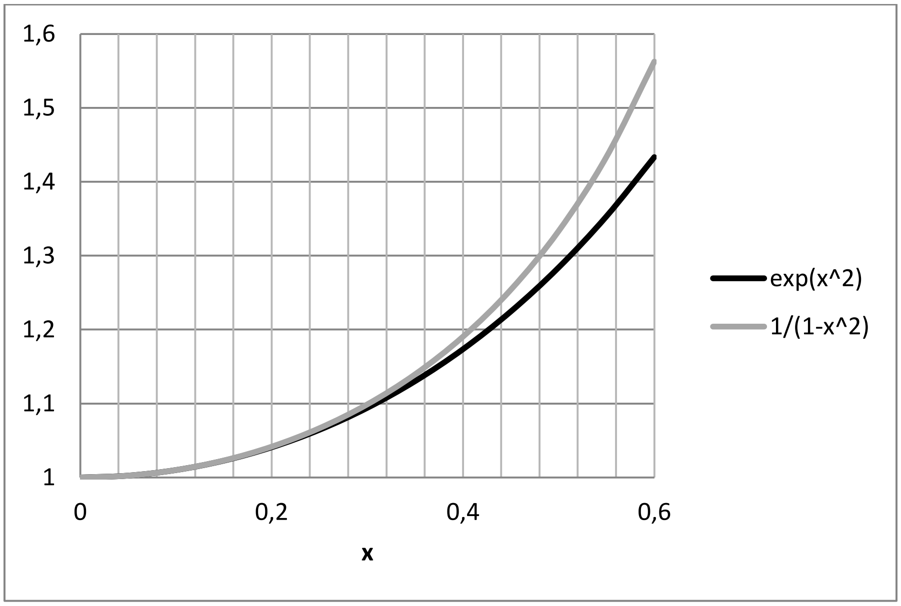

Figure 5. Curves of $f(x) = e^{x^2}$ and the approximation.

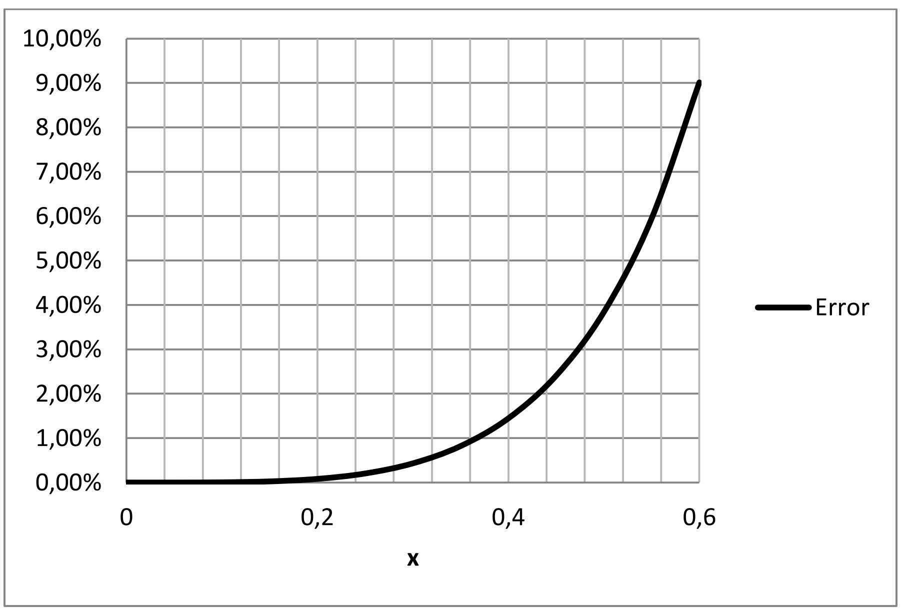

Figure 6. Error of the approximation of $f(x) = e^{x^2}$.

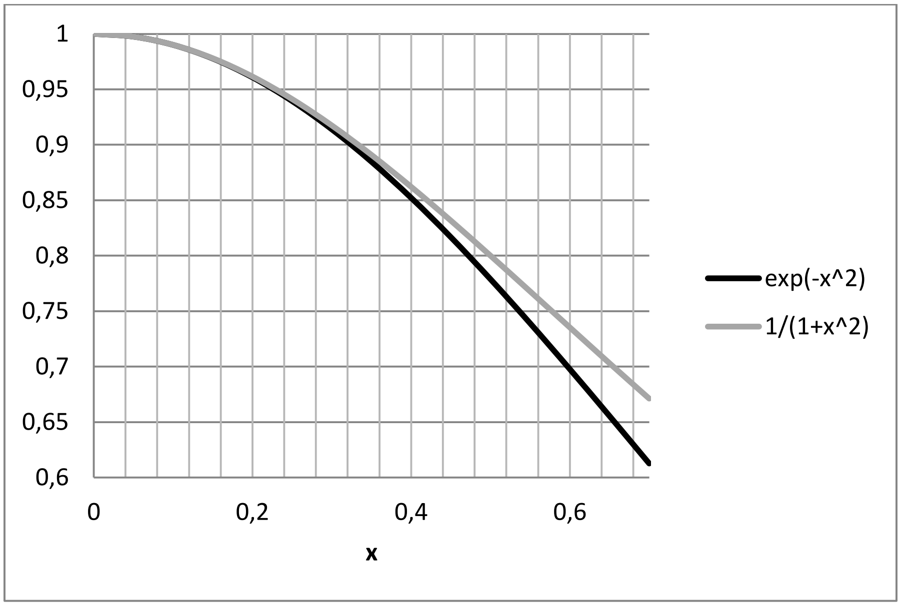

Figure 7. Curves of $f(x) = e^{-x^2}$ and the approximation.

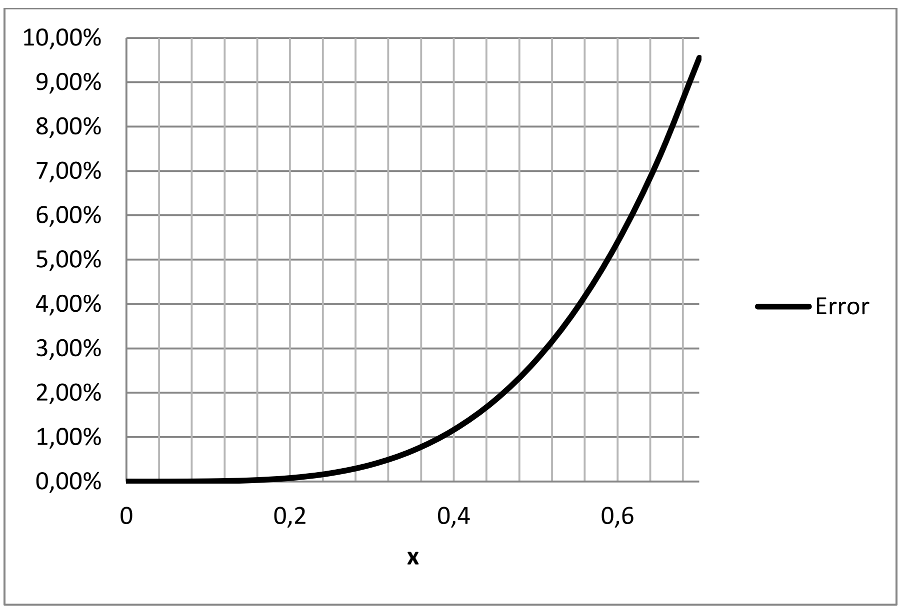

Figure 8. Error of the approximation of $f(x) = e^{-x^2}$.

The approximation has an error of less than 10% for exp(x) when $x < 0.35$ and for exp(-x) when $x < 0.5$; for exp(x^2) and exp(-x^2) the error is less than 10% for $x < 0.6$ and $x < 0.7$ respectively.

4. Conclusions

The approximation of the exponential function found by comparing solutions of first order linear ODE with its numerical approximations holds for small values of the independent variable with acceptable error for several engineering and science applications. This approximation is actually a truncated form of the continued fraction expression for exp(x) found by Wall [2]

$$e^x = \cfrac{1}{1 - \cfrac{x}{1 + \cfrac{x}{2 - \cfrac{x}{3 + \cdots}}}} \qquad (11)$$

… but arrived at by completely different means. Statistical analysis done by software of the error curves shown in Figures 2, 4, 6 and 8 shows that the error grows by $O(x^2)$, which concurs with the fact that the approximation for the derivative of a first order ODE in the implicit Euler method is actually a Taylor series truncated in its second term.

5. References

[1] De Castro, C. A. *Métodos numéricos básicos para ingeniería*. Online at http://www.bubok.es/libros/11909/Metodos-numericos-basicos-para-ingenieria

[2] Wall, H. S. *Analytic Theory of Continued Fractions.* New York: Chelsea, 1948.